TOTAL CURVATURE AND THE ISOPERIMETRIC INEQUALITY IN CARTAN-HADAMARD MANIFOLDS

MOHAMMAD GHOMI AND JOEL SPRUCK

Abstract. We prove that the total positive Gauss-Kronecker curvature of any closed hypersurface embedded in a complete simply connected manifold of nonpositive curvature M^n, $n \geq 2$, is bounded below by the volume of the unit sphere in Euclidean space $\mathbf{R}^n$. This yields the optimal isoperimetric inequality for bounded regions of finite perimeter in M, via Kleiner's variational approach, and thus settles the Cartan-Hadamard conjecture. The proof employs a comparison formula for total curvature of level sets in Riemannian manifolds, and estimates for smooth approximation of the signed distance function. Immediate applications include sharp extensions of the Faber-Krahn and Sobolev inequalities to manifolds of nonpositive curvature.

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1. Introduction

The classical isoperimetric inequality [14,23,39,95,119] states that in Euclidean space $\mathbf{R}^n$, of dimension $n \geq 2$, spheres form unique enclosures of least perimeter for any given volume. More formally, for any bounded set $\Omega \subset \mathbf{R}^n$,

$$\text{per}(\Omega) \geq n\,\omega_n^{\frac{1}{n}}\,\text{vol}(\Omega)^{\frac{n-1}{n}}, \tag{1}$$

where *per* stands for perimeter, *vol* is the volume, and $\omega_n := \pi^{n/2}/(n/2)!$ is the volume of the unit ball in $\mathbf{R}^n$. Furthermore, equality holds only if Ω is a ball. We extend these facts to *Cartan-Hadamard manifolds* M^n, i.e., complete simply connected Riemannian spaces of nonpositive curvature:

Theorem 1.1. *The isoperimetric inequality (1) holds for all bounded sets Ω of finite perimeter in Cartan-Hadamard manifolds M^n, $n \geq 2$. Furthermore, equality holds if and only if Ω is isometric to a ball in $\mathbf{R}^n$.*

This result settles a problem which has been widely known as the *Cartan-Hadamard conjecture* [8, 59, 88, 102, 106]. It may be traced back to Weil [145] [20, p. 347] who established the above theorem for $n = 2$ in 1926, and Beckenbach-Rado [18] who rediscovered the same result in 1933. In 1975 Aubin [7] conjectured that the above theorem holds for $n \geq 3$, as did Gromov [83, 85], and Burago-Zagaller [33, 34] a few years later. Prior to this work, only the cases $n = 3$ and 4 of the theorem had been established, by Kleiner [101] in 1992, and Croke [50] in 1984 respectively, using different methods. See [128, Sec. 3.3.2] and [134] for alternative proofs for $n = 3$, and [102] for another proof for $n = 4$. Other related studies and references may be found in [57,58,87,115,116,135].

Our proof of Theorem 1.1 generalizes Kleiner's approach, which is based on estimating the *total positive curvature* of closed hypersurfaces Γ in M. This quantity is defined as $\mathcal{G}_+(\Gamma) := \int_{\Gamma_+} GK d\sigma$, where GK denotes the Gauss-Kronecker curvature of Γ, and $\Gamma_+ \subset \Gamma$ is the region where $GK \geq 0$. Kleiner showed that when $n = 3$,

$$\mathcal{G}_+(\Gamma) \geq n\omega_n, \tag{2}$$

via the Gauss-Bonnet theorem and a convex hull argument. This inequality is easily seen to hold in $\mathbf{R}^n$, since the Gauss map of Γ covers the unit sphere $\mathbf{S}^{n-1}$, which has area $n\omega_n$. The central result of this work is that:

Theorem 1.2. *The total curvature inequality (2) holds for all closed embedded $\mathcal{C}^{1,1}$ hypersurfaces Γ in Cartan-Hadamard manifolds M^n, $n \geq 2$.*

The study of total positive curvature goes back to Alexandrov [2] and Nirenberg [117], and its relation to isoperimetric problems has been well-known [45, 46]. The minimizers for this quantity, which are called *tight hypersurfaces*, have been extensively studied since

Chern-Lashoff [43, 44]; see [37] for a survey, and [26, 27, 54, 55, 138] for other studies in Cartan-Hadmard manifolds or the hyperbolic space.

Theorem 1.1 follows from Theorem 1.2 via the well-known variational method involving the isoperimetric profile (Section 11). Our main task then is to prove Theorem 1.2 (Section 10). To this end we may assume, by Kleiner's convex hull argument (Section 9), that Γ is convex, i.e., it bounds a convex set Ω in M. We may further assume that the (signed) distance function u of Γ is convex, by replacing Γ with an outer parallel hypersurface of Ω in $M \times \mathbf{R}$ (Section 3). The main plan after that will be to use u to push Γ into Ω, without increasing $\mathcal{G}_+(\Gamma)$, until Γ collapses to a point. As M is locally Euclidean to first order, we will then obtain (2) as desired. So we develop a formula (Section 4) for comparing the curvature of level sets of $\mathcal{C}^{1,1}$ functions on M. This result will show (Section 4.3) that Γ may be moved inward through level sets of u with no increase in $\mathcal{G}_+(\Gamma)$, until it reaches the singularities of u, or the cut locus of Γ (Section 2). In particular, if the cut locus is a single point (i.e., Γ is a geodesic sphere) then we are done; otherwise, we will use the inf-convolution (Section 7) to approximate u by a family v^r of $\mathcal{C}^{1,1}$ convex functions with a unique minimum point x_0 (Section 10.1). We will control the rate at which the curvature of level sets of v^r blow up, as $v^r \to u$, via the theory of semiconcave functions and proximal maps (Sections 7 and 10.4), Reilly type integral formulas (Sections 5 and 10.5), and results on the structure of the cut locus (Sections 6 and 10.6). Finally, applying the comparison formula to the level sets of v^r, as they shrink to x_0, will complete the proof.

In short, the main theme of this work is the interplay between the curvature and distance function of Riemannian submanifolds. Smooth approximation of this function, with proper control over the first two derivatives, will provide the key to proving Theorem 1.2 and therefore Theorem 1.1, via our comparison formula. In addition to the techniques mentioned above, convolution in the sense of Greene and Wu (Sections 4 and 8), Federer's notion of positive reach (Sections 2 and 8), and Riccati's equation for curvature of tubes (Sections 3, 9, 11) will be featured along the way. A number of our intermediate results, particularly Theorem 4.9 (the comparison formula), Theorem 6.1 (structure of cut locus), Theorem 8.1 (continuity of total curvature), and Theorem 10.13 (Reilly type formula) may be of independent interest. In addition, our method might be considered as a type of degenerate flow which reduces total curvature. There is a more conventional geometric flow, by harmonic mean curvature, which also shrinks convex hypersurfaces to a point in Cartan-Hadamard manifolds [146]; however, it is not known how that flow effects total curvature; see also [4].

The isoperimetric inequality has several well-known applications [13, 121, 122], due to its relations with many other important inequalities [39, 141]. For instance Theorem

1.1 yields the following extension of the classical Sobolev inequality from the Euclidean space to Cartan-Hadamard manifolds [58, 66, 119], [113, App.1]. Indeed it was in this context where the Cartan-Hadamard conjecture was first proposed [7]:

Corollary 1.3. *(Sobolev Inequality) Let M^n be a Cartan-Hadamard manifold. Then for all Lipschitz functions with compact support in M,*

$$\left(\int_M f^{\frac{n}{n-1}}\, d\mu\right)^{\frac{n-1}{n}} \leq \frac{1}{n\omega_n^{\frac{1}{n}}} \int_M |\nabla f|\, d\mu. \tag{3}$$

Conversely, if (3) holds, then the isoperimetric inequality (1) holds for all bounded sets Ω of finite perimeter in M.

See [59, 105, 106] for a host of other Sobolev type inequalities on Cartan-Hadamard manifolds which follow from Theorem 1.1, and [48, 93, 147] for related studies. The isoperimetric inequality also has deep connections to spectral analysis. A fundamental result in this area is the Faber-Krahn inequality [20, 38, 91] which was established in 1920's [62, 103, 104] in Euclidean space, as had been conjectured by Rayleigh in 1877 [123]. By Theorem 1.1, this inequality may now be generalized to Cartan-Hadamard manifolds as well [38]:

Corollary 1.4. *(Faber-Krahn Inequality) Let λ_1 denote the first Dirichlet eigenvalue of a bounded domain Ω with Lipschitz boundary in a Cartan-Hadamard manifold M^n. Then,*

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

where B is a ball in $\mathbf{R}^n$ with $\mathrm{vol}(B) = \mathrm{vol}(\Omega)$; furthermore, equality holds only if Ω is isometric to B.

We should mention that the Cartan-Hadamard conjecture has a stronger form [7, 34, 85], sometimes called the *generalized Cartan-Hadamard conjecture* [102], which states that if the sectional curvatures of M in Theorem 1.1 are bounded above by $k \leq 0$, then the perimeter of Ω cannot be smaller than that of a ball of the same volume in the hyperbolic space of constant curvature k. The generalized conjecture has been proven only for $n = 2$ by Bol [25], and $n = 3$ by Kleiner [101]; see also Kloeckner-Kuperberg [102] for partial results for $n = 4$, and Johnson-Morgan [115] for a result on small volumes. The methods developed in this paper to prove Theorem 1.1 likely have some bearing on the generalized conjecture as well, although we do not directly address that problem here. The rest of this paper is organized as follows. In Sections 2 and 3 we develop the basic regularity and convexity properties of distance functions and hypersurfaces which will be needed throughout the paper. Then in Section 4 we establish the comparison formula mentioned above. Sections 5 to 9 will be devoted to other intermediate results and various estimates needed to apply the comparison formula to the proof of Theorem 1.2, which will be presented in Section 10. Finally, Theorem 1.1 will be proved in Section 11, with the aid of Theorem 1.2.

