Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

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1 Introduction

In this paper we study constant mean curvature graphs in $\mathbb{M} \times \mathbb{R}$ where $\mathbb{M} = \mathbb{H}^2$ or $\mathbb{S}^2$ the hyperbolic plane of curvature -1 or the 2-sphere of curvature 1. Let D be a bounded domain in $\mathbb{M}$ with piecewise smooth boundary. We prescribe continuous data on the smooth arcs of boundary D, which may be $+\infty$ on some arcs and $-\infty$ on others. Our purpose is to solve the Dirichlet problem for the constant mean curvature equation in D, assuming the prescribed boundary data.. More precisely, we seek a smooth function u defined in D, whose graph $S = \{(x, u(x)) : x \in D\}$ in $\mathbb{M} \times \mathbb{R}$ (with the standard product metric) has mean curvature H and which takes on the prescribed data on ∂D. The domains D we consider and the geometric properties that guarantee the existence of a solution to this Dirichlet problem, are given in section 7.

When $\mathbb{M} = \mathbb{R}^2$ and H=0, this is the classical theory of Jenkins-Serrin [3]. This theory was extended to $H \neq 0$ by Spruck [9], again for $\mathbb{M} = \mathbb{R}^2$.

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For $H = 0$, the Jenkins-Serrin theory was extended to $\mathbb{M} = \mathbb{H}^2$ by Nelli and Rosenberg [5] and to arbitrary Riemannian surfaces $\mathbb{M}$ by Pinheiro [4]. Our methods also work in this case.

We shall always assume that $H > 0$ with respect to the upward unit normal of S. Then if u tends to $+\infty$ for any approach to a boundary arc γ, then necessarily its curvature $\kappa(\gamma) = 2H$ is constant, while if u tends to $-\infty$ on γ then $\kappa(\gamma) = -2H$ (see Theorem 4.8). Thus we must deal with non-convex domains D with ∂D piecewise C^2 and consisting of three set of open arcs $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ satisfying $\kappa(A_i) = 2H$, $\kappa(B_i) = -2H$ and $\kappa(C_i) \geq 2H$ respectively. The generalized Dirichlet problem is to find a solution of (2.1) in D taking on the boundary values $+\infty$ on the A_i, $-\infty$ on the B_i and arbitrary continuous boundary data on the C_i . A precise definition of the admissible domains D is given in Definition 7.1 of section 7. When the family $\{C_i\}$ is non-empty, we give in Theorem 7.11 necessary and sufficient conditons that there exist a unique solution. These are the generalized Jenkins-Serrin flux conditions (as in [9]) formulated in terms of admissible polygons (see Definiton 7.4 of section 7). When the family $\{C_i\}$ is empty we give in Theorem 7.12 necessary and sufficent conditions for "Scherk type" solutions to exist; these are unique up to a vertical translation (see Theorem 7.13). Examples of this type are constructed for $\mathbb{H}^2$ for $H \leq \frac{1}{2}$ in section 8.

The proof of these results is long and involved and essentially follows that of [9] using the existence theory and interior gradient estimates of [10] as basic tools. As in all previous proofs, the key idea is to study the flux of monotone increasing and decreasing sequences of solutions along arcs where they diverge. This is carried out in sections 5 and 6. Section 2 contains an important general maximum principle which in conjunction with the special barriers of section 4 allows us to analyse the flux. Section 3 formulates a general Perron existence theorem which is a basic tool for the existence theory of section 7. In some sense, the Scherk type examples with $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ are the most interesting as they can be constructed over arbitrarily large domains. We hope in future work to use these to construct interesting complete graphs.

2 The mean curvature equation and a general maximum principle

Let $ds^2 = \sum_{i,j=1}^{2} \sigma_{ij} dx_i dx_j$ is a local Riemannian metric on $\mathbb{M}$ and denote by (σ^{ij}) the inverse matrix of (σ_{ij}). Then $\mathbb{M} \times \mathbb{R}$ is given the product metric $ds^2 + dt^2$ where t is a coordinate for $\mathbb{R}$ and (see [10]) if $S = \{\text{graph u}\}$ has constant mean curvature H, the height function $u(x) \in C^2(\Omega)$ satisfies the divergence form equation

$$Mu := \text{div} \, \frac{\nabla u}{W} = 2H \ , \ W = (1 + |\nabla u|^2)^{1/2} \tag{2.1}$$

where the divergence and gradient ∇u are taken with respect to the metric on $\mathbb{M}$. Equivalently, equation (2.1) can be written in non-divergence form

$$Mu = \frac{1}{W} \sum_{i,j=1}^{2} g^{ij} D_i D_j u = 2H \ , \tag{2.2}$$

where D denotes covariant differentiation on $\mathbb{M}$ and

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2} \ , \ u^i = \sum_{j=1}^{2} \sigma^{ij} u_j \ .$$

The main result of this section is a general maximum principle for sub and super solutions of the mean curvature operator for boundary data with a finite number of discontinuities. We follow the argument of [9] which holds in the general setting with little change. For the convenience of the reader we sketch the argument which uses the divergence structure.

Lemma 2.1. *Let u^1 and u^2 be functions in $C^2(\Omega)$, $\Omega \subset \mathbb{M}$ and set $W_k = W(\nabla u^k)$, $k = 1, 2$. Then*

$$< \nabla u^1 - \nabla u^2, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} > \quad \geq 0 \tag{2.3}$$

with equality at a point if and only if $\nabla u^1 = \nabla u^2$

Proof. The downward unit normal $\mathcal{N}_i$ to $S_i = \text{graph } u^i$ is given by

$$\mathcal{N}_i = \frac{\nabla u^i}{W_i} - \frac{1}{W_i} e \ , \ \ e = \frac{\partial}{\partial t} \ .$$

Hence using $< W_i \mathcal{N}_i, e >= -1$,

$$\begin{aligned}
< \ & \nabla u^1 - \nabla u^2, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} >=< W_1\mathcal{N}_1 - W_2\mathcal{N}_2, \ \mathcal{N}_1 - \mathcal{N}_2 + (\frac{1}{W_1} - \frac{1}{W_2})e > \\
= \ & < W_1\mathcal{N}_1 - W_2\mathcal{N}_2, \mathcal{N}_1 - \mathcal{N}_2 >= (W_1 + W_2)(1- < \mathcal{N}_1, \mathcal{N}_2 >) \\
= \ & \frac{W_1 + W_2}{2} ||\mathcal{N}_1 - \mathcal{N}_2||^2 \geq 0 \ .
\end{aligned}$$

(Here, the inner products are in the metric of $\mathbb{M} \times \mathbb{R}$.)

Theorem 2.2. *(General maximum principle) Let u^1 and u^2 satisfy $Mu^1 \geq nH \geq Mu^2$ in a bounded domain $D \subset \mathbb{M}$. Suppose that $\liminf(u^2 - u^1) \geq 0$ for any approach to ∂D with the possible exception of a finite number of points of ∂D. Then $u^2 \geq u^1$ with strict inequality unless $u^2 \equiv u^1$.*

Proof. Let N and ϵ be positive constants with N large and ϵ small. Define

$$\varphi = \begin{cases} N - \epsilon & \text{if } u^1 - u^2 \geq N \\ u^1 - u^2 - \epsilon & \text{if } \epsilon < u^1 - u^2 < N \\ 0 & \text{if } u^1 - u^2 \leq \epsilon \end{cases}$$

Then φ is Lipschitz and $0 \leq \varphi < N$ and $\nabla\varphi = \nabla u^1 - \nabla u^2$ in the set where $\epsilon < u^1 - u^2 < N$ and $\nabla\varphi = 0$ almost everywhere in the complement of this set. About each exceptional boundary point P_i, $i = 1, \ldots, k$ we construct a ball of radius ϵ and obtain a domain $D_\epsilon \subset D$ bounded by spherical boundaries

$C_i,\ i=1,\ldots,k$ and a subset of ∂D which we simply denote by Γ. Evidently, $\varphi \equiv 0$ in a neighborhood of Γ. Now let

$$J = \int_{\sum C_i} \varphi < \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2}, \nu > dS$$

where ν is the exterior normal to ∂D_ϵ. Then

$$J \leq 2\omega_{n-1}\epsilon^{n-1} kN \ , \tag{2.4}$$

where ω_{n-1} is the volume of the unit $(n-1)$ sphere. Since

$$\text{div } (\varphi(\frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2})) =< \nabla\varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} > +\varphi(\text{div}\frac{\nabla u^1}{W_1} - \text{div}\frac{\nabla u^2}{W_2})$$

almost everywhere in D, we can apply the divergence theorem and obtain

$$J = \int_{D_\epsilon} (< \nabla\varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} > +\varphi(\text{div}\frac{\nabla u^1}{W_1} - \text{div}\frac{\nabla u^2}{W_2}))\ dV \tag{2.5}$$

By our assumptions, the last term of the integrand in (2.5) is non-negative so by (2.3),(2.4),(2.5) we have

$$0 \leq \int_{D_\epsilon} < \nabla\varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} > \ dV \leq 2\omega_{n-1}\epsilon^{n-1}kN. \tag{2.6}$$

As ϵ decreases to zero, the sets D_ϵ are expanding and thus $\nabla u^1 \equiv \nabla u^2$ in the set $\{0 < u^1 - u^2 < N\}$. Since N is arbitrary, $\nabla u^1 \equiv \nabla u^2$ whenever $u^1 > u^2$. It follows that $u^1 \equiv u^2 +$ positive constant in any nontrivial component of the set $\{u^1 > u^2\}$. Suppose there exists one such component. Then the maximum principle ensures $u^1 \equiv u^2 +$ positive constant in D and by the assumptions of the theorem, the constant must be non-positive, a contradiction.

Remark 2.3. *With a little more effort using a covering argument as in [6], we can allow the exceptional set E of points on ∂D to have one dimensional Hausdorff measure $H_1(E) = 0$.*

3 Existence and compactness theorems

In this section we state the basic existence and compactness theorems we shall use in our study of the Dirichlet problem with infinite boundary data.

Theorem 3.1. Compactness theorem *Let $\{u_n\}$ be a uniformly bounded sequence of solutions of (2.1) in a bounded domain D. Then there exists a subsequence which comverges uniformly on compact subsets (in any topology) to a solution of (2.1) in D.*

Theorem 3.1 follows from the interior gradient estimate of [10] (see (6.22) at the beginnng of section 6 for a precise statement)and a standard argument (see Corollary 16.7 and Theorem 16.8 of [2]) using Schauder theory and the Arzela-Ascoli theorem.

We next turn to existence theorems. For simplicity, we will assume $\mathbb{M} = \mathbb{H}$ or $\mathbb{S}$ and consider only bounded piecewise C^2 domains D. For $P \in \partial D$ define the outer curvature $\hat{\kappa}(P)$ to be the supremum of all (inward) normal curvatures of C^2 curves passing through P and locally supporting D. If no such curve exists we define $\hat{\kappa}(P)$ to be $-\infty$. Note that $\hat{\kappa}(P) = \kappa(P)$ at all regular points of ∂D.