2. Regularity and Singular Points of the Distance Function

Throughout this paper, M denotes a complete connected Riemannian manifold of dimension $n \geq 2$ with metric $\langle \cdot, \cdot \rangle$ and corresponding distance function $d\colon M \times M \to \mathbf{R}$. For any pairs of sets X, $Y \subset M$, we define

$$d(X, Y) := \inf\{d(x, y) \mid x \in X,\ y \in Y\}.$$

Furthermore, for any set $X \subset M$, we define $d_X\colon M \to \mathbf{R}$, by

$$d_X(\,\cdot\,) := d(X, \,\cdot\,).$$

The *tubular neighborhood* of X with radius r is then given by $U_r(X) := d_X^{-1}([0, r))$. Furthermore, for any $t > 0$, the level set $d_X^{-1}(t)$ will be called a *parallel hypersurface* of X at distance t. A function $u\colon M \to \mathbf{R}$ is Lipschitz with constant L, or L-*Lipschitz*, if for all pairs of points x, $y \in M$, $|u(x) - u(y)| \leq L\, d(x, y)$. The triangle inequality and Rademacher's theorem quickly yield [60, p.185]:

Lemma 2.1 ([60])**.** *For any set $X \subset M$, d_X is 1-Lipschitz. In particular d_X is differentiable almost everywhere.*

For any point $p \in M$ and $X \subset M$, we say that $p^\circ \in X$ is a *footprint* of p on X provided that

$$d(p, p^\circ) = d_X(p),$$

and the distance minimizing geodesic connecting p and p° is unique. In particular note that every point of X is its own footprint. The following observation is well-known when $M = \mathbf{R}^n$. It follows, for instance, from studying super gradients of semiconcave functions [36, Prop. 3.3.4 & 4.4.1]. These arguments extend well to Riemannian manifolds [111, Prop. 2.9], since local charts preserve both semiconcavity and generalized derivatives. For any function $u\colon M \to \mathbf{R}$, we let ∇u denote its gradient.

Lemma 2.2 ([36, 111])**.** *Let $X \subset M$ be a closed set, and $p \in M \setminus X$. Then*

(i) d_X is differentiable at p if and only if p has a unique footprint on X.
(ii) If d_X is differentiable at p, then $\nabla d_X(p)$ is tangent to the distance minimizing geodesic connecting p to its footprint on X, and $|\nabla d_X(p)| = 1$.
(iii) d_X is $\mathcal{C}^1$ on any open set in $M \setminus X$ where d_X is pointwise differentiable.

Throughout this paper, Γ will denote a closed embedded topological hypersurface in M. Furthermore we assume that Γ bounds a designated *domain* Ω, i.e., a connected open set with compact closure $\mathrm{cl}(\Omega)$ and boundary

$$\partial\Omega = \Gamma.$$

The *(signed) distance function* $\widehat{d}_\Gamma \colon M \to \mathbf{R}$ of Γ (with respect to Ω) is then given by

$$\widehat{d}_\Gamma(\,\cdot\,) := d_\Omega(\,\cdot\,) - d_{M\setminus\Omega}(\,\cdot\,).$$

In other words, $\widehat{d}_\Gamma(p) = -d_\Gamma(p)$ if $p \in \Omega$, and $\widehat{d}_\Gamma(p) = d_\Gamma(p)$ otherwise. The level sets $\widehat{d}_\Gamma^{-1}(t)$ will be called *outer parallel hypersurfaces* of Γ if $t > 0$, and *inner parallel hypersurfaces* if $t < 0$. Let $\mathrm{reg}(\widehat{d}_\Gamma)$ be the union of all open sets in M where each point has a unique footprint on Γ. Then the *cut locus* of Γ is defined as

$$\mathrm{cut}(\Gamma) := M \setminus \mathrm{reg}\big(\widehat{d}_\Gamma\big).$$

For instance when Γ is an ellipse in $\mathbf{R}^2$, $\mathrm{cut}(\Gamma)$ is the line segment in Ω connecting the focal points of the inward normals (or the cusps of the evolute) of Γ, see Figure 1. Note

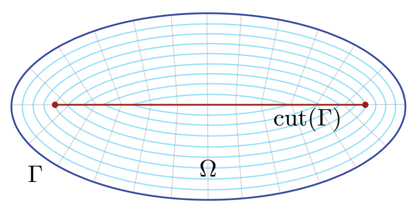

FIGURE 1.

that the singularities of parallel hypersurfaces of Γ all lie on $\mathrm{cut}(\Gamma)$. Since d_Γ may not be differentiable at any point of Γ, we find it more convenient to work with $\widehat{d}_\Gamma$ instead. Part (iii) of Lemma 2.2 may be extended as follows:

Lemma 2.3. *If Γ is $\mathcal{C}^1$, then $\widehat{d}_\Gamma$ is $\mathcal{C}^1$ on $M \setminus \mathrm{cut}(\Gamma)$ with $|\nabla \widehat{d}_\Gamma| = 1$.*

Proof. By Lemma 2.2, $\widehat{d}_\Gamma$ is $\mathcal{C}^1$ on $(M \setminus \Gamma) \setminus \mathrm{cut}(\Gamma)$. Thus it remains to consider the regularity of $\widehat{d}_\Gamma$ on $\Gamma \setminus \mathrm{cut}(\Gamma)$. To this end let $p \in \Gamma \setminus \mathrm{cut}(\Gamma)$, and U be a convex open neighborhood of p in M which is disjoint from $\mathrm{cut}(\Gamma)$. Then each point of U has a unique footprint on $\Gamma \cap U$, and thus U is fibrated by geodesic segments orthogonal to $\Gamma \cap U$. For convenience, we may assume that all these segments have the same length. Now let $\Gamma_\varepsilon := (\widehat{d}_\Gamma)^{-1}(\varepsilon)$ where $\varepsilon > 0$ is so small that Γ_ε intersects U. Then each point of $\Gamma_\varepsilon \cap U$ has a unique foot print on $\Gamma \cap U$. Furthermore, by Lemma 2.2, $\Gamma_\varepsilon \cap U$ is a $\mathcal{C}^1$ hypersurface, since $\widehat{d}_\Gamma$ is $\mathcal{C}^1$ on $U \setminus \Gamma$ and has nonvanishing gradient there. So $\Gamma_\varepsilon \cap U$ is orthogonal to the geodesic segments fibrating U. Since these segments do not intersect each other, U is disjoint from $\mathrm{cut}(\Gamma_\varepsilon)$. So $\widehat{d}_{\Gamma_\varepsilon}$ is $\mathcal{C}^1$ on $U \setminus \Gamma_\varepsilon$ by Lemma 2.2. Finally note that $\widehat{d}_\Gamma = \widehat{d}_{\Gamma_\varepsilon} - \varepsilon$ on U, which completes the proof. □

The *medial axis* of Γ, $\mathrm{medial}(\Gamma)$, is the set of points in M with multiple footprints on Γ. Note that

$$\mathrm{cut}(\Gamma) = \mathrm{cl}\big(\mathrm{medial}(\Gamma)\big). \tag{4}$$

For instance, when Γ is an ellipse in $\mathbf{R}^2$, $\mathrm{medial}(\Gamma)$ is the relative interior of the segment connecting its foci. Let $\mathrm{sing}(\widehat{d}_\Gamma)$ denote the set of *singularities* of $\widehat{d}_\Gamma$ or points of M where $\widehat{d}_\Gamma$ is not differentiable. Then

$$\mathrm{medial}(\Gamma) = \mathrm{sing}\big(\widehat{d}_\Gamma\big), \qquad \text{and} \qquad \mathrm{cut}(\Gamma) = \mathrm{cl}\big(\mathrm{sing}(\widehat{d}_\Gamma)\big).$$

We say that a (geodesic) sphere $S \subset \mathrm{cl}(\Omega)$ is *maximal* if it is not contained in a larger sphere which also lies in $\mathrm{cl}(\Omega)$. The set of centers of maximal spheres contained in $\mathrm{cl}(\Omega)$ is called the *skeleton* of Ω.

Lemma 2.4.

$$\mathrm{medial}(\Gamma) \cap \Omega \quad \subset \quad \mathrm{skeleton}(\Omega) \quad \subset \quad \mathrm{cl}\big(\mathrm{medial}(\Gamma) \cap \Omega\big).$$

Proof. The first inclusion is immediate. To see the second inclusion, let $x \in \mathrm{skeleton}(\Omega)$. Then there exists a maximal sphere S in $\mathrm{cl}(\Omega)$ centered at x. By (4), it suffices to show that $x \in \mathrm{cut}(\Gamma)$. Suppose that $x \notin \mathrm{cut}(\Gamma)$. Then, by Lemma 2.2, d_Γ is $\mathcal{C}^1$ in a neighborhood U of x. Furthermore ∇d_Γ does not vanish on U, and its integral curves are distance minimizing geodesics connecting points of U to their unique footprints on Γ. It follows then that the geodesic connecting x to its footprint in Γ, may be extended at x to a longer distance minimizing geodesic. This contradicts the maximality of S and completes the proof. □

The inclusion relations in Lemma 2.4 are in general strict, even when $M = \mathbf{R}^n$ [40]. There is a vast literature on the singularities of the distance function, due to its applications in a number of fields, including computer vision, and connections to Hamilton-Jacobi equations; see [6, 53, 107, 111, 112] for more references and background. Lemma 2.3 may be extended as follows:

Lemma 2.5 ([69, 111])**.** *For $k \geq 2$, if Γ is $\mathcal{C}^k$, then $\widehat{d}_\Gamma$ is $\mathcal{C}^k$ on $M \setminus \mathrm{cut}(\Gamma)$.*

This fact has been well-known for $M = \mathbf{R}^n$ and $k \geq 2$, as it follows from the basic properties of the normal bundle of M, and applying the inverse function theorem to the exponential map, e.g. see [69] or [72, Sec. 2.4]. For Riemannian manifolds, the lemma has been established in [111, Prop. 4.3], via essentially the same exponential mapping argument in [69].

For the purposes of this work, we still need to gather finer information about Lipschitz regularity of derivatives of $\widehat{d}_\Gamma$. To this end we invoke Federer's notion of *reach* [64, 140] which may be defined as

$$\mathrm{reach}(\Gamma) := d\big(\Gamma, \mathrm{cut}(\Gamma)\big).$$

In particular note that $\mathrm{reach}(\Gamma) = r$ if and only if there exists a geodesic ball of radius r *rolling freely* on each side of Γ in M, i.e., through each point p of Γ there passes the boundaries of geodesic balls B, B' of radius r such that $B \subset \mathrm{cl}(\Omega)$, and $B' \subset M \setminus \Omega$. We say that Γ is $\mathcal{C}^{1,1}$, if it is $\mathcal{C}^{1,1}$ in local charts, i.e., for each point $p \in M$ there exists a neighborhood U of p in M, and a $\mathcal{C}^\infty$ diffeomorphism $\phi\colon U \to \mathbf{R}^n$ such that $\phi(\Gamma)$ is $\mathcal{C}^{1,1}$ in $\mathbf{R}^n$. A function $u\colon M \to \mathbf{R}$ is called *locally* $\mathcal{C}^{1,1}$ on some region X, if it is $\mathcal{C}^{1,1}$ in local charts covering X. If X is compact, then we simply say that u is $\mathcal{C}^{1,1}$ near X.

Lemma 2.6. *The following conditions are equivalent:*

(i) $\mathrm{reach}(\Gamma) > 0$.
(ii) Γ *is* $\mathcal{C}^{1,1}$.
(iii) $\widehat{d}_\Gamma$ *is* $\mathcal{C}^{1,1}$ *near* Γ.

Proof. For $M = \mathbf{R}^n$, the equivalence (i)$\Leftrightarrow$(ii) is due to [73, Thm. 1.2], since Γ is a topological hypersurface by assumption, and the positiveness of reach, or more specifically existence of local support balls on each side of Γ, ensures that the tangent cones of Γ are all flat. The general case then may be reduced to the Euclidean one via local charts. Indeed local charts of M preserve the $\mathcal{C}^{1,1}$ regularity of Γ by definition. Furthermore, the positiveness of reach is also preserved, as we demonstrate in the next paragraph.

Let (U, ϕ) be a local chart of M around a point p of Γ. We may assume that $\phi(U)$ is a ball B in $\mathbf{R}^n$. Furthermore, since Γ is a topological hypersurface, we may assume that Γ divides U into a pair of components by the Jordan Brouwer separation theorem. Consequently $\phi(\Gamma \cap U)$ divides B into a pair of components as well, which we call the sides of $\phi(\Gamma)$. The image under ϕ of the boundary of the balls of some constant radius which roll freely on each side of Γ in M generate closed $\mathcal{C}^2$ surfaces S_x, S_x' on each side of every point x of $\phi(\Gamma)$. Let $B' \subset B$ be a smaller ball centered at $\phi(p)$, and X be the connected component of $\phi(\Gamma \cap U)$ in B' which contains $\phi(p)$. Furthermore, let κ be the supremum of the principal curvatures of S_x, S_x', for all $x \in X$. Then $\kappa < \infty$, since X has compact closure in B and the principal curvatures of S_x, S_x' vary continuously, owing to the fact that ϕ is $\mathcal{C}^2$. It is not difficult then to show that the reach of S_x, S_x' is uniformly bounded below, which will complete the proof. Alternatively, we may let (U, ϕ) be a normal coordinate chart generated by the exponential map. Then for U sufficiently small, S_x and S_x' will have positive principal curvatures. So, by Blaschke's rolling theorem [24, 30], a ball rolls freely inside S_x, S'_x and consequently on each side of $\phi(\Gamma \cap U)$ near $\phi(p)$. Hence $\phi(\Gamma \cap U)$ has positive reach near $\phi(p)$, as desired.

It remains then to establish the equivalence of (iii) with (i) or (ii). First suppose that (iii) holds. Let $p \in \Gamma$ and U be neighborhood of p in M such that $u := \widehat{d}_\Gamma$ is $\mathcal{C}^{1,1}$ on U. By Lemma 2.2, $|\nabla u| \equiv 1$ on $U \setminus \Gamma$. Also note that each point of Γ is a limit of points of $U \setminus \Gamma$, since by assumption Γ is a topological hypersurface. Thus, since u is $\mathcal{C}^1$ on U, it follows that $|\nabla u| \neq 0$ on U. In particular, $\Gamma \cap U$ is a regular level set of u on U, and is $\mathcal{C}^1$ by the inverse function theorem. Let $\phi\colon U \to \mathbf{R}^n$ be a diffeomorphism. Then $\phi(\Gamma \cap U)$ is a regular level set of the locally $\mathcal{C}^{1,1}$ function $u \circ \phi^{-1}\colon \mathbf{R}^n \to \mathbf{R}$. In particular the unit normal vectors of $\phi(\Gamma \cap U)$ are locally Lipschitz continuous, since they are given by $\nabla(u \circ \phi^{-1})/|\nabla(u \circ \phi^{-1})|$. So $\phi(\Gamma \cap U)$ is locally $\mathcal{C}^{1,1}$. Hence Γ is locally $\mathcal{C}^{1,1}$, and so we have established that (iii)$\Rightarrow$(ii). Conversely, suppose that (ii) and therefore (i) hold. Then any point $p \in \Gamma$ has an open neighborhood U in M where each point has a unique footprint on M. Thus, by Lemma 2.3, u is $\mathcal{C}^1$ on U and its gradient vector field is tangent to geodesics orthogonal to Γ. So, for ε small, each level set $u^{-1}(\varepsilon) \cap U$ has positive reach and is therefore $\mathcal{C}^{1,1}$ by (ii). Via local charts we may transfer this configuration to $\mathbf{R}^n$, to generate a fibration of $\mathbf{R}^n$ by $\mathcal{C}^{1,1}$ hypersurfaces which form the level sets of $u \circ \phi^{-1}$. Since $\nabla(u \circ \phi^{-1})/|\nabla(u \circ \phi^{-1})|$ is orthogonal to these level sets, it follows then that $\nabla(u \circ \phi^{-1})$ is locally Lipschitz. Thus $u \circ \phi^{-1}$ is locally $\mathcal{C}^{1,1}$ which establishes (iii) and completes the proof. □

The following proposition for $M = \mathbf{R}^n$ is originally due to Federer [64, Sec. 4.20]; see also [56, p. 365], [36, Sec. 3.6], and [49]. In [111, Rem. 4.4], it is mentioned that Federer's result should hold in all Riemannian manifolds. Indeed it follows quickly from Lemma 2.6:

Proposition 2.7. *$\widehat{d}_\Gamma$ is locally $\mathcal{C}^{1,1}$ on $M \setminus \mathrm{cut}(\Gamma)$. In particular if Γ is $\mathcal{C}^{1,1}$, then $\widehat{d}_\Gamma$ is locally $\mathcal{C}^{1,1}$ on $U_r(\Gamma)$ for $r := \mathrm{reach}(\Gamma)$.*

Proof. For each point $p \in M \setminus \mathrm{cut}(\Gamma)$, let α_p be the (unit speed) geodesic in M which passes through p and is tangent to $\nabla\widehat{d}_\Gamma(p)$. By Lemma 2.2, α_p is a trajectory of the gradient field $\nabla\widehat{d}_\Gamma$ near p. It follows that these geodesics fibrate $M\setminus\mathrm{cut}(\Gamma)$. Consequently the level set $\{\widehat{d}_\Gamma = \widehat{d}_\Gamma(p)\}$ has positive reach near p, since it is orthogonal to the gradient field. So, by Lemma 2.6, $\widehat{d}_\Gamma$ is $\mathcal{C}^{1,1}$ near p, which completes the proof. □

We will also need the following refinement of Proposition 2.7, which gives an estimate for the $\mathcal{C}^{1,1}$ norm of $\widehat{d}_\Gamma$ near Γ, depending only on reach(Γ) and the sectional curvature K_M of M; see also Lemma 7.4 below. Here ∇^2 denotes the Hessian.

Proposition 2.8. *Suppose that* $r := \mathrm{reach}(\Gamma) > 0$*, and* $K_M \geq -C$*, for* $C \geq 0$*, on* $U_r(\Gamma)$*. Then, for* $\delta := r/2$*,*

$$\left|\nabla^2 \widehat{d_\Gamma}\right| \leq \sqrt{C}\coth\left(\sqrt{C}\delta\right)$$

almost everywhere on $U_\delta(\Gamma)$*.*

Proof. By Proposition 2.7 and Rademacher's theorem, $\widehat{d_\Gamma}$ is twice differentiable at almost every point of $U_\delta(\Gamma)$. Let $p \in U_\delta(\Gamma)$ be such a point. Then the eigenvalues of $\nabla^2\widehat{d_\Gamma}(p)$, except for the one in the direction of $\nabla\widehat{d_\Gamma}(p)$ which vanishes, are the principal curvatures of the level set $\Gamma_p := \{\widehat{d_\Gamma} = \widehat{d_\Gamma}(p)\}$. Since by assumption a ball of radius r rolls freely on each side of Γ, it follows that a ball of radius δ rolls freely on each side of Γ_p. Thus the principal curvatures of Γ_p at p are bounded above by those of spheres of radius δ in $U_r(\Gamma)$, which are in turn bounded above by $\sqrt{C}\coth(\sqrt{C}\delta)$ due to basic Riemannian comparison theory [100, p. 184]. □

3. Hypersurfaces with Convex Distance Function

A set $X \subset M$ is *(geodesically) convex* provided that every pair of its points may be joined by a unique geodesic in M, and that geodesic is contained in X. Furthermore, X is *strictly convex* if ∂X contains no geodesic segments. In this work, a *convex hypersurface* is the boundary of a compact convex subset of M with nonempty interior. In particular Γ is convex if Ω is convex. A function $u\colon M \to \mathbf{R}$ is *convex* provided that its composition with parametrized geodesics in M is convex, i.e., for every geodesic $\alpha\colon [t_0, t_1] \to M$,

$$u \circ \alpha\big((1-\lambda)t_0 + \lambda t_1\big) \leq (1-\lambda)\, u \circ \alpha(t_0) + \lambda\, u \circ \alpha(t_1),$$

for all $\lambda \in [0,1]$. We assume that all parametrized geodesics in this work have unit speed. We say that u is *strictly convex* if the above inequality is always strict. Furthermore, u is called *concave* if $-u$ is convex. When u is $\mathcal{C}^2$, then it is convex if and only if $(u \circ \alpha)'' \geq 0$, or equivalently the Hessian of u is positive semidefinite. We may also say that u is convex on a set $X \subset M$ provided that u is convex on all geodesic segments of M contained in X. For basic facts and background on convex sets and functions in general Riemannian manifolds see [143], for convex analysis in Cartan-Hadamard manifolds see [22, 136], and more generally for Hadamard or CAT(0) spaces (i.e., metric spaces of nonpositive curvature), see [11, 17, 29, 108]. In particular it is well-known that if M is a Cartan-Hadamard manifold, then $d\colon M \times M \to \mathbf{R}$ is convex [29, Prop. 2.2], which in turn yields [29, Cor. 2.5]:

Lemma 3.1 ([29])**.** *If* M *is a Cartan-Hadamard manifold, and* $X \subset M$ *is a convex set, then* d_X *is convex.*

So it follows that geodesic spheres are convex in a Cartan-Hadamard manifold as they are level sets of the distance function from one point. Let $X \subset M$ be a bounded convex set with interior points. If $M = \mathbf{R}^n$, then it is well-known that $\widehat{d}_{\partial X}$ is convex on X and therefore on all of M [56, Lemma 10.1, Ch. 7]. More generally $\widehat{d}_{\partial X}$ will be convex on X as long as the curvature of M on X is nonnegative [131, Lem. 3.3]. However, if the curvature of M is strictly negative on X, then $\widehat{d}_{\partial X}$ may no longer be convex. This is the case, for instance, when X is the region bounded in between a pair of non-intersecting geodesics in the hyperbolic plane. See [84, p. 44] for a general discussion of the relation between convexity of parallel hypersurfaces and the sign of curvature of M. Therefore we are led to make the following definition. We say that a hypersurface Γ in M is distance-convex or *d-convex* provided that $\widehat{d}_\Gamma$ is convex on Ω.