Theorem 3.2. Existence theorem *Suppose that $\hat{\kappa}(P) \geq 2H$ for all $P \in \partial D$ except for a finite set E of exceptional corner points of ∂D.*
a. If $E = \emptyset$ there is a unique solution of (2.1) in D taking on arbitrarily assigned continuous boundary data on $\partial D \setminus E$.
b. If $E \neq \emptyset$ the same result holds if either
i. $M = H^2$ and $H \leq \frac{1}{2}$, or
ii. there is a bounded subsolution of (2.1) in D.

Remark 3.3. *Theorem 3.2 is unusual in that existence and uniqueness holds for certain special non-convex domains. Note that D may even be multiply* *connected. A simple example is provided by an annulus with outer boundary a circle and inner boundary a quadrilateral composed of four (or an even number) of circular arcs (convex toward the annulus). If all the circles have curvature at least 2H, Theorem 3.2 guarantees a unique solution for arbitrary continuous boundary data. Note that the solution is not required to be continous at the corner points and in general cannot be so. By a simple approximation argument we can also allow a finite number of discontinuities in the boundary data on $\partial D \setminus E$, that is on the part of the boundary satisfying the generalized curvature condition.*

Example 3.4. Lens domain *A simple but useful example is provided by a lens domain L bounded by an arc γ of curvature 2H and its geodesic reflection γ^*. In S^2 for all $H > 0$ and in H^2 for $H > \frac{1}{2}$ we must restict the length of γ to half that of the geodesic circle of curvature 2H. However in $\mathbb{H}$ for $0 < H \leq \frac{1}{2}$ there are no restrictions needed. In all cases, a subsolution of (2.1)exists, so there is existence and uniqueness for the Dirichlet problem in the lens domain.*

Lemma 3.5. *Theorem 3.2 holds when $\partial D \in C^{2+\alpha}$ and $\kappa(P) \geq 2H$ for all $P \in \partial D$.*

Proof. The existence theory developed in [10](specifically Theorems 1.4 and 5.4) along with a special local barrier construction for $\mathbb{H}$ in case $H < \frac{1}{2}$ (see Corollary 4.3 of the next section) gives the existence of a unique solution $u \in C^{2+\alpha}(\overline{D})$ of (2.1).

Using Lemma 3.5 we construct local barriers needed in the proof of Theorem 3.2.

Corollary 3.6. *Let γ be an arc of constant curvature 2H and Q an interior point of γ. Let $\mathcal{N}_\delta \subset B_\delta(Q)$ be a smooth (convex) domain obtained by smoothing the convex domain bounded by γ and $\partial B_\delta(Q)$. Consider smooth* *boundary data* $f \leq 0$ *on* $\partial\mathcal{N}_\delta$ *with* $f \equiv 0$ *in a neighborhood of* Q *and* $f \equiv -M$ *on a neighborhood of* $\overline{\partial B_\delta(Q)}$. *Then there is a smooth solution* w^- *of (2.1) in* $\overline{\mathcal{N}_\delta}$ *with boundary values* f.

The family w^- and $w^+ = -w^-$ depending on M, δ are local lower and upper barriers at Q.

Proof of Theorem 3.2:

For the existence in case a., we approximate D by smooth (convex) domains $D_n \subset D$ satisfying $\kappa(\partial D_n) \geq 2H$ by rounding each corner point of ∂D. Extend the boundary data f continuously to all of D and let f_n be smooth functions in $\overline{D_n}$ converging uniformly to f. Then Theorems 1.4 and 5.4 of [10] give a unique smooth solution u_n in D_n with $u_n = f$ on ∂D_n and each u_n is uniformly bounded independent of n (see also [1]). Thus by the compactness theorem, a subsequence of the u_n converges uniformly on compact subsets to a solution u of (2.1) in D. It remains to show that $u = f$ on ∂D. Fix $P \in \partial D,\ Q \in \partial D_n$ with $d(Q, P) < \delta$. Given $\varepsilon > 0$ choose $\delta > 0$ such that $|f_n(x) - f_n(Q)| < \varepsilon$ and $|f_n(Q) - f(P)| < \varepsilon$ if $d(x, Q) < \delta$ and n large. Now let γ be an arc of constant curvature 2H that supports ∂D_n at Q and let $w^+ = -w^- = w(x)$ be the upper and lower barriers in $\mathcal{N}_\delta$ given by Corollary 3.6 with $M = 2 \sup u_n$. Then by the maximum principle

$$-w(x) - 3\varepsilon \leq u_n(x) - f(P) \leq w(x) + 3\varepsilon \text{ in } \mathcal{N}_\delta \cap D_n \ ,$$

Hence there is a uniform modulus of continuity $\sigma(t)$ so that $|u_n(x) - f(P)| < \sigma(d(x, P))$ for any $P \in \partial D$ and $x \in D_n$. It follows that $u \in C(\overline{D})$ and $u = f$ on ∂D.

For part b., we use Perron's method as in [8]. We note that the existence of a bounded subsolution in case $M = H^2$ and $H \leq \frac{1}{2}$ is automatic since there are global radial solutions. Otherwise, we assume such a bounded subsolution exists. Note that the constants are always supersolutions. This being the case there is a Perron solution u in D which achieves the assigned boundary data on $\partial D - \backslash E$ because of the existence of local barriers.

In all cases, the solution is unique by the General maximum principle, Theorem 2.2.

4 Special barriers and the asymptotic behavior of solutions with infinite boundary values

We begin this section with the construction of special barriers in case $\mathbb{M} = \mathbb{H}$ and $H \leq \frac{1}{2}$. For the half-space model of $\mathbb{H} = \{(x,y) : y \geq 0\}$ with metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, note that

$$(4.7) \qquad Mu = y^2 \left\{ \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+y^2(u_x^2+u_y^2)}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+y^2(u_x^2+u_y^2)}}\right) \right\}$$

Lemma 4.1. *For $u = f(\phi(x) - y)$,*

(4.8)

$$Mu = \frac{y^2}{(1+y^2f'^2(1+\phi'^2))^{\frac{3}{2}}} \left\{ f'\phi'' + f''(1+\phi'^2) + y^2f'^3\phi'' + yf'^3(1+\phi'^2) \right\}$$

In particular for $\phi(x) = 2H + \sqrt{1-x^2}$ and $f(t) = -\frac{1}{a}\log|t|$ for $0 < |t| < \frac{\epsilon_0}{a}$ (ϵ_0 sufficiently small independent of $a > 0$)

$$(4.9) \qquad Mu = 2H \left\{ 1 + \frac{a^2t}{2Hy}\left(\sqrt{1-x^2} - \frac{y}{a^2}\right) + O((at)^2) \right\}$$

where $t = 2H + \sqrt{1-x^2} - y$. Hence u tends to $+\infty$ on the curve $C = \{(x,y) : y = \phi(x)\}$ and in a neighborhood of $(0, 1+2H)$, if $a > \sqrt{1+2H}$, u is a subsolution for $t > 0$ and a supersolution for $t < 0$ while if $a < \sqrt{1+2H}$, u is a supersolution for $t > 0$ and a subsolution for $t < 0$.

Proof. Equation (4.8) is a straightforward computation. To derive (4.9) we make a Taylor expansion about $t = 0$ (using $f' = -\frac{1}{at}$, $f'' = af'^2$, $\phi'' = -(1+\phi^2)^{\frac{3}{2}} = (1-x^2)^{-\frac{3}{2}}$):

$$\begin{aligned} Mu &= (1 - \frac{3}{2}\frac{1-x^2}{y^2}a^2t^2 + O(a^4t^4))(2H - t + \frac{\sqrt{1-x^2}}{y}a^2t + \frac{a^2t^2}{y}) \\ &= 2H + (a^2\frac{\sqrt{1-x^2}}{y} - 1)t + O(a^2t^2) \end{aligned}$$

Remark 4.2. *For $2H < 1$ the curve $C = \{(x,y) : y = \phi(x)\}$ is a so called equidistant circle and has constant curvature 2H. (For 2H=1 the curve C is half a horocycle .) Note that the translates $C_t = \{(x,y) : y = \phi(x) - t)\}$, $t > 0$ are also equidistant circles of smaller curvature $2H - t$*

Corollary 4.3. *Let D be a domain in H^2 bounded in part by an arc Γ with $\hat{\kappa} \geq 2H$ (for $H \leq \frac{1}{2}$). Given $M > 0$ large, let C_{t_0} be a supporting equidistant circle C at $P \in \Gamma$ of constant curvature $2H - t_0$ ($0 < t_0 \leq \frac{\epsilon_0}{2a}e^{-aM}$) For C' a translate of C of curvature $2H - e^{aM}t_0$, let $N(t_0, M)$ be the subdomain of $\overline{D}$ bounded by Γ and C'. Then there is a subsolution $w < 0$ of (2.1) in N with $w(P) = 0$ and $w = -M$ on C'.*

Proof. Let P=(0,2H) and $\{y = \phi(x)\}$ be the equidistant circle of Remark 4.2 so that C_{t_0} is the translate $\{y = \phi(x) - t_0\}$. Then $w = -\frac{1}{a}\log\frac{\phi(x)-y}{t_0}$ has the required properties.

We next construct barriers (with infinite slope) using the (oriented) distance function $d(x)$ to an arc Γ in M. Let $w = h(d)$; then as in [10] (with n=2) from the non-divergence form (2.2) of the equation

$$\begin{aligned} Mw &= \frac{1}{W}(\sigma^{ij} - \frac{w^iw^j}{W^2})(h'D_iD_jd + h''D_idD_jd) \\ &= \frac{1}{\sqrt{1+h'^2}}(h'\Delta d + \frac{h''}{(1+h'^2)}) \\ &= \frac{h''}{(1+h'^2)^{\frac{3}{2}}} - \frac{h'}{\sqrt{1+h'^2}}\kappa(x), \end{aligned} \tag{4.10}$$

where $\kappa(x)$ is the inward curvature of the level set of $d(x)$ passing through x.

Lemma 4.4. *i. Let Γ be a C^2 arc with $\kappa(\Gamma) \leq 2\alpha < 2H$ with respect to the "interior" unit normal $-n$. Then in a sufficiently small neighborhood N of any interior point of Γ, there is a supersolution w of (2.1) with exterior normal derivative $\frac{\partial w}{\partial n} = +\infty$ on Γ.*
ii. If $\kappa(\Gamma) \leq -2\alpha < -2H$ with respect to the "interior" unit normal $-n$, then there is a a subsolution w of (2.1) with exterior normal derivative $\frac{\partial w}{\partial n} = -\infty$ on Γ.

Proof. Let $h(t) = -\sqrt{\frac{2t}{\epsilon}}$. Then $\frac{h''}{(1+h'^2)^{\frac{3}{2}}} = \epsilon(1+2\epsilon t)^{-\frac{3}{2}}$, $\frac{h'}{\sqrt{1+h'^2}} = -(1+2\epsilon t)^{-\frac{1}{2}}$, so from (4.10) $w = h(d)$ satisfies

$$Mw = \epsilon(1+2\epsilon d)^{-\frac{3}{2}} + (1+2\epsilon d)^{-\frac{1}{2}}\kappa(x) \leq (1+2\epsilon d)^{-\frac{1}{2}}(2\alpha + 2\epsilon) < 2H$$

for ϵ and d small enough, proving i. For part ii. let $w = -h(d)$. Then

$$Mw = -\epsilon(1+2\epsilon d)^{-\frac{3}{2}} - (1+2\epsilon d)^{-\frac{1}{2}}\kappa(x) \geq (1+2\epsilon d)^{-\frac{1}{2}}(2\alpha - 2\epsilon) > 2H$$

for ϵ and d small enough, proving ii.

Remark 4.5. *For $\epsilon = \frac{1}{2}|\alpha - H|$ the barriers of the lemma are defined in an annulus $\{0 < d(x) < \frac{\epsilon}{C}\}$ of fixed size For $\mathbb{M} = \mathbb{R}^2, \mathbb{S}$ and $\mathbb{H}$ (here we must restrict $H > \frac{1}{2}$), Delaunay's onduloids provide explicit such barriers which are radial. For later use we note also that if Γ has constant curvature -2α, then $Mw \leq 2\alpha = 2H + 2(\alpha - H)$. Hence if α is very close to H, w is an approximate supersolution.*

These special barriers are important because of the following well-known variant of the maximum principle due to Finn.

Lemma 4.6. *Let D be a domain whose boundary is the union of two closed sets γ_1 and γ_2 with $\gamma_2 \in C^1$. Let $u_1 \in C^2(D) \cap C^1(\gamma_2)$ and $u_2 \in C^2(D) \cap C^0(\overline{D})$ satisfy $Mu_1 \geq Mu_2$ in D and suppose $\frac{\partial u_2}{\partial \nu} = +\infty$ (ν is the outer normal) at every point of γ_2. Then if* $\liminf\,(u_2 - u_1) \geq 0$ *for any approach to γ_1, then $u_2 \geq u_1$ in D.*

Corollary 4.7. *Let u be a solution of (2.1) in a domain D and let P be a regular point of ∂D. If $\kappa(P) < 2H$ then u cannot tend to $+\infty$ at P. If $\kappa(P) < -2H$ then u cannot tend to $-\infty$ at P.*

Proof. Suppose $\kappa(P) < 2H$. Then there is an arc C_0 of curvature $< 2H$ internally tangent to ∂D at P. Let C_t ,$0 < t < \delta$ be inward parallel arcs to C, with δ so small that all have curvature $< 2H$. We can then choose a domain N_t (contained in D) bounded by C_t , C_δ and two geodesics joining them that are strictly interior to D (independent of t). We can also arrange that N_t is sufficiently small that the supersolution w of Lemma 4.4 is well-defined for $\Gamma = C_t$. Set $M = \sup_{\partial N_t \setminus C_t} u + \sup_{N_t} w$. Then $u \leq w + M$ in N_t by Lemma 4.6. Letting $t \to 0$ implies $u(P) < \infty$. An analogous argument in case $\kappa(P) < -2H$ shows $u(P) > -\infty$.

We can now characterize those arcs on which solutions can tend to $\pm\infty$.

Theorem 4.8. *Let D be a domain bounded in part by a C^2 arc γ and let u be a solution of (2.1). If u tends to $+\infty$ for any approach to interior points of γ, then $\kappa(\gamma) = 2H$, while if u tends to $-\infty$ for any approach to interior points of γ, then $\kappa(\gamma) = -2H$.*

Proof. Suppose $u \to +\infty$ on γ. Then by Corollary 4.7 $\kappa(\gamma) \geq 2H$. Now suppose $\kappa > 2H$ on some subarc $\gamma' \subset \gamma$. Then by shrinking γ' if necessary, we can find a subdomain $\Delta \subset D$ bounded by γ' and an arc Γ with curvature (with respect to the interior of Δ) satisfying $\kappa(\Gamma) < -2H$. The maximum principle Lemma 4.6 and part ii of Lemma 4.4 implies $u > w + M$ in Δ for any constant M, a contradiction. Hence $\kappa(\gamma) \equiv 2H$. Now suppose $u \to -\infty$ for any approach to interior points of γ. Then by Corollary 4.7 $\kappa(\gamma) \geq -2H$. If $\kappa > -2H$ on some subarc $\gamma' \subset \gamma$. Then by shrinking γ' if necessary, we can find a subdomain $\Delta \subset D$ bounded by γ' and an arc Γ with curvature (with respect to the interior of Δ) satisfying $\kappa(\Gamma) < 2H$. Then Lemma 4.6 and part i of Lemma 4.4 implies $u < w - M$ in Δ for any constant M, a contradiction. Thus $\kappa(\gamma) \equiv -2H$.

The proof of Theorem 4.8 contains the following useful result.

Lemma 4.9. *Let u be a solution of (2.1) in a domain D bounded in part by an arc γ and suppose $m \leq u \leq M$ on γ. Then there is a constant c=c(D) such that for any compact C^2 subarc $\gamma' \subset \gamma$,*
(i) if $\kappa \geq 2H$ on γ' with strict inequality except for isolated points , there is a neighborhood Δ of γ' in $\overline{D}$ such that $u \geq m - c$ in Δ.
(ii) if $\kappa > -2H$ on γ', there is a neighborhood Δ of γ' in $\overline{D}$ such that $u \leq M + c$ in Δ.

5 Flux formula on boundary arcs

In this section we collect some flux formulas we will need to develop our existence theory for solutions with infinite boundary values. These formulas are exactly the same as in section 4 of [9] with almost identical proof. The only essential difference is that we use the barrier in part ii of Lemma 4.4 instead of Delaunay solutions. Therefore we only briefly indicate the idea of the proofs which are already contained in the proof of the general maximum principle Theorem 2.2.

Let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution of (2.1) in a domain D . Then integrating (2.1) over D gives

$$2H\mathcal{A}(D) = \int_{\partial D} < \frac{\nabla u}{W}, \nu > ds \tag{5.11}$$

where $\mathcal{A}(D)$ is the area of D and ν is the outer normal to ∂D. The right hand integral is called the flux of u across ∂D. Let γ be a subarc of ∂D (homeomorphic to $[0,1]$). Even if u is not differentiable on γ we can define the flux of u across γ as follows.

Definition 5.1. *Choose Γ to be a simple smooth curve in D so that $\gamma \cup \Gamma$ bounds a simply connected domain Δ_Γ. We then define the flux of u across γ to be*

$$F_u(\gamma) = 2H\mathcal{A}(\Delta_\Gamma) - \int_\Gamma < \frac{\nabla u}{W}, \nu > ds \tag{5.12}$$

The last integral in (5.12) is well-defined as an improper integral. To see that this definition is independent of Γ, let Γ' be another choice of curve and consider the 2-chain R with oriented boundary $\Gamma' - \Gamma$. Using (2.1) and the divergence theorem on R gives

$$2H\mathcal{A}(\Delta_{\Gamma'}) - 2H\mathcal{A}(\Delta_\Gamma) = \int_{\Gamma'} < \frac{\nabla u}{W}, \nu > ds - \int_\Gamma < \frac{\nabla u}{W}, \nu > ds \ .$$

Therefore the definition makes sense. If $u \in C^1(D \cup \gamma)$, then $F_u(\gamma) = \int_\gamma < \frac{\nabla u}{W}, \nu > ds$.

The following lemma is now obvious.

Lemma 5.2. *Let u be a solution of (2.1) in a domain D and let Γ be a piecewise C^1 curve in $\overline{D}$. Then*

$$2H\mathcal{A}(D) = \int_{\partial D} < \frac{\nabla u}{W}, \nu > ds \ \text{and} \ \left| \int_\Gamma < \frac{\nabla u}{W}, \nu > ds \right| \leq |\Gamma| \ .$$

Lemma 5.3. *Let D be a domain bounded in part by a piecewise C^2 arc Γ satisfying $\hat{\kappa}(\Gamma) \geq 2H$. Let u be a solution of (2.1) in D which is continuous on Γ. Then*

$$\left| \int_\Gamma < \frac{\nabla u}{W}, \nu > ds \right| < |\Gamma| \ . \tag{5.13}$$

Proof. It suffices to prove (5.13) for a small subarc γ of Γ. To this end let $P \in \Gamma$ and let $D_\epsilon = D \cap B_\epsilon(P)$. Then by Theorem 3.2 there is a solution v of (2.1) in D_ϵ with $v = u + 1$ on γ and $v = u$ on the remainder of the boundary. Set $w = v - u$. Then as in the proof of the general maximum principle,

$$0 < \int_{D_\epsilon} < \nabla w, \frac{\nabla v}{W_v} - \frac{\nabla u}{W_u} > dV = \int_\gamma < \frac{\nabla v}{W_v} - \frac{\nabla u}{W_u}, \nu > ds \ .$$

Hence $F_u(\gamma) < F_v(\gamma) \leq |\gamma|$.

Lemma 5.4. *Let D be a domain bounded in part by an arc γ and let u be a solution of (2.1) in D. Then*
(i) if u tends to $+\infty$ on γ, we have $\int_\gamma < \frac{\nabla u}{W}, \nu > ds = |\gamma|$.
(ii) if u tends to $-\infty$ on γ, we have $\int_\gamma < \frac{\nabla u}{W}, \nu > ds = -|\gamma|$.

Proof. We sketch the proof of (i) which is a slight modification of Lemma 4.3 of [9]; part (ii) is similar. By Theorem 4.8, $\kappa(\gamma) = 2H$. Fix $\epsilon > 0$ and let δ be arbitrarily small. Let γ' be a subarc of γ of length a small multiple of $\sqrt{\epsilon}$. Choose an arc Λ of constant curvature $-2\alpha = -(2H + 4\epsilon)$ (with respect to its normal pointing exterior of D) so that the distance between Λ and γ' satisfies $\delta < d(\Lambda, \gamma') \leq \delta + C|\gamma'|^2$. Then by by Remark 4.5, the subsolution w of Lemma 4.4 part (ii) (with Γ replaced by Λ) is well-defined in an annulus of size proportional to ϵ (with γ' strictly interior) and w is an approximate supersolution with flux

$$F_w(\gamma') \geq (1 - \epsilon(\delta + C|\gamma'|^2))|\gamma'| \ . \tag{5.14}$$

Now choose a simple arc Γ joining the endpoints of γ' so that $\gamma' \cup \Gamma$ bounds a simply connected domain Δ which is contained in the annulus. Set $\varphi = u_1 - u_2$ where $u_1 = u$ and $u_2 = w$. We may assume $\varphi > 0$. Let M_0 be chosen so large that the the linear measure of $\Gamma \cap \{\varphi > M_0\}$ is less than $\frac{\epsilon}{8}$. Let Γ_0 be the subarc of Γ with endpoints at arc length distance $\frac{\epsilon}{16}$ from the endpoints of Γ. For $M > \sup_{\Gamma_0} \varphi$, let Δ_M be the component of the set $\Delta \cap \{\varphi < M\}$ containing Γ_0 in its closure. Then Δ_M will be bounded by a curve γ_M with endpoints on Γ close to those of γ' (on which $\varphi = M$), by subarcs $\Gamma_M \subset \Gamma$ and possibly by other curves $\tilde{\gamma_M}$ (with endpoints on Γ) on which $\varphi = M$. Further we may always choose M so that $\partial \Delta_M$ is piecewise smooth.