As far as we know, d-convex hypersurfaces have not been specifically studied before; however, as we show below, they are generalizations of the well-known h-convex or horo-convex hypersurfaces [28, 51, 68, 89, 97], which are defined as follows. A *horosphere*, in a Cartan-Hadamard manifold, is the limit of a family of geodesic spheres whose radii goes to infinity, and a *horoball* is the limit of the corresponding family of balls (thus horospheres are generalizations of hyperplanes in $\mathbf{R}^n$). The distance function of a horosphere, which is known as a *Busemann function*, has been extensively studied. In particular it is well-known that it is convex and $\mathcal{C}^2$ [11, Prop. 3.1 & 3.2]. A hypersurface Γ is called *h-convex* provided that through each of its points there passes a horosphere which contains Γ, i.e., Γ lies in the corresponding horoball. The convexity of the Busemann function yields:

Lemma 3.2. *In a Cartan-Hadamard manifold, every $\mathcal{C}^{1,1}$ h-convex hypersurface Γ is d-convex.*

Proof. For points $q \in \Gamma$, let S_q be the horosphere which passes through q and contains Γ. For points $p \in \Omega$, let p° be the footprint of p on Γ, and let S_{p° be the horosphere which passes through p° and contains Γ. Then

$$\widehat{d}_\Gamma(p) = -d(p, \Gamma) = -d(p, p^\circ) = -d(p, S_{p^\circ}) = \widehat{d}_{S_{p^\circ}}(p).$$

On the other hand, since Γ lies inside S_q, for any point $p \in \Omega$, we have $d(p, \Gamma) \leq d(p, S_q)$. Thus

$$\widehat{d}_\Gamma(p) = -d(p, \Gamma) \geq -d(p, S_q) = \widehat{d}_{S_q}(p).$$

So we have shown that

$$\widehat{d}_\Gamma = \sup_{q \in \Gamma} \widehat{d}_{S_q},$$

on Ω. Since $\widehat{d}_{S_q}$ (being a Busemann function) is convex, it follows then that $\widehat{d}_\Gamma$ is convex on Ω, which completes the proof. □

The converse of the above lemma, however, is not true. For instance, for any geodesic segment in the hyperbolic plane, there exists $r > 0$, such that the tubular hypersurface of radius r about that segment (which is d-convex by Lemma 3.1) is not h-convex. So in summary we may record that, in a Cartan-Hadamard manifold,

$$\big\{h\text{-convex hypersurfaces}\big\} \subsetneq \big\{d\text{-convex hypersurfaces}\big\} \subsetneq \big\{\text{convex hypersurfaces}\big\}.$$

The main aim of this section is to relate the total curvature of a convex hypersurface in an n-dimensional Cartan-Hadamard manifold to that of a d-convex hypersurface in an $(n+1)$-dimensional Cartan-Hadamard manifold. First note that if M is a Cartan-Hadamard manifold, then $M \times \mathbf{R}$ is also a Cartan-Hadamard manifold, which contains M as a totally geodesic hypersurface. For any convex hypersurface $\Gamma \subset M$, bounding a convex domain Ω, and $\varepsilon > 0$, let $\widetilde{\Gamma}_\varepsilon$ be the parallel hypersurface of Ω in $M \times \mathbf{R}$ of distance ε. Then $\widetilde{\Gamma}_\varepsilon$ is a d-convex hypersurface in $M \times \mathbf{R}$ by Lemma 3.1. Note also that $\widetilde{\Gamma}_\varepsilon$ is $\mathcal{C}^{1,1}$ by Lemma 2.6, so its total curvature is well-defined. In the next proposition we will apply some facts concerning evolution of the second fundamental form of parallel hypersurfaces and tubes, which is governed by Riccati's equation. A standard reference here is Gray [78, Chap. 3]; see also [12, 100]. We will use some computations from [71] on Taylor expansion of the second fundamental form. For more extensive computations see [110].

First let us fix our basic notation and sign conventions with regard to computation of curvature. Let Γ be a $\mathcal{C}^{1,1}$ closed embedded hypersurface in M, bounding a designated domain Ω of M as we discussed in Section 2. Then the *outward normal* ν of Γ is a unit normal vector field along Γ which points away from Ω. Let p be a twice differentiable point of Γ, and $T_p\Gamma$ denote the tangent space of Γ at p. Then the *shape operator* $\mathcal{S}_p \colon T_p\Gamma \to T_p\Gamma$ of Γ at p with respect to ν is defined as

$$\mathcal{S}_p(V) := \nabla_V \nu, \tag{5}$$

for $V \in T_p\Gamma$. Note that in a number of sources, including [71, 78] which we refer to for some computations, the shape operator is defined as $-\nabla_V \nu$. Thus our principal curvatures will have opposite signs compared to those in [71, 78], which will effect the appearance of Riccati's equation below. The eigenvalues and eigenvectors of $\mathcal{S}_p$ then define the *principal curvatures* $\kappa_i(p)$ and *principal directions* $E_i(p)$ of Γ at p respectively. So we have

$$\kappa_i(p) = \big\langle \mathcal{S}_p(E_i(p)), E_i(p)\big\rangle = \big\langle \nabla_i \nu, E_i(p)\big\rangle.$$

The *Gauss-Kronecker* curvature of Γ at p is given by

$$GK(p) := \det(\mathcal{S}_p) = \prod_{i=1}^{n-1} \kappa_i(p). \tag{6}$$

Finally, *total Gauss-Kronecker* curvature of Γ is defined as

$$\mathcal{G}(\Gamma) := \int_\Gamma GK d\sigma.$$

We will always assume that the shape operator of Γ is computed with respect to the outward normal. Thus when Γ is convex, its principal curvatures will be nonnegative. The main result of this section is as follows. For convenience we assume that Γ is $\mathcal{C}^2$, which will be sufficient for our purposes; however, the proof can be extended to the $\mathcal{C}^{1,1}$ case with the aid Lemma 9.4 which will be established later.

Proposition 3.3. *Let Γ be a $\mathcal{C}^2$ convex hypersurface in a Cartan-Hadamard manifold M, bounding a convex domain Ω, and $\widetilde{\Gamma}_\varepsilon$ be the parallel hypersurface of Ω at distance ε in $M \times \mathbf{R}$. Then, as $\varepsilon \to 0$,*

$$\frac{\mathcal{G}(\widetilde{\Gamma}_\varepsilon)}{(n+1)\omega_{n+1}} \quad \longrightarrow \quad \frac{\mathcal{G}(\Gamma)}{n\omega_n}.$$

In particular, if $\mathcal{G}(\widetilde{\Gamma}_\varepsilon) \geq (n+1)\omega_{n+1}$, then $\mathcal{G}(\Gamma) \geq n\omega_n$.

Recall that in $\mathbf{R}^n$ the total curvature of any convex hypersurface is equal to the volume of its Gauss map, or the area of $\mathbf{S}^{n-1}$, which is $n\omega_n$. So Proposition 3.3 holds immediately when $M = \mathbf{R}^n$, since then $\mathcal{G}(\widetilde{\Gamma}_\varepsilon) = (n+1)\omega_{n+1}$ and $\mathcal{G}(\Gamma) = n\omega_n$. The proof in the general case follows from tube formulas and properties of the gamma and beta functions as we describe below (see also Note 3.4 which eliminates the use of special functions).

Proof of Proposition 3.3. For every point $q \in \widetilde{\Gamma}_\varepsilon$ let p be its (unique) footprint on $\mathrm{cl}(\Omega) = \Omega \cup \Gamma$. If $p \in \Omega$, then there exists an open neighborhood U of p in $\widetilde{\Gamma}_\varepsilon$ which lies on $M \times \{\varepsilon\}$ or $M \times \{-\varepsilon\}$. So $GK^\varepsilon(q) = 0$, since each hypersurface $M \times \{t\} \subset M \times \mathbf{R}$ is totally geodesic. Thus the only contribution to $\mathcal{G}(\widetilde{\Gamma}_\varepsilon)$ comes from points $q \in \widetilde{\Gamma}_\varepsilon$ whose footprint $p \in \Gamma$. This portion of $\widetilde{\Gamma}_\varepsilon$ is the outer half of the tube of radius ε around Γ, which we denote by $\mathrm{tube}^+_\varepsilon(\Gamma)$, see Figure 2, and will describe precisely below. So we have

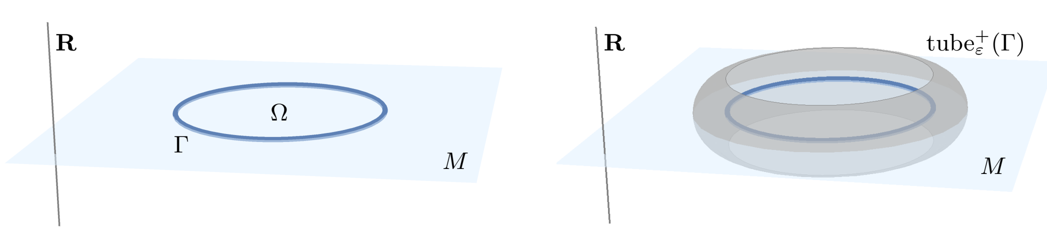

FIGURE 2.

$$\mathcal{G}(\widetilde{\Gamma}_\varepsilon) = \mathcal{G}(\mathrm{tube}^+_\varepsilon(\Gamma)).$$