Now let

$$J = \int_{\Delta_M} < \nabla \varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} > dV$$

Since

$$\text{div } \varphi(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}) =< \nabla \varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} > + \varphi(\text{div}\frac{\nabla u_1}{W_1} - \text{div}\frac{\nabla u_2}{W_2})$$

in D, we can apply the divergence theorem and obtain

$$\begin{aligned} J &= M \int_{\gamma_M} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds + \int_{\Gamma_M} \varphi < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds \\ &+ M \int_{\tilde{\gamma_M}} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds + O(M\epsilon|\Delta|) \\ &= J_1 + J_2 + J_3 + O(M\epsilon|\Delta|). \end{aligned} \tag{5.15}$$

We can estimate J_2 as follow:

$$J_2 \leq \frac{\epsilon}{4} M + 2|\Gamma| M_0 \tag{5.16}$$

To estimate J_3 we complete $\tilde{\gamma}_M$ to closed curves by subarcs $\tilde{\Gamma}_M$ of Γ_M; then

$$J_3 = M \int_{\tilde{\gamma_M} + \tilde{\Gamma}_M} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds - M \int_{\tilde{\Gamma_M}} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds$$

Since u_2 is a subsolution and u_1 is a solution of (2.1),

$$J_3 \leq \frac{\epsilon}{4} M \ . \tag{5.17}$$

From (5.15), (5.16), (5.17) and the non-negativity of J, we find

$$0 \leq \int_{\gamma_M} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds + \frac{\epsilon}{2} + 2|\Gamma| \frac{M_0}{M} + O(\epsilon|\Delta|) \ . \tag{5.18}$$

Consider now the domain Δ'_M bounded by γ' , γ_M and parts of Γ. Then

$$\begin{aligned} \int_{\gamma'} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds \ &\geq \ - \int_{-\gamma_M} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds \\ &- \ \frac{\epsilon}{2} - \epsilon |\Delta'_M| \ . \end{aligned} \tag{5.19}$$

Combining (5.18) and (5.19) (remembering the difference in orientation of the common term) gives

$$\int_{\gamma'} < \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu > ds \geq -\epsilon - 2|\Gamma| \frac{M_0}{M} + O(\epsilon|\Delta|) \ . \tag{5.20}$$

We can now let $M \to \infty$ and $|\Delta| \to 0$ in (5.20) to obtain

$$F_u(\gamma') \geq F_w(\gamma') - \epsilon$$

Recalling (5.14) and letting $\delta \to 0$ gives

$$[F_u(\gamma') \geq |\gamma'| - K\epsilon |\gamma'|^3 - \epsilon \tag{5.21}$$

Note that we cannot let ϵ tend to zero in (5.21) since $|\gamma'| = \sqrt{\frac{\epsilon}{K}}\ |\gamma|$. Instead we divide our original arc γ into $N = \sqrt{\frac{K}{\epsilon}}$ small pieces γ'. Then summing (5.21) gives

$$F_u(\gamma) \geq |\gamma| - \epsilon^2 |\gamma|^3 - \sqrt{K\epsilon} \ .$$

Since ϵ is arbitrary, $F_u(\gamma) = |\gamma|$.

The following lemma is a simple extension of Lemma 5.4.

Lemma 5.5. *Let D be a domain bounded in part by an arc γ and let $\{u_n\}$ be a sequence of solutions of (2.1) in D with each u_n continuous on γ. Then (i) if the sequence tends to $+\infty$ uniformly on compact subsets of γ while remaining uniformly bounded on compact subsets of D, we have*

$$\lim_{n\to\infty}\int_\gamma < \frac{\nabla u_n}{W_n}, \nu > ds = |\gamma| \ .$$

(ii) if the sequence tends to $-\infty$ uniformly on compact subsets of γ while remaining uniformly bounded on compact subsets of D, we have

$$\lim_{n\to\infty}\int_\gamma < \frac{\nabla u_n}{W_n}, \nu > ds = -|\gamma| \ .$$

.

We also will need the following variant of Lemma 5.5.

Lemma 5.6. *Let D be a domain bounded in part by an arc γ with $\kappa(\gamma) = 2H$ and let $\{u_n\}$ be a sequence of solutions of (2.1) in D with each u_n continuous on γ. Then if the sequence diverges to $-\infty$ uniformly on compact subsets of D while remaining uniformly bounded on compact subsets of γ, we have*

$$\lim_{n\to\infty}\int_\gamma < \frac{\nabla u_n}{W_n}, \nu > ds = |\gamma| \ .$$

We shall omit the proof of Lemmas 5.5 and 5.6 which uses the technique of Lemma 5.4. See the proof of Lemma 4.5 of [9] for more details.

6 The local Harnack inequality and monotone convergence theorem

Let u be a non-negative or non-positive solution of (2.1) in $B_\rho(P)$ for $\rho < R(P)$, the injectivity radius at P. Then it follows from Theorem 1.1 of [10] that

$$|\nabla u(P)| \leq f(\frac{|u(P)|}{\rho}) \ , \ f(t) = e^{C(1+t)^2} \tag{6.22}$$

where C depends on ρ and the local geometry of $\mathbb{M}$ but is independent of u. Following Serrin [8] we prove

Theorem 6.1. Local Harnack inequality *There exists a function $\Phi(t, r)$ such that*

$$|u(Q)| \leq \Phi(m, r) \tag{6.23}$$

where $m = |u(P)|$ and $r = d(P, Q)$ the distance from P to Q. For each fixed t, $\Phi(t, r)$ is a continuous strictly increasing function defined on an interval $0 \leq r < \rho(t)$ with

$$\Phi(t, 0) = t \ , \ \Phi(t, r) \to \infty \ \textit{as} \ r \to \rho(t) \ ,$$

where $\rho(t)$ is a continuous strictly decreasing function tending to zero as t tends to infinity.

Proof. Suppose $u \geq 0$ (otherwise let $v = -u$). Consider points Q lying on a fixed geodesic emanating from P and let $u(r)$ denote the values of u along this ray. Then from (6.22),

$$\frac{du}{dr} \leq f(\frac{u(r)}{\rho - r}) \ .$$

Define $\Phi(t, r)$ by the conditions

$$\frac{d\Phi}{dr} = f(\frac{\Phi}{\rho - r}) \ , \Phi(t, 0) = t \ . \tag{6.24}$$

Then $u(r) \leq \Phi(m, r)$ whenever Φ is well-defined. The solution of (6.24) is implicitly defined by $\Phi = (\rho - r)v$ and

$$\int_{v_0}^{v} \frac{dv}{f(v) + v} = \log \frac{1}{\rho - r} \ , \ v_0 = \frac{t}{\rho} \ .$$

Since the integral converges as v tends to infinity, Φ is defined only on an interval $0 \leq r < \rho(t)$ where clearly $\rho(t) \to 0$ as $t \to \infty$. This completes the proof.

The existence theorems of the following sections depend upon the limiting behavior of monotone increasing and monotone decreasing sequences of solutions in a fixed domain. As in the Euclidean case, we have the following consequence of Theorems 4.8 and 6.1.

Theorem 6.2. Monotone convergence theorem *Let $\{u_n\}$ be a monotonically increasing or decreasing sequence of solutions of (2.1) in a fixed domain D. If the sequence is bounded at a single point of D, there exists a non-empty open set $U \subset D$ such that $\{u_n\}$ converges to a solution in U. The convergence is uniform on compact subsets of U and the divergence is uniform on compact subsets of $V = D \setminus U$. If V is nonempty, ∂V consists of arcs of curvature $\pm 2H$ and parts of ∂D. These arcs are convex to U for increasing sequences and concave to U for decreasing sequences.*

In particular, no component of V can consist of a single interior arc. We have more information about the set V if we have more knowledge about the boundary behavior of the sequence $\{u_n\}$ and the curvature of ∂D.

Lemma 6.3. *Let D be a domain bounded in part by an arc C with $\hat{\kappa}(C) \geq 2H$. Let $\{u_n\}$ be an increasing or decreasing sequence of solutions of (2.2) in D with each u_n continuous in $D \cup C$. Suppose γ is an interior arc of D of curvature 2H forming part of the boundary of V. Then γ cannot terminate at an interior point of C if $\{u_n\}$ either diverges on C to $\pm\infty$ or remains uniformly bounded on compact subsets of C.*

Proof. Let γ be an arc in ∂V as in Lemma 6.3 that terminates at an interior point P of C. By considering only a small neighborhood of P, we may assume that C is C^2. By Theorem 4.8 the sequence $\{u_n\}$ cannot diverge to $-\infty$ on C. Moreover, if the curvature of C is not identically $2H$ and $\{u_n\}$ remains uniformly bounded on compact subsets of C, Lemma 4.9 insures that a neighborhood of C is either contained in U or V, a contradiction. Hence assume $\kappa(C) \equiv 2H$. Suppose $\{u_n\}$ diverges to $+\infty$ on C and there exists exactly one such γ terminates at P. Let Q be a point of γ close to P and choose R on C close to P so that the geodesic segment $\overline{RQ}$ lies in U. Let T be the triangle formed by $\overline{RQ}$ and the constant curvature 2H arcs $\overset{\frown}{QP}$ and $\overset{\frown}{PR}$. Then by (5.11)

$$2H\mathcal{A}(T) = F_{u_n}(\overset{\frown}{QP}) + F_{u_n}(\overset{\frown}{PR}) + F_{u_n}(\overline{RQ}) \tag{6.25}$$

while by Lemma 5.5

$$\lim_{n\to\infty} F_{u_n}(\overset{\frown}{QP}) = |\ \overset{\frown}{QP}\ |\ ,\ \ \lim_{n\to\infty} F_{u_n}(\overset{\frown}{PR}) = |\ \overset{\frown}{PR}\ |\ . \tag{6.26}$$

From (6.25), (6.26) and Lemma 5.3 we can conclude

$$\frac{2H\mathcal{A}(T)}{|\overline{RQ}|} \geq \frac{|\ \overset{\frown}{QP}\ | + |\ \overset{\frown}{PR}\ |}{|\overline{RQ}|} - 1 \tag{6.27}$$

Keeping P fixed, we move Q to Q' and R to R' along the same arcs so that $|\ \overset{\frown}{Q'P}\ | = \lambda|\ \overset{\frown}{QP}\ |$ and $|\ \overset{\frown}{PR'}\ | = \lambda|\ \overset{\frown}{PR}\ |$ and form the triangle T' by joining Q' to R' by a geodesic. Then the left hand side of (6.27) tends to zero as $\lambda \to 0$ while the right hand side of (6.27) remains uniformly positive, a contradiction. The only other possibility is that two arcs γ_1 and γ_2 terminate at P. Then again we can find a triangle $T \subset U$ whose edges are two constant curvature 2H arcs and a geodesic segment as before (perhaps with $\partial T \cap C = \{P\}$). The same argument gives a contradiction.

In case the sequence remains uniformly bounded on compact subsets of C and there is exactly one γ, we choose R on C so that T is contained in V. By Lemma 4.4 the sequence must be divergent to $-\infty$ in V. We now reach a contradiction as above by using Lemma 5.5. If there are two arcs terminating at P, then V is necessarily the convex lens domain formed by γ_1 and γ_2. Choose the point Q on γ_1 and R on γ_2 and form T in V . Then (6.25), (6.26) and (6.27) still hold and we reach a contradiction as before.

7 Existence and uniqueness

In this section we prove existence theorems for constant mean curvature graphs in which the boundary values $\pm\infty$ are allowed on entire arcs of the boundary. As we have seen in Theorem 4.8, these arcs must be of curvature $\pm 2H$ with respect to the domain.

Definition 7.1. Admissible domain *We say a bounded domain D is* admissible *if it is simply connected and ∂D is piecewise C^2 and consists of three set of C^2 open arcs $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ satisfying $\kappa(A_i) = 2H$, $\kappa(B_i) = -2H$ and $\kappa(C_i) \geq 2H$ respectively (with respect to the interior of D). We suppose that no two of the arcs A_i and no two of the arcs B_i have a common endpoint. In addition if the family $\{B_i\}$ is nonempty, we assume that the set D^* formed by replacing each arc B_i by B_i^*, the geodesic reflection of B_i across its endpoints, is a well-defined domain. Finally we assume that there is a bounded subolution of (2.1) in D^* (recall this is automatic if $\hat{\kappa}(\partial D^*) \geq 2H$ or if $D \subset H^2$ and $H \leq \frac{1}{2}$) so that by Theorem 3.2, the Dirichlet problem with finite data is well-posed in D^*)*

Remark 7.2. *Note that we do not insist that $\hat{\kappa}(P) \geq 2H$ for P an endpoint of the A_i or B_i so our definition allows admissible domains to be nonconvex.*

Definition 7.3. Dirichlet Problem *Given an admissible domain D, the generalized Dirichlet problem is to find a solution of (2.1) in D which assumes the value $+\infty$ on each A_i, $-\infty$ on each B_i and assigned continuous data on each of the open arcs C_i. Note that the continuous data is allowed to become unbounded at the endpoints.*

We shall need one more definition before we can start considering some special cases of the Dirichlet problem.

Definition 7.4. Admissible polygon *Let D be an admissible domain. We say that $\mathcal{P}$ is an admissible polygon if $\mathcal{P}$ is a simple domain contained in $\overline{D}$ with $\partial\mathcal{P}$ piecewise smooth consisting of arcs of constant curvature $\kappa = \pm 2H$ with vertices chosen from among the endpoints of the families $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$. For an admissible $\mathcal{P}$, let α and β be the total length of the arcs in $\partial\mathcal{P}$ belonging to $\{A_i\}$ and $\{B_i\}$ respectively. Finally, let ℓ be the perimeter of $\mathcal{P}$ and $\mathcal{A}(P)$ be the area of $\mathcal{P}$.*

Proposition 7.5. *Consider the Dirichlet problem in an admissible domain D and suppose the family $\{B_i\}$ is empty, $\kappa(C_i) > 2H$ and the assigned data f on the arcs C_i are bounded below. Then there exists a solution if and only if*

$$2\alpha < \ell + 2H\mathcal{A}(\mathcal{P}) \text{ for all admissible polygons } \mathcal{P}. \tag{7.28}$$

Proof. Let u_n be the solution of (2.1) in D such that

$$u_n = \begin{cases} n & \text{on } \cup A_i \\ \min(n, f) & \text{on } \cup C_i \end{cases}$$

Such a solution exists and is unique by Theorem 3.2. Moreover by the general maximum principle, the sequence $\{u_n\}$ is monotone increasing so the monotone convergence theorem 6.2 applies. Suppose V is nonempty. By Lemma 6.3 an interior arc of D which bounds V must be of curvature 2H and can terminate only at an endpoint of some A_i or C_i. Moreover by Lemma 4.9, a neighborhood of each C_i is contained in U. Therefore the boundary of each component of V is an admissible polygon $\mathcal{P}$ with vertices among those of the $A_i$and C_i. Also the curvature 2H arcs forming the boundary which are not among the A_i are concave to V. By Lemma 5.2 applied to each u_n in $\mathcal{P}$,

$$2H\mathcal{A}(\mathcal{P}) = F_n(\partial\mathcal{P} - \Sigma' A_i) + F_n(\Sigma' A_i) \tag{7.29}$$

where $\Sigma' A_i$ is the union of the arcs A_i which are part of $\mathcal{P}$. Then by Lemma 5.5

$$(7.30) \qquad \lim_{n\to\infty} F_n(\partial\mathcal{P} - \Sigma' A_i) = -(\ell - \alpha) \ .$$

But $|F_n(\Sigma' A_i)| \leq \alpha$, hence $\ell - \alpha \leq \alpha - 2H\mathcal{A}(\mathcal{P})$, contradicting our assumption (7.28). Thus V is empty and the sequence converges uniformly on compact subsets of D to a solution u. Since each u_n is uniformly bounded in a neighborhood of each C_i by Lemma 4.9, a standard barrier argument shows that $u = f$ on $\cup C_i$. Using (7.29) and Lemmas 5.3 and 5.4, the necessity of (7.28) is clear. This completes the proof.

Similarly, we have

Proposition 7.6. *Consider the Dirichlet problem in an admissible domain D and suppose the family $\{A_i\}$ is empty, $\kappa(C_i) > 2H$ and the assigned data f on the arcs C_i are bounded above. Then there exists a solution if and only if*

$$(7.31) \qquad 2\beta < \ell - 2H\mathcal{A}(\mathcal{P}) \ \ \textit{for all admissible polygons } \mathcal{P}.$$

The proof is completely analogous using monotone decreasing sequences in D^*, which must diverge in a neighborhood of each B_i^* by Lemma 4.9. The simple case of one B and one C already illustrates the importance of considering the domain D^*, which is a key idea.

We now use Propositions 7.5 and 7.6 to construct some useful barriers and to remove the assumption $\kappa(C_i) > 2H$.

Example 7.7. *Let $B = B_\delta(P)$ be a ball of small radius δ and let Q and R be "antipodal" points on ∂B. Choose points Q_1 and Q_2 on ∂B and symmetric with respect to the geodesic through QPR. Now let B_1 be an arc of curvature*

-2H (as seen from P) joining Q_1 and Q_2 and set $A_1 = B_1^*$. Let R_1 and R_2 on ∂B be reflections of Q_1 and Q_2 (with respect to the geodesic orthogonal to QPR through P)and define B_2 and A_2. Then for δ small compared with H, the domain B^+ bounded by A_1 , A_2 and parts of ∂B satisfies the conditions of Proposition 7.5 and similarly the domain B^- bounded by B_1 , B_2 and parts of ∂B satisfies the conditions of Proposition 7.6. Let u^+ be the solution of (2.1) in D^+ with boundary values $+\infty$ on $A_1 \cup A_2$ and the constant value M on the remainder of the boundary. Similarly let u^- be the solution of (2.1) in D^- with boundary values $-\infty$ on $B_1 \cup B_2$ and the constant value -M on the remainder of the boundary.

Using these examples we prove

Proposition 7.8. *Let D be a domain bounded in part by an arc γ and let $\{u_n\}$ be a sequence of solutions of (2.1) in D which converge uniformly on compact subsets of D to a solution u. Suppose each u_n is continuous in $D \cup \gamma$. Then*

(i). Suppose the boundary values of u_n converge uniformly on compact subset of γ to a bounded limit f. If $\kappa(\gamma) \geq 2H$, then u is continuous in $D \cup \gamma$ and $u = f$ on γ.

(ii). If $\kappa(\gamma) = 2H$ and the boundary values of u_n diverge uniformly to $+\infty$ on compact subsets of γ, then u takes on the boundary values $+\infty$ on γ.

(iii). If $\kappa(\gamma) = -2H$ and the boundary values of u_n diverge uniformly to $-\infty$ on compact subsets of γ, then u takes on the boundary values $-\infty$ on γ.

Proof. (i). It suffices to prove that the sequence $\{u_n\}$ is uniformly bounded in the intersection of D with a neighborhood of any interior point P of γ. Orient the ball B of Example 7.7 so that geodesic joining QPR is tangent to ∂D. We may choose the points Q_i $i = 1, 2$ and δ small so that the boundary arc joining Q_2 and R_2 lies is in a compact subset of D. Then if M is large enough,

$$u_n \leq u^+ \text{ in } D \cap B^+ \quad \text{and} \quad u_n \geq u^- \text{ in } D \cap B^- .$$

Therefore the sequence is uniformly bounded in a neighborhood of P.
(ii). Let P be an interior point of γ, Similarly as in (i), we obtain that there exists M large enough so that $u_n \geq -M$ in $N = B_\epsilon(P) \cap D$. Let v_m be the solution of (2.1) in N with boundary values m on $\gamma \cap B_\epsilon(P)$ and -M on the remaining boundary. By the general maximum principle $u_n \geq v_m$ for n sufficiently large so $u \geq v_m$ in N. In particular, $u(P) > m$ for every m and u must take on the value $+\infty$ at P.
(iii). Again for P interior to γ, $u_n \leq M$ in $N = B_\epsilon(P) \cap D$. Let v_m be the solution of (2.1) in N with boundary values -m on $\gamma \cap B_\epsilon(P)$ and M on the remaining boundary. By the general maximum principle $u_n \leq v_m$ for n sufficiently large so $u \leq v_m$ in N. Since the v_m are monotonically decreasing (and converges to a solution with boundary values $+-\infty$ on $\gamma \cap B_\epsilon(P)$), u must take on the value $-\infty$ at P.

We can now extend Propositions 7.5 and 7.6 to allow the arcs C_i to satisfy $\kappa(C_i) \geq 2H$. The only change needed in the proof of Proposition 7.5 is to use part (i) of Proposition 7.8 to show that the solution takes on the required boundary data on the arcs C_i. The extension of Proposition 7.6 is more delicate since if $\kappa(C_i) = 2H$ for some i, we do not know that the sequence is bounded below in a neighborhood of C_i. However by Lemma 6.3, a neighborhood of C_i is either contained in U or V. We have already handled the former case. In the latter case, consider a component of V whose boundary is an admissible polygon $\mathcal{P}$ (with vertices among those of the B_i and C_i) containing a subset $\Sigma' C_i$ of the C_i of curvature 2H and a subset $\Sigma' B_i$ of the B_i. The interior arcs of D which are in $\mathcal{P}$ are convex to

V. By Lemma 5.2 applied to each u_n in $\mathcal{P}$,

$$(7.32) \qquad 2H\mathcal{A}(\mathcal{P}) = F_n(\Sigma' B_i) + F_n(\Sigma' C_i) + F_n(\partial\mathcal{P} - \Sigma' B_i - \Sigma' C_i)$$

Then by Lemma 5.5

$$(7.33) \qquad \lim_{n\to\infty} F_n(\partial\mathcal{P} - \Sigma' B_i - \Sigma' C_i) = \ell - \beta - \Sigma'|C_i| \ .$$

and by Lemma 5.6

$$(7.34) \qquad \lim_{n\to\infty} F_n(\Sigma' C_i) = \Sigma'|C_i|$$

But $|F_n(\Sigma' B_i)| \leq \beta$, hence

$$2H\mathcal{A}(\mathcal{P}) \geq -\beta + \Sigma'|C_i| + (\ell - \Sigma'|C_i| - \beta) = \ell - 2\beta \ ,$$

contradicting our assumption (7.31). Thus V is empty and the sequence converges uniformly on compact subsets of D to a solution u. Finally, we use parts (i) and (iii) of Proposition 7.8 to show that our solution achieves the boundary values. We state these results as

Theorem 7.9. *Consider the Dirichlet problem in an admissible domain D and suppose the family $\{B_i\}$ is empty and the assigned data f on the arcs C_i are bounded below. Then there exists a solution if and only if*

$$(7.35) \qquad 2\alpha < \ell + 2H\mathcal{A}(\mathcal{P}) \ \textit{ for all admissible polygons } \mathcal{P}.$$

Theorem 7.10. *Consider the Dirichlet problem in an admissible domain D and suppose the family $\{A_i\}$ is empty and the assigned data f on the arcs C_i are bounded above. Then there exists a solution if and only if*

$$(7.36) \qquad 2\beta < \ell - 2H\mathcal{A}(\mathcal{P}) \ \textit{ for all admissible polygons } \mathcal{P}.$$

In the following theorem we allow both families $\{A_i\}$ and $\{B_i\}$ to occur and allow the data f on the $\{C_i\}$ to be unbounded both above and below as we approach the endpoints.