Furthermore recall that $\omega_n = \pi^{n/2}/G(n/2+1)$, where G is the gamma function. In particular, $G(1/2) = \sqrt{\pi}$, $G(x+1) = xG(x)$, and $G(n) = (n-1)!$, which yields

$$\alpha_n := \frac{(n+1)\omega_{n+1}}{n\omega_n} = \frac{G(\frac{1}{2})G(\frac{n}{2})}{G(\frac{1}{2}+\frac{n}{2})} = B\left(\frac{1}{2}, \frac{n}{2}\right) = \int_{-\pi/2}^{\pi/2} \cos^{n-1}(\theta)\, d\theta, \tag{7}$$

where B is the beta function (see [5, Sec. 1.1] for the basic properties of gamma and beta functions). Thus it suffices to show that, as $\varepsilon \to 0$,

$$\mathcal{G}\big(\mathrm{tube}^+_\varepsilon(\Gamma)\big) \to \alpha_n\, \mathcal{G}(\Gamma).$$

To this end let ν denote the outward unit normal of Γ with respect to Ω in M, $\nu^\perp$ be a unit normal vector orthogonal to M in $M \times \mathbf{R}$, and define $f^\varepsilon \colon \Gamma \times \mathbf{R} \to M \times \mathbf{R}$ by

$$f^\varepsilon(p,\theta) := \exp_p\big(\varepsilon \nu_p(\theta)\big), \qquad \nu_p(\theta) := \cos(\theta)\nu(p) + \sin(\theta)\nu^\perp(p),$$

where exp is the exponential map of $M \times \mathbf{R}$. Then we set

$$\mathrm{tube}^+_\varepsilon(\Gamma) := f^\varepsilon\big(\Gamma \times [-\pi/2, \pi/2]\big).$$

Note that $\mathrm{tube}^+_\varepsilon(\Gamma) \subset d_\Gamma^{-1}(\varepsilon)$, where d_Γ denotes the distance function of Γ in $M \times \mathbf{R}$. Thus, since M is $\mathcal{C}^2$, d_Γ is $\mathcal{C}^2$ [69, Thm. 1] which yields that $\mathrm{tube}^+_\varepsilon(\Gamma)$ is $\mathcal{C}^2$. So the shape operator of $\mathrm{tube}^+_\varepsilon(\Gamma)$ is well-defined. By [71, Cor. 2.2], this shape operator, at the point $f^\varepsilon(p,\theta)$, is given by

$$\mathcal{S}^\varepsilon_{p,\theta} = \begin{pmatrix} \mathcal{S}_{p,\theta} + \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & 1/\varepsilon + \mathcal{O}(\varepsilon) \end{pmatrix}, \tag{8}$$

where $\mathcal{O}(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\mathcal{S}_{p,\theta}$ denotes the shape operator of Γ at p in the direction $\nu_p(\theta)$ (note that the shape operators in this work, as defined by (5), have the opposite sign compared to those in [71]). The eigenvalues of $\mathcal{S}_{p,\theta}$ are $\kappa_i(p)\cos(\theta)$ where $\kappa_i(p)$ are the principal curvatures of Γ at p. Thus it follows that the Gauss-Kronecker curvature of $\mathrm{tube}^+_\varepsilon(\Gamma)$ at the point $f^\varepsilon(p,\theta)$ is given by

$$GK^\varepsilon(p,\theta) = \det\big(\mathcal{S}^\varepsilon_{p,\theta}\big) = \frac{1}{\varepsilon}\det\big(\mathcal{S}_{p,\theta}\big) + \mathcal{O}(1) = \frac{1}{\varepsilon} GK(p)\cos^{n-1}(\theta) + \mathcal{O}(1), \tag{9}$$

where $\mathcal{O}(1)$ converges to a constant as $\varepsilon \to 0$. Furthermore, we claim that

$$\mathrm{Jac}(f^\varepsilon)_{(p,\theta)} = \varepsilon + \mathcal{O}(\varepsilon^2). \tag{10}$$

Then it follows that, as $\varepsilon \to 0$,

$$\mathcal{G}\big(\mathrm{tube}^+_\varepsilon(\Gamma)\big) = \int_{\mathrm{tube}^+_\varepsilon(\Gamma)} GK^\varepsilon d\mu_\varepsilon \to \int_{-\pi/2}^{\pi/2}\int_{p\in\Gamma} GK(p)\cos^{n-1}(\theta)\, d\mu d\theta = \alpha_n\, \mathcal{G}(\Gamma),$$

as desired. So it remains to establish (10). To this end we will apply the fact that, due to Riccati's equation [78, Thm. 3.11 & Lem. 3.12],

$$\mathrm{Jac}(f^\varepsilon)_{(p,\theta)} = \varepsilon\, \Theta(\varepsilon),$$

where Θ is given by

$$\frac{\Theta'(\varepsilon)}{\Theta(\varepsilon)} = -\frac{1}{\varepsilon} + \mathrm{trace}(\mathcal{S}^\varepsilon_{p,\theta}), \qquad \Theta(0) = 1, \tag{11}$$

(again note that our shape operator has the opposite sign to that in [78]). Next observe that by (8)

$$\mathrm{trace}(\mathcal{S}^\varepsilon_{p,\theta}) = \mathrm{trace}(\mathcal{S}_{p,\theta}) + \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon).$$

So we may rewrite (11) as

$$\frac{\Theta'(\varepsilon)}{\Theta(\varepsilon)} = \mathrm{trace}(\mathcal{S}_{p,\theta}) + \mathcal{O}(\varepsilon) = \mathcal{O}(1).$$

Hence, we obtain

$$\Theta(\varepsilon) = \Theta(0)e^{\int_0^\varepsilon \mathcal{O}(1)dt} = e^{\mathcal{O}(\varepsilon)} = 1 + \mathcal{O}(\varepsilon),$$

which in turn yields

$$\mathrm{Jac}(f^\varepsilon)_{(p,\theta)} = \varepsilon\big(1 + \mathcal{O}(\varepsilon)\big) = \varepsilon + \mathcal{O}(\varepsilon^2),$$

as desired. □

Note 3.4. Let $M = \mathbf{R}^n$, $\Gamma = \mathbf{S}^{n-1}$, and $\widetilde{\Gamma}_\varepsilon$ be as in the statement of Proposition 3.3. Then the map $f^\varepsilon(p,\theta)$ in the proof of Proposition 3.3 simplifies to $p + \varepsilon\nu_p(\theta)$, and we quickly obtain

$$\mathcal{G}(\widetilde{\Gamma}_\varepsilon) = \left(\int_{-\pi/2}^{\pi/2} \cos^{n-1}(\theta)\, d\theta\right) \mathcal{G}(\Gamma).$$

Since $\mathcal{G}(\widetilde{\Gamma}_\varepsilon) = (n+1)\omega_{n+1}$ and $\mathcal{G}(\Gamma) = n\omega_n$, it follows that

$$\frac{(n+1)\omega_{n+1}}{n\omega_n} = \int_{-\pi/2}^{\pi/2} \cos^{n-1}(\theta)\, d\theta,$$

which establishes (7) without the need to use the gamma and beta functions.

4. A Comparison Formula for Total Curvature of Level Sets

In this section we establish an integral formula for comparing the total curvature of regular level sets of $\mathcal{C}^{1,1}$ functions on Riemannian manifolds. Some applications of this formula, and its connections with quermassintegrals, will also be discussed in Section 4.3.

4.1. **Divergence of the Hessian cofactor.** Here we assume that u is a $\mathcal{C}^{1,1}$ function on a Riemannian manifold M, so that it is twice differentiable almost everywhere, and derive a basic identity for the cofactor operator associated to the Hessian of u. This operator is a special case of a more general device, the Newton operator, which will be discussed in Section 5. To start, let ∇ be the *covariant derivative* on M. The *gradient* of u, ∇u, is then given by

$$\langle \nabla u(p), V\rangle := \nabla_V u,$$

for tangent vectors $V \in T_pM$. Next (at a twice differentiable point p) we define the *Hessian operator* $\nabla^2 u$ of u as the self-adjoint linear map on T_pM given by

$$\nabla^2 u(V) := \nabla_V(\nabla u).$$

The *Hessian* of u in turn will be the corresponding symmetric bilinear form on T_pM,

$$\mathrm{Hess}_u(V,W) := \left\langle \nabla^2 u(V), W\right\rangle = \langle \nabla_V(\nabla u), W\rangle\,.$$

Let E_i denote a smooth orthonormal frame field in a neighborhood U of p, and set $\nabla_i := \nabla_{E_i}$. Then $\nabla u = u_i E_i$ on U, and $\nabla^2 u(V) = u_{ij}V^j E_i$ at p, where

$$u_i := \nabla_i u = \langle \nabla u, E_i\rangle, \qquad \text{and} \qquad u_{ij} := \mathrm{Hess}_u(E_i, E_j).$$

In general $u_{ij} = \nabla_j u_i - \langle \nabla_j E_i, E_k\rangle u_k$. We may assume, however, that $(\nabla_j E_i)_p := \nabla_{E_j(p)} E_i = 0$, i.e., E_i is a local *geodesic frame* based at p. Then

$$u_{ij}(p) = (\nabla_j u_i)_p. \tag{12}$$

The *cofactor* of a square matrix (a_{ij}) is the matrix $(\overline{a}_{ij})$ where $\overline{a}_{ij}$ is the (i,j)-*signed minor* of (a_{ij}), i.e., $(-1)^{i+j}$ times the determinant of the matrix obtained by removing the i^{th} row and j^{th} column of (a_{ij}). We define the self-adjoint operator $\mathcal{T}^u \colon T_pM \to T_pM$ by setting

$$(\mathcal{T}^u_{ij}) := \mathrm{cofactor}(u_{ij}) = (\overline{u}_{ij}).$$

Note that, when $\nabla^2 u$ is nondegenerate, $(\nabla^2 u)^{-1}(V) = u^{ij}V^j E_i$, where $(u^{ij}) := (u_{ij})^{-1}$. In that case,

$$\mathcal{T}^u(V) = \det(\nabla^2 u)(\nabla^2 u)^{-1}(V) = \mathcal{T}^u_{ij} V^j E_i,$$

and $(\mathcal{T}^u_{ij}) = \det(\nabla^2 u)(u^{ij})$. In Section 5 we will show that $\mathcal{T}^u = \mathcal{T}^u_{n-1}$ is one of the Newton operators associated to $\nabla^2 u$, which appear in the well-known works of Reilly [124, 125]. We are interested in $\mathcal{T}^u$ since it can be used to compute the curvature of the level sets of u, as discussed below.