Theorem 7.11. *Consider the Dirichlet problem in an admissible domain D and suppose the family $\{C_i\}$ is nonempty. Then there exists a solution if and only if*

$$2\alpha < \ell + 2H\mathcal{A}(\mathcal{P}) \ \ \textit{and} \ \ 2\beta < \ell - 2H\mathcal{A}(\mathcal{P}) \tag{7.37}$$

for all admissible polygons $\mathcal{P}$.

Proof. By Theorem 7.9 the first condition of (7.37) guarantees the existence of a solution u^+ of (2.1) in D^* such that

$$u^+ = \begin{cases} +\infty & \text{on } \cup A_i \\ 0 & \text{on } \cup B_i^* \\ \max(f, 0) & \text{on } \cup C_i \end{cases}$$

Similarly by Theorem 7.10, the second condition of (7.37) guarantees the existence of a solution u^- of (2.1) in D^* such that

$$u^- = \begin{cases} -\infty & \text{on } \cup B_i \\ 0 & \text{on } \cup A_i \\ \min(f, 0) & \text{on } \cup C_i \end{cases}$$

Now let u_n be the solution of (2.1) in D^* such that

$$u_n = \begin{cases} n & \text{on } \cup A_i \\ -n & \text{on } \cup B_i^* \\ f_n & \text{on } \cup C_i \end{cases}$$

where f_n is the truncation of f above by n and below by -n.

By the general maximum principle,

$$u_n \leq u^+ \ \text{ in } D^* \quad \text{and} \quad u^- \leq u_n \ \text{ in } D\ .$$

Therefore the sequence $\{u_n\}$ is uniformly bounded on compact subsets of D and so a subsequence converges uniformly on compact subsets to a solution u in D. By Proposition 7.8, u takes on the assigned boundary data. The necessity of the conditions (7.37) follows essentially as in the Theorems 7.9

and 7.10.

We turn next to the important case when the family $\{C_i\}$ is empty.

Theorem 7.12. *Consider the Dirichlet problem in an admissible domain D and suppose the family $\{C_i\}$ is empty. Then there exists a solution if and only if*

$$\alpha = \beta + 2H\mathcal{A}(D) \tag{7.38}$$

and

$$2\alpha < \ell + 2H\mathcal{A}(\mathcal{P}) \ \ \textit{and} \ \ 2\beta < \ell - 2H\mathcal{A}(\mathcal{P}) \tag{7.39}$$

for all other admissible polygons $\mathcal{P}$.

Proof. Let v_n be the solution of (2.1) in D^* with boundary values n on each A_i and 0 on each B_i^*. For $0 < c < n$ we define for $n \geq 1$

$$E_c = \{v_n - v_0 > c\} \cap \quad F_c = \{v_n - v_0 < c\};$$

we suppress the dependence of these sets on n. Let E_c^i and F_c^i denote respectively the components of E_c and F_c whose closure contains respectively A_i and B_i^*. By the general maximum principle, $E_c = \cup E_c^i$ and $F_c = \cup F_c^i$. If c is sufficiently close to n, the sets $\{E_c^i\}$ will be distinct and disjoint (to see this, note that we can seperate any two of the A_i by a curve joining two of the B_i^* on which $v_n - v_0$ is bounded away from n). Now define $\mu(n)$ to be the infimum of the constants c such that the sets $\{E_c^i\}$ are distinct and disjoint. The sets $\{E_\mu^i\}$ will again be distinct although there must be at least one pair (i, j) , $i \neq j$ such that $\overline{E}_\mu^i \cap \overline{E}_\mu^j$ is nonempty. This implies that given any F_μ^i there is some F_μ^j distinct from it. Now let u_i^+ , $i = 1, \ldots, k$ be the solution of (2.1) in D^* taking on the boundary values $+\infty$ on A_i and 0 on the remaining boundary. This solution exist by Theorem 7.9 since the solvability condition 7.35 follows trivially from (7.38), (7.39). Also let u_i^- be the solution of (2.1) in the domain $\tilde{D}$ bounded by $\cup A_i, B_i^*, \cup_{j\neq i} B_j$ taking on the boundary values $-\infty$ on $\cup_{j\neq i} B_j$ and 0 on the remainder of the boundary. We claim this solution exists by Theorem 7.10. To verify (7.36), we need only consider admissible polygons $\tilde{\mathcal{P}}$ in $\tilde{D}$ which contain the lens domain L formed by B_i and B_i^*. Let $\mathcal{P}$ be the corresponding admissible polygon for D formed by deleting L. By (7.38), (7.39) we have

$$2\beta = 2(|B_i| + \Sigma'_{j\neq i}|B_j|) \leq \ell - 2H\mathcal{A}(\mathcal{P}) \ ,$$

or equivalently

$$2\tilde{\beta} = 2\Sigma'_{j\neq i}|B_j| \leq \tilde{\ell} - 2H\mathcal{A}(\tilde{\mathcal{P}}) + (2H\mathcal{A}(L) - 2|B_i|) \ .$$

However since a solution (for example v_n) exists in L, we have by Lemma 5.3 $2H\mathcal{A}(L) < 2|B_i|$ so conditon (7.36) is satisfied.

We now set

$$u^+ = \min_i u_i^+ \ \text{ in } D^* \ \text{ and } \ u^- = \min_i u_i^- \ \text{ in } D$$

We note that if we compare each u_i^+ to a fixed bounded subsolution in D^* (which exists since D is admissible), then by the general maximum principle there is a constant $N > 0$ such that $u_i^+ > -N, \ i = 1, \ldots, k$. Finally we set $u_n = v_n - \mu(n)$.

We now claim that

$$u_n \leq u^+ + M \text{ in } D^* \quad \text{and} \quad u_n \geq u^- - M \ \text{ in } D \ .$$

where $M = N + \sup_{D^*} |v_0|$. S Suppose $u_n > v_0$ at some point P. Then $v_n - v_0 > \mu(n)$ at P so that P is in some E_μ^i. Applying the general maximum principle in the domain E_μ^i, we obtain

$$u_n \leq u_i^+ + N + \sup_{E_\mu^i} |v_0| \leq u^+ + M \quad \text{at P} \ .$$

On the other hand, suppose $u_n < v_0$ at some point $P \in D$. Then $v_n - v_0 > \mu(n)$ at P so that P is in some F^i_μ. By what has been shown above, there is a corresponding $j = j(i)$ so that $F^i_\mu \cap F^j_\mu = \emptyset$. Applying the general maximum principle in F^i_μ, we obtain

$$u_n \geq u_j^- - \sup_{F^i_\mu} |v_0| \geq u^- - M \ .$$

Therefore the claim is justified and the sequence $\{u_n\}$ is uniformly bounded on compact subsets of D. By the compactness principle, a subsequence $\{u_n\}$ converges uniformly on compact subsets of D to a solution u. We still must show that u takes on the required boundary values. We observe that a subsequence $\mu(n)$ diverges to $+\infty$ otherwise we can extract a subsequence converging to a finite limit μ_0. Each u_n would then be bounded below in D^* uniformly in n, and the boundary values of u_n would tend uniformly on compact subsets of $\cup B_i^*$ to $-\mu_0$ and diverge uniformly to $+\infty$ on $\cup A_i$. Once again we could find a subsequence converging uniformly on compact subsets to a solution in D^*. By Propositon 7.8 we would have

$$v = \begin{cases} +\infty & \text{on } \cup A_i \\ -\mu_0 & \text{on } \cup B_i^* \end{cases}$$

We can now obtain a contradiction to (7.38) by a flux argument. By Lemma 5.2,

$$2H\mathcal{A}(D) = F_v(\Sigma A_i) + F_v(\Sigma B_i) \ , \tag{7.40}$$

while by Lemmas 5.3 and 5.4

$$|F_v(\Sigma B_i)| < \beta \ \text{ and } \ F_v(\Sigma A_i) = \alpha \ . \tag{7.41}$$

Combining (7.40) and (7.41) gives $\alpha - \beta < 2H\mathcal{A}(D)$, a contradiction. In the same way, we see $n - \mu(n)$ diverges to $+\infty$. Summing up we have shown that the boundary values of u_n, namely, $-\mu(n)$ on $\cup B_i^*$ and $n - \mu(n)$ on $\cup A_i$

diverge to $-\infty$ and $+\infty$ respectively. A now familiar argument shows that u_n diverges to $-\infty$ on $\cup B_i$. The necessity of the conditions (7.38) and (7.39) is straigtforward. The theorem is proved.

We end the section with a maximum principle that is valid for solutions with infinite boundary values. This result immediately proves uniqueness for the Dirichlet problem in an admissible domain if the family $\{C_i\}$ is nonempty and uniqueness up to translation if the family $\{C_i\}$ is empty.

Theorem 7.13. *Let D be a domain whose boundary contains two families $\{A_i\}$ and $\{B_i\}$of arcs satisfying $\kappa(A_i) = 2H$ and $\kappa(B_i) = -2H$. Let E be a finite set of exceptional points on ∂D and let C denote the set of points in $\partial D \setminus E$ which are not in the closure of the families $\{A_i\}$ and $\{B_i\}$. Let u^1 and u^2 be solutions of (2.1) in D taking on the values $+\infty$ on each A_i and $-\infty$ on each B_i. If C is nonempty, assume* $\limsup u^1 - u^2 \leq 0$ *for any approach to points of C. If C is empty, assume $u^1 \leq u^2$ at some point P in D. Then in either case $u^1 \leq u^2$ in D.*

Proof. For simplicity we give the proof in case E is empty; the general case requires only a slight modification. As in the proof of the general maximum principle, let N and ε be positive constants with N large and ε small. Define

$$\varphi = \begin{cases} N - \varepsilon & \text{if } u^1 - u^2 \geq N \\ u^1 - u^2 - \varepsilon & \text{if } \varepsilon < u^1 - u^2 < N \\ 0 & \text{if } u^1 - u^2 \leq \varepsilon \end{cases}$$

Then φ is Lipschitz and vanishes in a neighborhood of any point of C. Moreover, $0 \leq \varphi < N$ and $\nabla\varphi = \nabla u^1 - \nabla u^2$ in the set where $\varepsilon < u^1 - u^2 < N$ and $\nabla\varphi = 0$ almost everywhere in the complement of this set. Let $D_{\varepsilon,\delta}$ denote the set of points in D whose distance from ∂D is at least δ and whose distance from the endpoints of $\{A_i\}$ and $\{B_i\}$ is at least ε, where $\delta < \varepsilon$. For small δ, $\Gamma := \partial D_{\varepsilon,\delta}$ consists of arcs A_i' parallel to A_i, B_i' parallel to B_i , C' parallel to