We say that $\Gamma := \{u = u(p)\}$ is a *regular level set* of u near p, if u is $\mathcal{C}^1$ on a neighborhood of p and $\nabla u(p) \neq 0$. Then $\nabla u/|\nabla u|$ generates a normal vector field on Γ near p. If we let E_ℓ be the principal directions of Γ at p, then the corresponding principal curvatures of Γ with respect to $\nabla u/|\nabla u|$ are given by

$$\kappa_\ell = \left\langle \nabla_\ell \left(\frac{\nabla u}{|\nabla u|} \right), E_\ell \right\rangle = \frac{\langle \nabla_\ell(\nabla u), E_\ell \rangle}{|\nabla u|} = \frac{\mathrm{Hess}_u(E_\ell, E_\ell)}{|\nabla u|} = \frac{u_{\ell\ell}}{|\nabla u|}. \tag{13}$$

Note that the above formula demonstrates the well-known fact that the second fundamental form of Γ at p is given by the restriction of the Hessian of u to $T_p\Gamma$. Using this formula, we can show:

Lemma 4.1. *Let $\Gamma := \{u = u(p)\}$ be a level set of u which is regular near p, and suppose that Γ is twice differentiable at p. Then the Gauss Kronecker curvature of Γ at p with respect to $\nabla u/|\nabla u|$ is given by*

$$GK = \frac{\langle \mathcal{T}^u(\nabla u), \nabla u \rangle}{|\nabla u|^{n+1}}.$$

Proof. Let E_i be an orthonormal frame for T_pM such that E_ℓ, $\ell = 1, \dots, n-1$ are principal directions of Γ at p. Then the $(n-1) \times (n-1)$ leading principal submatrix of (u_{ij}) will be diagonal. Thus,

$$\mathcal{T}^u_{nn} = \overline{u}_{nn} = \prod_{\ell=1}^{n-1} u_{\ell\ell}.$$

Furthermore, since E_n is orthogonal to Γ, and Γ is a level set of u, ∇u is parallel to $\pm E_n$. So $u_n = \langle \nabla u, E_n \rangle = \pm |\nabla u|$. Now, using (13), we have

$$\frac{\langle \mathcal{T}^u(\nabla u), \nabla u \rangle}{|\nabla u|^{n+1}} = \frac{\mathcal{T}^u_{ij} u_j u_i}{|\nabla u|^{n+1}} = \frac{\mathcal{T}^u_{nn} u_n u_n}{|\nabla u|^{n+1}} = \frac{\mathcal{T}^u_{nn}}{|\nabla u|^{n-1}} = \prod_{\ell=1}^{n-1} \frac{u_{\ell\ell}}{|\nabla u|} = GK.$$

□

Let V be a vector field on U. Since $(\nabla_j E_i)_p = 0$, the divergence of the vector field $\mathcal{T}^u(V)$ at p is given by

$$\mathrm{div}_p\big(\mathcal{T}^u(V)\big) = \big(\nabla_i(\mathcal{T}^u_{ij} V^j)\big)_p. \tag{14}$$

The *divergence* $\mathrm{div}(\mathcal{T}^u)$ of $\mathcal{T}^u$ is defined as follows. If $\mathcal{T}^u$ is viewed as a bilinear form or $(0,2)$ tensor, then $\mathrm{div}(\mathcal{T}^u)$ generates a one-form or $(0,1)$ tensor given by $\langle \mathrm{div}(\mathcal{T}^u), \cdot \rangle$, where

$$\mathrm{div}_p(\mathcal{T}^u) := (\nabla_i \mathcal{T}^u_{ij})_p \, E_j(p). \tag{15}$$

In other words, with respect to our frame E_i, $\mathrm{div}_p(\mathcal{T}^u)$ is a vector whose i^{th} coordinate is the divergence of the i^{th} column of $\mathcal{T}^u$ at p.

Lemma 4.2. *If u is three times differentiable at p, $\nabla u(p) \neq 0$, and $\nabla^2 u(p)$ is nondegenerate, then*

$$\mathrm{div}\left(\mathcal{T}^u \left(\frac{\nabla u}{|\nabla u|^n} \right) \right) = \left\langle \mathrm{div}(\mathcal{T}^u), \frac{\nabla u}{|\nabla u|^n} \right\rangle. \tag{16}$$

Proof. By (14) and (15), it suffices to check that, at the point p,

$$\nabla_i\Big(\mathcal{T}^u_{ij}\frac{u_j}{|\nabla u|^n}\Big) = (\nabla_i\mathcal{T}^u_{ij})\frac{u_j}{|\nabla u|^n}.$$

This follows from (12) via Liebnitz rule, since

$$\mathcal{T}^u_{ij}\nabla_i\left(\frac{u_j}{|\nabla u|^n}\right) = \mathcal{T}^u_{ij}\left(\frac{u_{ji}}{|\nabla u|^n} - n\frac{u_j u_k u_{ki}}{|\nabla u|^{n+2}}\right) = n\frac{\det(u_{ij})}{|\nabla u|^n} - n\frac{u_j u_k \delta_{kj}\det(u_{ij})}{|\nabla u|^{n+2}} = 0.$$

□

Lemmas 4.1 and 4.2 will be extended in Section 5 to general Newton operators and symmetric functions of principal curvatures.

4.2. **Derivation of the comparison formula.** Here we apply the divergence identity (16) developed in the last section to obtain the comparison formula via Stokes' theorem. Let Γ be a closed embedded $\mathcal{C}^{1,1}$ hypersurface in a Riemannian manifold M bounding a domain Ω. Recall that the outward normal of Γ is the unit normal vector field ν along Γ which points away from Ω, and if p is a twice differentiable point of Γ, we will assume that the Gauss-Kronecker curvature $GK(p)$ of Γ is computed with respect to ν according to (6). We say that p is a *regular point* of a function u on M provided that u is $\mathcal{C}^1$ on an open neighborhood of p and $\nabla u(p) \neq 0$. Furthermore, x is a *regular value* of u provided that every $p \in u^{-1}(x)$ is a regular point of u. Then $u^{-1}(x)$ will be called a *regular level set* of u. In this section we assume that Γ is a regular level set of u, and γ is another regular level set bounding a domain $D \subset \Omega$. We assume that u is $\mathcal{C}^{2,1}$ on $\mathrm{cl}(\Omega) \setminus D$ and ∇u points outward along Γ and γ with respect to their corresponding domains. Furthermore we assume that $|\nabla u| \neq 0$ and $\nabla^2 u$ is nondegenerate at almost every point p in $\mathrm{cl}(\Omega) \setminus D$. Below we will assume that local calculations always take place at such a point p with respect to a geodesic frame based at p, as defined in Section 4.1, and often omit the explicit reference to p. Throughout the paper $d\mu$ denotes the n-dimensional Riemannian volume measure on M, and $d\sigma$ is the $(n-1)$-dimensional volume or hypersurface area measure.

Lemma 4.3.

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = \int_{\Omega\setminus D}\left\langle \mathrm{div}(\mathcal{T}^u), \frac{\nabla u}{|\nabla u|^n}\right\rangle d\mu.$$

Proof. By Lemma 4.2 and the divergence theorem,

$$\begin{aligned}\int_{\Omega\setminus D}\left\langle \mathrm{div}(\mathcal{T}^u), \frac{\nabla u}{|\nabla u|^n}\right\rangle d\mu &= \int_{\Omega\setminus D}\mathrm{div}\left(\mathcal{T}^u\left(\frac{\nabla u}{|\nabla u|^n}\right)\right) d\mu \\ &= \int_{\Gamma\cup\gamma}\left\langle \mathcal{T}^u\left(\frac{\nabla u}{|\nabla u|^n}\right), \nu\right\rangle d\sigma,\end{aligned}$$

where ν is the outward normal to $\partial(\Omega \setminus D) = \Gamma \cup \gamma$. Now Lemma 4.1 completes the proof since by assumption $\nu = \nabla u/|\nabla u|$ on Γ and $\nu = -\nabla u/|\nabla u|$ on γ. □

In the next computation we will need the formula

$$\nabla_i \det(\nabla^2 u) = \mathcal{T}^u_{r\ell} u_{r\ell i} = \det(\nabla^2 u) u^{r\ell} u_{r\ell i}, \tag{17}$$

where $u_{rki} := \nabla_i u_{rk}$. Further note that by the definition of the Riemann tensor R in local coordinates:

$$u_{rik} - u_{rki} = \nabla_k \nabla_i u_r - \nabla_i \nabla_k u_r = R_{kir\ell} u_\ell, \tag{18}$$

where we have used the fact that $R^\ell_{kir} = R_{kirm} g^{m\ell} = R_{kir\ell}$, since $g^{m\ell} := \langle E_m, E_\ell \rangle = \delta_{m\ell}$. Note that in formulas below we use the *Einstein summation convention*, i.e., we assume that any term with repeated indices is summed over that index with values ranging from 1 to n, unless indicated otherwise. The next observation relates the divergence of the Hessian cofactor to a trace or contraction of the Riemann tensor:

Lemma 4.4. *For any orthonormal frame E_i at a point $p \in \Omega$,*

$$\big\langle \operatorname{div}(\mathcal{T}^u), \nabla u \big\rangle = \frac{R\big(\mathcal{T}^u(\nabla u), \mathcal{T}^u(E_i), E_i, \nabla u\big)}{\det(\nabla^2 u)} = \frac{R\big(\mathcal{T}^u(\nabla u), E_i, \mathcal{T}^u(E_i), \nabla u\big)}{\det(\nabla^2 u)}.$$

Proof. Differentiating both sides of $u^{ir} u_{rk} = \delta_{ik}$, we obtain $\nabla_i u^{ij} = -u^{ir} u^{kj} u_{rki}$. This together with (17) and (18) yields that

$$\begin{aligned} \nabla_i \mathcal{T}^u_{ij} &= \nabla_i (u^{ij} \det(\nabla^2 u)) \\ &= -u^{ir} u^{kj} u_{rki} \det(\nabla^2 u) + u^{ij} \det(\nabla^2 u) u^{r\ell} u_{r\ell i} \\ &= \det(\nabla^2 u) u^{kj} u^{ir} R_{kir\ell} u_\ell, \end{aligned}$$

where passing from the second line to the third proceeds via reindexing $i \to k$, $\ell \to i$, in the second term of the second line. Thus by (15)

$$\langle \operatorname{div}(\mathcal{T}^u), \nabla u \rangle = \nabla_i \mathcal{T}^u_{ij} u_j = \det(\nabla^2 u) u^{kj} u^{ir} R_{kir\ell} u_\ell u_j.$$

It remains then to work on the right hand side of the last expression. To this end recall that $u^{ij} E_j = \mathcal{T}^u_{ij} E_j / \det(\nabla^2 u) = \mathcal{T}^u(E_i)/\det(\nabla^2 u)$. Thus

$$\begin{aligned} \det(\nabla^2 u) u^{kj} u^{ir} R_{kir\ell} u_\ell u_j &= \det(\nabla^2 u) R(u^{kj} E_k u_j, E_i, u^{ir} E_r, u_\ell E_\ell) \\ &= \frac{R(\mathcal{T}^u(\nabla u), E_i, \mathcal{T}^u(E_i), \nabla u)}{\det(\nabla^2 u)}. \end{aligned} \tag{19}$$

Note that we may move u_{ir}, on the right hand side of the first inequality in the last expression, next to E_i, which will have the effect of moving $\mathcal{T}^u$ over to the second slot of R in the last line of the expression. □

Combining Lemmas 4.3 and 4.4 we obtain the basic form of the comparison formula:

Corollary 4.5. *Let E_i be any choice of an orthonormal frame at each point $p \in \Omega \setminus D$. Then*
$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = \int_{\Omega\setminus D} \frac{R\big(\mathcal{T}^u(\nabla u), \mathcal{T}^u(E_i), E_i, \nabla u\big)}{|\nabla u|^n \det(\nabla^2 u)}\, d\mu.$$

Next we will express the integral in Corollary 4.5 with respect to a suitable local frame. To this end we need to gather some basic facts from matrix algebra:

Lemma 4.6. *Let A be an $n\times n$ symmetric matrix, with diagonal $(n-1)\times(n-1)$ leading principal submatrix, given by*
$$\begin{pmatrix} b_1 & & 0 & a_1 \\ & \ddots & & \vdots \\ 0 & & b_{n-1} & a_{n-1} \\ a_1 & \cdots & a_{n-1} & a \end{pmatrix},$$
and let $\overline{A} = (\overline{a}_{ij})$ denote the cofactor matrix of A. Then

(i) $\overline{a}_{in} = -a_i \Pi_{\ell\neq i} b_\ell$ *for* $i < n$.
(ii) $\overline{a}_{ij} = a_i a_j \Pi_{\ell\neq i,j} b_\ell$ *for* $i, j < n$, $i \neq j$.
(iii) $\overline{a}_{ii} = a\Pi_{l\neq i} b_\ell - \sum_{k\neq i} a_k^2\ \Pi_{\ell\neq k,i} b_\ell$ *for* $i < n$.
(iv) $\det(A) = a\Pi_\ell b_\ell - \sum a_k^2\ \Pi_{\ell\neq k} b_\ell$
(v) For fixed $b_1, \dots, b_{n-1}, |a|$ tending to infinity, and $|a_i| < C$ (independent of a), the eigenvalues of A satisfy $\lambda_\alpha = b_\alpha + o(1)$ for $\alpha < n$ and $\lambda_n = a + \mathcal{O}(1)$, where the $o(1)$ and $\mathcal{O}(1)$ are uniform depending only on $b_1, \dots, b_{n-1}$ and C. In particular,
$$\det(A) = a \prod_i b_i + \mathcal{O}(1).$$

Proof. Parts (i), (ii), and (iii) follow easily by induction, and part (iv) follows from part (i) by the cofactor expansion of $\det(A)$ using the last column. Finally, part (v) is provided by [35, Lem. 1.2]. □

Let p be a regular point of a function u on M. We say that $E_1, \dots, E_n \in T_pM$ is a *principal frame* of u at p provided that
$$E_n = -\frac{\nabla u(p)}{|\nabla u(p)|},$$
and $E_1, \dots, E_{n-1}$ are principal directions of the level set $\{u = u(p)\}$ at p with respect to $-E_n$. Then the corresponding principal curvatures and the Gauss-Kronecker curvature of $\{u = u(p)\}$ will be denoted by $\kappa_i(p)$ and $GK(p)$ respectively. By a principal frame for u over some domain we mean a choice of principal frame at each point of the domain.

Theorem 4.7 (Comparison Formula, First Version)**.** *Let u be a function on a Riemannian manifold M, and Γ, γ be a pair of its regular level sets bounding domains Ω, and D*

respectively, with $D \subset \Omega$. Suppose that ∇u points outward along Γ and γ with respect to their corresponding domains. Further suppose that u is $\mathcal{C}^{2,1}$ on $\mathrm{cl}(\Omega) \setminus D$, and almost everywhere on $\mathrm{cl}(\Omega) \setminus D$, $\nabla u \neq 0$, and $\nabla^2 e^u$ is nondegenerate. Then,

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = -\int_{\Omega \setminus D} R_{rnrn} \frac{GK}{\kappa_r}\, d\mu + \int_{\Omega \setminus D} R_{rkrn} \frac{GK}{\kappa_r \kappa_k} \frac{u_{nk}}{|\nabla u|} d\mu,$$

where all quantities are computed with respect to a principal frame of u, and $k \leq n-1$.

Proof. Let $w := \phi(u) := (e^{hu} - 1)/h$ for $h > 0$. Then $\nabla^2 w$ will be nondegenerate almost everywhere. So we may apply Corollary 4.5 to w to obtain

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = \int_{\Omega \setminus D} \frac{R\big(\mathcal{T}^w(\nabla w), E_i, \mathcal{T}^w(E_i), \nabla w\big)}{\det(\nabla^2 w) |\nabla w|^n} d\mu. \tag{20}$$

Let p be a point of the level set $\{w = \phi(t)\}$, and E_α, $\alpha = 1, \ldots, n-1$ be principal directions of $\{w = \phi(t)\}$ at p. Since w is constant on $\{w = \phi(t)\}$, $w_i(p) = 0$ for $i < n$ and $|w_n| = |\nabla w|$. Consequently, the integrand in the right hand side of (20) at p is given by

$$\frac{\det(\nabla^2 w) w^{kj} w^{ir} R_{kir\ell} w_\ell w_j}{|\nabla w|^n} = \frac{\det(\nabla^2 w) w^{kn} w^{ir} R_{kirn}}{|\nabla w|^{n-2}}. \tag{21}$$

Next note that $(w_{ij}) = \phi'(u)(a_{ij})$ where, $a_{ij} = u_{ij} + h u_i u_j$. Again we have $u_i(p) = 0$ for $i < n$. Also recall that, by (13), $u_{kk} = |\nabla u| \kappa_k$. Furthermore note that $\nabla u = -u_n$. Thus it follows that

$$(a_{ij}) = \begin{pmatrix} |\nabla u|\kappa_1 & & \text{\Large 0} & u_{1n} \\ & \ddots & & \vdots \\ \text{\Large 0} & & |\nabla u|\kappa_{n-1} & u_{(n-1)n} \\ u_{n1} & \cdots & u_{(n-1)n} & u_{nn} + h|\nabla u|^2 \end{pmatrix}.$$

Let $(\overline{a}_{ij})$ be the cofactor matrix of (a_{ij}). Since $(w_{ij}) = \phi'(u)(a_{ij})$, it follows that the cofactor matrix of (w_{ij}) is given by $\det(\nabla^2 w) w^{ij} = \phi'(u)^{n-1} \overline{a}_{ij}$. Then, the right hand side of (21) becomes:

$$\frac{\det(\nabla^2 w) w^{kn} w^{ir} R_{kirn}}{|\nabla w|^{n-2}} = \frac{\phi'(u)^{2n-2} \overline{a}_{kn} \overline{a}_{ir} R_{kirn}}{\det(\nabla^2 w) |\nabla w|^{n-2}} = \frac{\overline{a}_{kn} \overline{a}_{ir} R_{kirn}}{\det(a_{ij}) |\nabla u|^{n-2}}, \tag{22}$$

where in deriving the second equality we have used the facts that $|\nabla w| = \phi'(u)|\nabla u|$, and $\det(\nabla^2 w) = \phi'(u)^n \det(a_{ij})$. By Lemma 4.6 (as $h \to \infty$),

$$\overline{a}_{ij} = \begin{cases} -u_{in}\frac{GK}{\kappa_i}|\nabla u|^{n-2}, & \text{for } i < n \text{ and } j = n; \\ u_{in}u_{nj}\frac{GK}{\kappa_i\kappa_j}|\nabla u|^{n-3}, & \text{for } i \neq j \text{ and } i,j < n; \\ \big(u_{nn} + h|\nabla u|^2\big)\,\frac{GK}{\kappa_i}|\nabla u|^{n-2} + \mathcal{O}(1), & \text{for } i = j \text{ and } i,j < n; \\ GK|\nabla u|^{n-1}, & \text{for } i = j = n. \end{cases}$$

Observe that $\overline{a}_{ij}$ for $i \neq j$ or $i = j = n$ are independent of h. On the other hand, again by Lemma 4.6,

$$\det(a_{ij}) = \big(u_{nn} + h|\nabla u|^2\big) GK|\nabla u|^{n-1} + \mathcal{O}(1).$$

Therefore, the last term in (22) takes the form

$$\frac{\overline{a}_{kn}\overline{a}_{rr}R_{krrn}}{\det(a_{ij})|\nabla u|^{n-2}} + \mathcal{O}\left(\frac{1}{h}\right) \;=\; -R_{rnrn}\frac{GK}{\kappa_r} + R_{rkrn}\frac{GK}{\kappa_r\kappa_k}\frac{u_{nk}}{|\nabla u|} + \mathcal{O}\left(\frac{1}{h}\right).$$

where $k \leq n-1$. So, by the coarea formula, the right hand side of (20) becomes

$$\int_{\phi(t_0)}^{\phi(t_1)}\int_{\{w=s\}}\left(-R_{rnrn}\frac{GK}{\kappa_r} + R_{rkrn}\frac{GK}{\kappa_r\kappa_k}\frac{u_{nk}}{|\nabla u|} + \mathcal{O}\left(\frac{1}{h}\right)\right)\frac{d\sigma}{|\nabla w|}ds$$
$$= \int_{t_0}^{t_1}\int_{\{u=t\}}\left(-R_{rnrn}\frac{GK}{\kappa_r} + R_{rkrn}\frac{GK}{\kappa_r\kappa_k}\frac{u_{nk}}{|\nabla u|}\right)\frac{d\sigma}{|\nabla u|}dt$$

after the change of variable $s = \phi(t)$ and letting $h \to \infty$. □

Next we develop a more general version of Theorem 4.7, via integration by parts and a smoothing procedure, which may be applied to $\mathcal{C}^{1,1}$ functions, to functions with singularities, or to a sequence of functions whose derivatives might blow up over some region. The latter scenario will be the case where the main result of this section, Theorem 4.9 below, will be applied in Section 10.3. First we describe the smoothing procedure. Let $\rho(x) := d(x, x_0)$, for some $x_0 \in \Omega$, and set

$$\overline{u}^{\varepsilon}(x) := u(x) + \frac{\varepsilon}{2}\rho^2(x).$$

If u is convex, then $\overline{u}^{\varepsilon}$ will be strictly convex in the sense of Greene and Wu [79], and thus their method of smoothing by convolution will preserve convexity of u. This convolution is a generalization of the standard Euclidean version via the exponential map, and is defined as follows. Let $\phi : \mathbf{R} \to \mathbf{R}$ be a nonnegative $\mathcal{C}^{\infty}$ function supported in $[-1, 1]$ which is constant in a neighborhood of the origin, and satisfies $\int_{\mathbf{R}^n}\phi(|x|)dx = 1$. Then for any function $f\colon M \to \mathbf{R}$, we set

$$f \circ_{\lambda} \phi(p) := \frac{1}{\lambda^n}\int_{v\in T_pM}\phi\left(\frac{|v|}{\lambda}\right) f\big(\exp_p(v)\big)\, d\mu_p, \tag{23}$$

where $d\mu_p$ is the measure on $T_pM \simeq \mathbf{R}^n$ induced by the Riemannian measure $d\mu$ of M. We set

$$\widehat{u}^{\varepsilon}_{\lambda} := \overline{u}^{\varepsilon} \circ_{\lambda} \phi.$$