C (all at distance δ) and small spherical arcs of radius ε with centers at the endpoints of the A_i and B_i. Now let

$$J = \int_\Gamma \varphi < \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2}, \nu > ds,$$

where ν is the outer normal to Γ with respect to the domain $D_{\varepsilon,\delta}$. We can estimate J from above in two ways:

(7.42)

$$J = \int_\Gamma \varphi(1 - < \frac{\nabla u^2}{W_2}, \nu >) ds - \int_\Gamma \varphi(1 - < \frac{\nabla u^1}{W_1}, \nu >) ds \leq M \int_{\Gamma \backslash C'} (1 - < \frac{\nabla u^2}{W_2}, \nu >) ds$$

or

(7.43)

$$J = \int_\Gamma \varphi(1 + < \frac{\nabla u^1}{W_1}, \nu >) ds - \int_\Gamma \varphi(1 + < \frac{\nabla u^2}{W_2}, \nu >) ds \leq M \int_{\Gamma \backslash C'} (1 + < \frac{\nabla u^1}{W_1}, \nu >) ds$$

since for small enough δ, $\varphi \equiv 0$ in a neighborhood of C'. Now applying Lemmas 5.2 and 5.4 in the domain bounded by A_i, A_i' and the two spherical arcs of radius ε gives

$$\int_{A_i'} (1 - < \frac{\nabla u^j}{W_j}, \nu >) ds = O(\varepsilon) \ , \ j = 1, 2 \ . \tag{7.44}$$

Similarly considering the domain bounded by B_i, B_i' and the two spherical arcs of radius ε gives

$$\int_{B_i'} (1 + < \frac{\nabla u^j}{W_j}, \nu >) ds = O(\varepsilon) \ , \ j = 1, 2 \ . \tag{7.45}$$

where ν is the outer normal to B_i' with respect to the domain $D_{\varepsilon,\delta}$. Using (7.42, (7.43), (7.44) and (7.45) we find that $J = O(N\varepsilon)$. On the other hand by Lemma 2.1

$$J = \int_{D_{\varepsilon,\delta}} < \nabla\varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} > dV \geq 0 \ . \tag{7.46}$$

Now consider a sequence of values of ε decreasing to zero with corresponding values $\delta(\varepsilon)$ also decreasing to zero. Then $D_{\varepsilon,\delta}$ is increasing so that J is

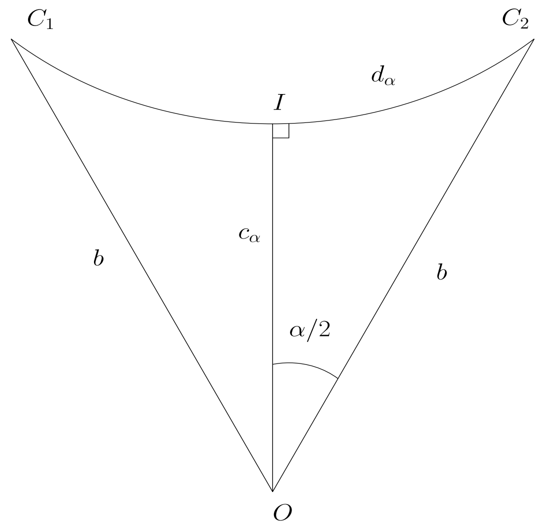

Figure 1: Triangle $T(\alpha)$

monotone increasing (recall Lemma 2.1). It follows that $\nabla u^1 \equiv \nabla u^2$ in the set where $0 < u^1 - u^2 < N$. Since N is arbitrary, $\nabla u^1 \equiv \nabla u^2$ in the set where $u^1 > u^2$. It follows that if the set $u^1 > u^2$ contains any nontrivial component, then $u^1 \equiv u^2$+positive constant in that component and so by the maximum principle, $u^1 \equiv u^2 +$ positive constant in D. By the assumptions of the theorem, the constant must be nonpositive, a contradiction. Thus $u^1 \leq u^2$ in D. The same proof holds if C is empty.

8 Examples of Scherk type

Suppose $\mathbb{M} = \mathbb{H}^2$ and $H \leqslant \frac{1}{2}$. We now construct polygons with an arbitrary number of sides that are admissible domains satisfying the Jenkins-Serrin conditions. Let S_θ be the sector of angle θ in a Euclidean circle of radius b and center O and denote by T_α the triangle (O, C_1, C_2), where the angle at O is α and the points C_1, C_2 are on the circle. Let I be the mid-point of $[C_1, C_2]$

and let $2d_\alpha := dist(C_1, C_2)$ and $c_\alpha := dist(O, I)$, see figure 1. We define a lens $L_H(2d)$, to be a region bounded by a geodesic of length $2d$ and a segment of curvature $\kappa = 2H$. When the geodesic is fixed, there are two lenses with this boundary. By $T(\alpha) + L_H(2d_\alpha)$ we mean $T(\alpha)$ union the lens $L_H(2d_\alpha)$ whose intersection with $T(\alpha)$ is the geodesic $[C_1, C_2]$. Also $T(\alpha) - L_H(2d_\alpha)$ is the complement in $T(\alpha)$ of the lens $L_H(2d_\alpha)$ having $[C_1, C_2]$ in its boundary and intersecting $T(\alpha)$ in a region; (cf. figure 2).

We consider the domain $D(\alpha, \beta) \subset S_{\alpha+\beta}$ defined as (see figure 3):

$$D(\alpha, \beta) = (T(\alpha) + L_H(2d_\alpha)) \cup (T(\beta) - L_H(2d_\beta)).$$

The boundary $\partial D(\alpha, \beta)$ consists of two geodesics of length b, and two arcs A, B of constant curvature $\kappa = 2H$, with curvature vectors pointing inside $D(\alpha, \beta)$ on A and outside $D(\alpha, \beta)$ on B.

Proposition 8.1. *Let $\theta \in [0, \pi]$. There exists $S_\alpha \cup S_\beta = S_\theta$, a partition of S_θ such that:*

$$|A| - |B| = 2H|D(\alpha, \beta)|$$

where $|D(\alpha, \beta)| = \mathrm{Area}(D(\alpha, \beta))$. The perimeter $|\partial D(\alpha, \beta)| = 2b + |A| + |B|$.

Proof. In this proof we will use several formulas of hyperbolic geometry which we derive in the appendix. We consider the function $F(\alpha, \beta) = 2H|D(\alpha, \beta)| + |B| - |A|$. By Lemma 9.3 (here we are using $H \leqslant \frac{1}{2}$), we have $F(t) = F(t\theta, (1-t)\theta))$ is a continuous function on $t \in [0, 1]$ with $F(0) = F(0, \theta) > 0$ and $F(1) = F(\theta, 0) < 0$.

Fix an integer n, and apply Proposition 8.1 to the sector $S_{\frac{2\pi}{n}}$. We do n consecutive reflections through the geodesic sides of $S_{\frac{2\pi}{n}}$ to obtain a simple closed curve $\partial D(n)$ composed of n arcs of constant curvature $\kappa = +2H$ and n arcs with curvature $\kappa = -2H$. Since the Euclidean circle centered at O

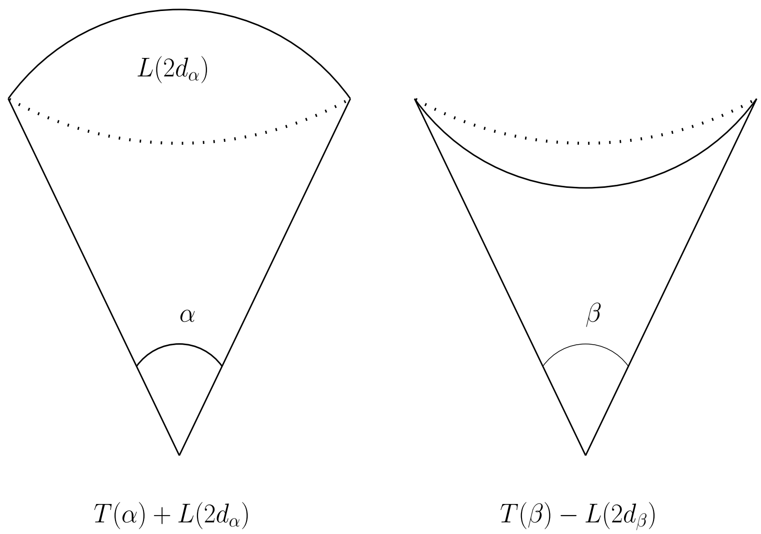

Figure 2: Domain $D(\alpha, 0)$ et $D(0, \beta)$

of radius b has curvature greater than one, such domains D are inscribed in this circle. Here the sign of κ is with respect to the disk $D(n)$ bounded by $\partial D(n)$. It is easy to see that $D(n)$ is an admissible domain (see definition 7.1) with only boundary components of type A_i and B_i. Thus to solve the Dirichlet problem with boundary data $+\infty$ on the A_i and $-\infty$ on the B_i of $\partial D(n)$, it suffices to show the conditions (7.38) and (7.39) of theorem 7.12 are satisfied.

Now we denote $D(n)$ by D and $\partial D(n)$ by ∂D. So we must show that for $\mathcal{P}$ an admissible polygon inscribed in D we have

$$n|A| = n|B| + 2H\mathcal{A}(D), \text{ for } \mathcal{P} = D, \tag{7.38}$$

and

$$2\lambda|A| < |\partial\mathcal{P}| + 2H\mathcal{A}(\mathcal{P}) \tag{7.39a}$$

$$2\mu|B| < |\partial\mathcal{P}| - 2H\mathcal{A}(\mathcal{P}) \tag{7.39b}$$

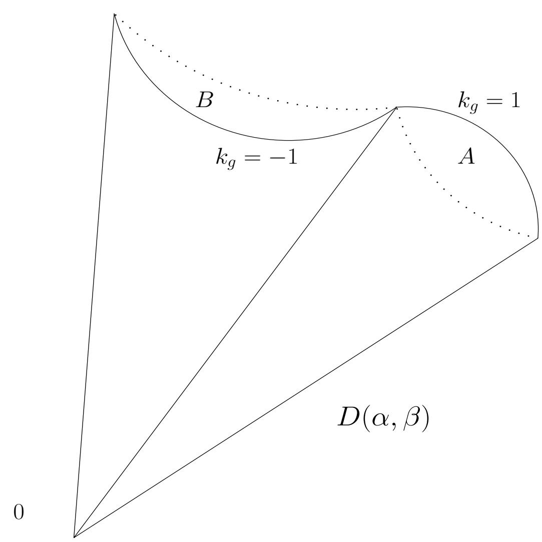

Figure 3: Domain $D(\alpha, \beta)$

for $\mathcal{P} \neq D$ and λ, μ the number of A_i and B_i's sides in $\partial\mathcal{P}$ respectively. The condition (7.38) is satisfied for $\mathcal{P} = D$ by construction: the perimeter of $\mathcal{P}$ is given by $|\partial D| = n|A| + n|B|$ and (by Proposition 8.1)

$$n(|A| - |B|) = 2H\mathcal{A}(D).$$

Now let $\mathcal{P}$ an admissible polygon in D, $\mathcal{P} \neq D$.

Definition 8.2. *An elementary admissible polygon $\mathcal{P}$ is an admissible polygon contained in a half-disk such that $(D - \mathcal{P})$ have only one connected component.*

We first check that the Jenkins-Serrin inequalities (7.39) are satisfied for elementary polygons. The elementary polygons have only one additional side C, the boundary of a lens which can be concave or convex with respect to $\mathcal{P}$. When it is convex (resp. concave) we denote the admissible polygon by $\mathcal{P}_i$ (resp. $\mathcal{P}_e$). We have three cases, depending on $(\lambda, \mu) = (k, k), (k+1, k)$

or $(k, k+1)$ with k chosen so that $\lambda\alpha + \mu\beta \leqslant \pi$. The perimeter is given by $|\partial\mathcal{P}| = \lambda|A| + \mu|B| + |C|$.

We remark that $|C|$ is the length of an equidistant curve which bound a lens associated to a geodesic of length $2d_{\lambda\alpha+\mu\beta}$. The relationship between the area is $\mathcal{A}(\mathcal{P}_i(\lambda,\mu)) = \mathcal{A}(\mathcal{P}_e(\lambda,\mu)) + 2\mathcal{A}(L_H(2d_{\lambda\alpha+\mu\beta}))$. Thus it suffices to prove

$$2\lambda|A| - |\partial\mathcal{P}(\lambda,\mu)| < \mathcal{A}(\mathcal{P}_e(\lambda,\mu)) < \mathcal{A}(\mathcal{P}_i(\lambda,\mu)) < |\partial\mathcal{P}(\lambda,\mu)| - 2\mu|B|$$

We have

$$\begin{aligned}\mathcal{A}(\mathcal{P}_i(\lambda,\mu)) &= k|D(\alpha,\beta)| + (\lambda-k)|D(\alpha,0)| + (\mu-k)|D(0,\beta)| \\ &\quad -|T(\lambda\alpha+\mu\beta)| + |L_H(2d_{\lambda\alpha+\mu\beta})|.\end{aligned}$$

By definition of $F(\alpha,\beta)$ and Lemma 9.3 of the appendix we have

$$\begin{aligned}2H\mathcal{A}(\mathcal{P}_i(\lambda,\mu)) - |\partial\mathcal{P}(\lambda,\mu)| + 2\mu|B| &= (\lambda-k)F(\alpha,0) + (\mu-k)F(0,\beta) \\ &\quad -F(0,\lambda\alpha+\mu\beta) < 0\end{aligned}$$

To check the sign, we remark $F(\alpha,0) < 0, F(0,\lambda\alpha+\mu\beta) > 0$ and since $d_\alpha < d_{\lambda\alpha+\mu\beta}$, we use $|A| < |C|$:

$$\begin{aligned}F(0,\beta) - F(0,\lambda\alpha+\mu\beta) &= -F(\alpha,0) - F(0,\lambda\alpha+\mu\beta) < (1-4H^2)(|A| - |C|) \\ &\quad +4H\arcsin(2H\tanh d_\alpha) - 4H\arcsin(2H\tanh d_{\lambda\alpha+\mu\beta}) < 0\end{aligned}$$

Similarly we have

$$\begin{aligned}2H\mathcal{A}(\mathcal{P}_e(\lambda,\mu)) + |\partial\mathcal{P}(\lambda,\mu)| - 2\lambda|A| &= (\lambda-k)F(\alpha,0) + (\mu-k)F(0,\beta) \\ &\quad -F(\lambda\alpha+\mu\beta,0) > 0\end{aligned}$$

To check the sign, we remark $F(0,\beta) > 0, F(\lambda\alpha+\mu\beta,0) < 0$ and $F(\alpha,0) - F(\lambda\alpha+\mu\beta,0) > 0$ by lemma 9.3.

Now let $\mathcal{P}$ be an admissible polygon. We can write $\mathcal{P} = D - \sum \mathcal{P}_i$ where each $\mathcal{P}_i$ is an elementary admissible polygon or its complement in D. To conclude we prove the following lemma:

Lemma 8.3. *We consider $\mathcal{P}$ an admissible polygon with $\mathcal{P} = D - \sum \mathcal{P}_i$ for a finite collection of admissible polygons $\mathcal{P}_i$ satisfying $|\partial\mathcal{P}_i| = \lambda_i|A| + \mu_i|B| + |C_i|$*

$$2\lambda_i|A| - |\partial\mathcal{P}_i| < 2H\mathcal{A}(\mathcal{P}_i) < |\partial\mathcal{P}_i| - 2\mu_i|B|.$$

Then

$$2\lambda|A| - |\partial\mathcal{P}| < 2H\mathcal{A}(\mathcal{P}) < |\partial\mathcal{P}| - 2\mu|B|.$$

Proof. We have $|\partial\mathcal{P}| = (n - \sum\lambda_i)|A| + (n - \sum\mu_i)|B| + \sum|C_i|$. Using $2H\mathcal{A}(P) = n(|A| - |B|)$ we have

$$n(|A| - |B|) + \sum 2\mu_i|B| - \sum|\partial\mathcal{P}_i| < 2H\mathcal{A}(P) - \sum 2H\mathcal{A}(\mathcal{P}_i)$$

$$2H\mathcal{A}(P) - \sum 2H\mathcal{A}(\mathcal{P}_i) < n(|A| - |B|) - \sum 2\lambda_i|A| + \sum|\partial\mathcal{P}_i|$$

which gives the result by observing $\lambda = n - \sum\lambda_i$ and $\mu = n - \sum\mu_i$.

9 Appendix

9.1 Hyperbolic geometry

Lemma 9.1. *In a Euclidean circle of radius $R = \tanh\frac{b}{2}$, centered at the origin O in the disk model of $\mathbb{H}$, we consider a triangle (O, C_1, C_2) where the angle at the center is α and $dist(O, C_1) = dist(O, C_2) = b$. The point I is the* *mid-point of* $[C_1, C_2]$. *With* b *fixed, we note* $dist(O, I) = c_\alpha, dist(C_1, I) = d_\alpha$ *and the area* $|T(\alpha)| = \text{Area}(O, C_1, C_2)$.

1) (Pythagoras) $\cosh b = \cosh d_\alpha \cosh c_\alpha$

2) $\sinh d_\alpha = \sinh b \sin(\alpha/2)$ *and* $\tanh c_\alpha = \tanh b \cos(\alpha/2)$

3) $\sin \dfrac{|T(\alpha)|}{2} = \dfrac{\sinh c_\alpha \sinh d_\alpha}{1 + \cosh b}$

Lemma 9.2. *We consider the half-lens* $L_H(2d)$ *bounded by a geodesic* γ *of length* $2d$ *and* A *a constant curvature* $\kappa = 2H$ *arc joining the end points of* γ. *Then the area is*

$$\mathcal{A}(L_H(2d)) = |L_H(2d)| = 2H|A| - 2\arcsin(2H \tanh d).$$

Proof. By Gauss-Bonnet, $|L_H(2d)| = 2H|A| - 2\alpha$ where $\pi - \alpha$ is the exterior angle at the edge of the lens. We use the half-space model of the hyperbolic space. The geodesic segment of hyperbolic length $2d$ is the vertical Euclidean segment between the points $p_- = (0, 1 - \tanh d)$ and $p_+ = (0, 1 + \tanh d)$ in $R^{+,3}$. We consider an arc of a circle with radius $R = 1/2H$, passing by p_- and p_+. Then the center of the circle is $O = (x, 1)$ with $x^2 + \tanh^2 d = 1/4H^2$. The hyperbolic curvature of the circle is $\kappa = 2H$ (the point O is at heigth 1). The lens $L_H(2d)$ is the region bounded by the arc and the vertical segment between p_- and p_+. Elementary trigonometry gives that $\sin \alpha = 2H \tanh d$.

Lemma 9.3. *We consider the function*

$$F(\alpha, \beta) = 2H(|T(\alpha)| + |L_H(2d_\alpha)| + |T(\beta)| - |L_H(2d_\beta)|) + |B| - |A|.$$

with $(\alpha, \beta) \in [0, \pi]^2$ *and* $\alpha + \beta = \theta \in [0, \pi]$. *Then we have* $F(0, \beta) > 0$ *and for* $0 < \alpha < \gamma \leqslant \pi$,

$$0 > F(\alpha, 0) > F(\gamma, 0).$$

Proof. We consider $F(\alpha,\beta)=2H(|T(\alpha)|+|L_H(2d_\alpha)|+|T(\beta)|-|L_H(2d_\beta)|)+|B|-|A|$ for some value $\alpha+\beta=\theta\in[0,\pi]$ and by the two lemmas above we have

$$F(\alpha,\beta)=4H\arcsin\left(\frac{\sinh c_\alpha\sinh d_\alpha}{1+\cosh b}\right)+4H\arcsin\left(\frac{\sinh c_\beta\sinh d_\beta}{1+\cosh b}\right)$$

$$-4H\arcsin(2H\tanh d_\alpha)+4H\arcsin(2H\tanh d_\beta)+(1-4H^2)(|B|-|A|)$$

When $\alpha=0$, we have $d_\alpha=0$ and then $F(0,\beta)>0$ when $\pi>\beta>0$. When $\beta=0$, we prove for $\alpha\in]0,\pi[$

$$F(\alpha,0)=4H\arcsin\left(\frac{\sinh c_\alpha\sinh d_\alpha}{1+\cosh b}\right)-4H\arcsin(2H\tanh d_\alpha)+(4H^2-1)|A|<0.$$

To see that we remark that $F(0,0)=0$ by construction and we prove that $F(\gamma,0)<F(\alpha,0)$ for $0<\alpha<\gamma<\pi$. We rewrite

$$\begin{aligned}
&F(\alpha,0)-F(\gamma,0)=(1-4H^2)(|A_\gamma|-|A_\alpha|)\\
&\quad+4H\arcsin\left(\tfrac{\sinh b}{1+\cosh b}\tanh d_\alpha\cos\alpha/2\right)-4H\arcsin\left(\tfrac{\sinh b}{1+\cosh b}\tanh d_\alpha\cos\gamma/2\right)\\
&\quad+4H\arcsin\left(\tfrac{\sinh b}{1+\cosh b}\tanh d_\alpha\cos\gamma/2\right)-4H\arcsin\left(\tfrac{\sinh b}{1+\cosh b}\tanh d_\gamma\cos\gamma/2\right)\\
&\quad+4H\arcsin\left(2H\tanh d_\gamma\right)-4H\arcsin\left(2H\tanh d_\alpha\right)
\end{aligned}$$

When $\alpha<\gamma$, we have $d_\alpha<d_\gamma$ and then $|A_\alpha|<|A_\gamma|$. Then the first line is positive. The second line of the equation above is positive ($\cos\gamma/2<\cos\alpha/2$). For the last two lines, we can see that the derivative of the function $g(t)=2\arcsin(ta)-2\arcsin(tb)$ has the sign of a^2-b^2. With $a=\tanh d_\alpha$ and $b=\tanh d_\gamma$, we have

$$g\left(\frac{\sinh b\cos\gamma/2}{1+\cosh b}\right)>g(1)>g(2H)$$

and $F(\alpha,0)$ is decreasing as desired.

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