The following result is established in [81, Thm. 2 & Lem. 3(3)], with reference to earlier work in [79, 80]. In particular see [80, p. 280] for how differentiation under the integral sign in (23) may be carried out via parallel translation.

Proposition 4.8. *(Greene-Wu [81]) For any continuous function $u\colon M \to \mathbf{R}$, $\varepsilon > 0$, and compact set $X \subset M$, there exists $\lambda > 0$ such that $\widehat{u}^{\varepsilon}_{\lambda}$ is $\mathcal{C}^{\infty}$ on an open neighborhood U of X, and $\widehat{u}^{\varepsilon}_{\lambda} \to \overline{u}^{\varepsilon}$ uniformly on U, as $\lambda \to 0$. Furthermore, if u is $\mathcal{C}^k$ on an open neighborhood of X, then $\widehat{u}^{\varepsilon}_{\lambda} \to \overline{u}^{\varepsilon}$ on U with respect to the $\mathcal{C}^k$ topology. Finally, if u is convex, then $\widehat{u}^{\varepsilon}_{\lambda}$ will be strictly convex with positive definite Hessian everywhere.*

Recall that, for any set $X \subset M$, $U_\theta(X)$ denotes the tubular neighborhood of radius θ about X. A *cutoff function* for $U_\theta(X)$ is any $\mathcal{C}^\infty$ function $\eta \geq 0$ on M which depends only on the distance $\widehat{r}(\,\cdot\,) := d_X(\,\cdot\,)$, is nondecreasing in terms of $\widehat{r}$, and satisfies

$$\eta(x) := \begin{cases} 0 & \text{if} \quad \widehat{r}(x) \leq \theta, \\ 1 & \text{if} \quad \widehat{r}(x) \geq 2\theta. \end{cases} \tag{24}$$

Since by Lemma 2.1 $\widehat{r}$ is Lipschitz, η is Lipschitz as well, and thus differentiable almost everywhere. At every differentiable point of η we have

$$\left\langle \mathcal{T}^u\left(\frac{\nabla u}{|\nabla u|^n}\right), \nabla\eta \right\rangle = \frac{\mathcal{T}^u_{ij}\eta_i u_j}{|\nabla u|^n} = \frac{\mathcal{T}^u_{in}\eta_i u_n}{|\nabla u|^n} = -\frac{\mathcal{T}^u_{in}\eta_i}{|\nabla u|^{n-1}} = -\frac{\mathcal{T}^u_{kn}\eta_k}{|\nabla u|^{n-1}} - \frac{\mathcal{T}^u_{nn}\eta_n}{|\nabla u|^{n-1}},$$

where $k \leq n-1$. Furthermore, by Lemma 4.6,

$$-\frac{\mathcal{T}^u_{kn}\eta_k}{|\nabla u|^{n-1}} = \frac{u_{nk}\eta_k}{|\nabla u|}\prod_{\ell\neq k}\kappa_\ell = \frac{u_{nk}\eta_k}{|\nabla u|}\frac{GK}{\kappa_k}, \qquad \text{and} \qquad -\frac{\mathcal{T}^u_{nn}\eta_n}{|\nabla u|^{n-1}} = -\eta_n GK.$$

So we obtain

$$\left\langle \mathcal{T}^u\left(\frac{\nabla u}{|\nabla u|^n}\right), \nabla\eta \right\rangle = \frac{u_{nk}\eta_k}{|\nabla u|}\frac{GK}{\kappa_k} - \eta_n GK.$$

Next recall that $\int \operatorname{div}(\eta X)d\mu = \int(\langle X, \nabla\eta\rangle + \eta\operatorname{div}(X))d\mu$, for any vector field X on M. Thus

$$\int \operatorname{div}\left(\eta\,\mathcal{T}^u\left(\frac{\nabla u}{|\nabla u|^n}\right)\right)d\mu = \int\left(\frac{u_{nk}\eta_k}{|\nabla u|}\frac{GK}{\kappa_k} - \eta_n GK\right)d\mu + \int \eta\operatorname{div}\left(\mathcal{T}^u\left(\frac{\nabla u}{|\nabla u|^n}\right)\right)d\mu. \tag{25}$$

We set

$$\mathcal{G}_\eta(\Gamma) := \int_\Gamma \eta\, GK\, d\sigma, \quad \text{and} \quad \mathcal{G}_\eta(\gamma) := \int_\gamma \eta\, GK\, d\sigma.$$

The following result generalizes the comparison formula in Theorem 4.7. Note in particular that our new comparison formula may be applied to convex functions, where the principal curvatures of level sets might vanish. So we will use the following conventions.

$$\frac{GK}{\kappa_r} := \prod_{i\neq r} \kappa_i, \qquad \text{and} \qquad \frac{GK}{\kappa_r\kappa_k} := \prod_{i\neq r,k} \kappa_i, \tag{26}$$

Now the terms GK/κ_r and $GK/(\kappa_r\kappa_k)$ below will always be well-defined.

Theorem 4.9 (Comparison Formula, General Version)**.** *Let u, Γ, γ, Ω, and D be as in Theorem 4.7, except that u is $\mathcal{C}^{1,1}$ on $(\Omega \setminus D) \setminus X$, for some (possibly empty) closed set $X \subset \Omega \setminus D$, and u is either convex or else $\nabla^2 u$ is nondegenerate almost everywhere on $(\Omega \setminus D) \setminus X$. Then, for any $\theta > 0$, and cutoff function η for $U_\theta(X)$,*

$$\mathcal{G}_\eta(\Gamma) - \mathcal{G}_\eta(\gamma) = \int_{\Omega\setminus D} \left(\eta_k \frac{GK}{\kappa_k}\frac{u_{nk}}{|\nabla u|} - \eta_n GK\right) d\mu + \int_{\Omega\setminus D} \eta \left(-R_{rnrn}\frac{GK}{\kappa_r} + R_{rkrn}\frac{GK}{\kappa_r\kappa_k}\frac{u_{nk}}{|\nabla u|}\right) d\mu,$$

where all quantities are computed with respect to a principal frame of u, and $k \leq n-1$.

Proof. Let $\widehat{u}^\varepsilon_\lambda$ be as in Proposition 4.8 with X in that theorem set to $\mathrm{cl}(\Omega) \setminus D$. Furthermore, let $\Gamma^\varepsilon_\lambda$ and $\gamma^\varepsilon_\lambda$ be regular level sets of $\widehat{u}^\varepsilon_\lambda$ close to Γ and γ respectively. Replace u by $\widehat{u}^\varepsilon_\lambda$ in (25) and follow virtually the same argument used in Theorem 4.7. Finally, letting λ and then ε go to 0 completes the argument. □

4.3. **Some special cases and applications.** Here we will record some consequences of the comparison formula developed in Theorem 4.9. Let

$$\sigma_r(x_1, \ldots, x_k) := \sum_{i_1<\cdots<i_r} x_{i_1} \ldots x_{i_r}, \tag{27}$$

denote the *elementary symmetric functions*. Furthermore, set $\kappa := (\kappa_1, \ldots, \kappa_{n-1})$, where κ_i are principal curvatures of level sets $\{u = u(p)\}$ at a regular point p of u which is twice differentiable. Then the r^{th} *generalized mean curvature* of $\{u = u(p)\}$ is given by

$$\sigma_r(\kappa) := \sigma_r(\kappa_1, \ldots, \kappa_{n-1}).$$

In particular note that $\sigma_{n-1}(\kappa) = GK$, and $\sigma_1(\kappa) = (n-1)H$, where H is the (normalized first) mean curvature of $\{u = u(p)\}$. The integrals of $\sigma_r(\kappa)$, which are called *quermassintegrals*, are central in the theory of mixed volumes [132, 142]. In the next two corollaries we adopt the same notation as in Theorem 4.9 and assume that $X = \emptyset$. In particular note that

$$\int_{\Omega\setminus D} \sigma_{n-2}(\kappa) d\mu = \int_{t_0}^{t_1} \int_{\{u=t\}} \sigma_{n-2}(\kappa) \frac{d\sigma}{|\nabla u|} dt.$$

If M has constant sectional curvature K_0, then $R_{ijk\ell} = K_0(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})$. Consequently Theorem 4.9 quickly yields:

Corollary 4.10. *If M has constant sectional curvature K_0, then*

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = -K_0 \int_{\Omega\setminus D} \sigma_{n-2}(\kappa) d\mu.$$

In particular, $\mathcal{G}(\Gamma) \geq \mathcal{G}(\gamma)$ if u has convex level sets and $K_0 \leq 0$.

A version of the last observation had been obtained earlier by Borbely [26]. Another important special case of Theorem 4.9 occurs when $|\nabla u|$ is constant on level sets of u, or $u_{kn} \equiv 0$ (for $k \leq n-1$), e.g., u may be the distance function of Γ, in which case recall that we say γ is an inner parallel hypersurface of Γ. The following result shows that a $\mathcal{C}^{1,1}$ d-convex hypersurface Γ in a Cartan-Hadamard manifold may be pushed inward by a short distance without any increase in its total curvature, as had been mentioned in the introduction.

Corollary 4.11. *Suppose that $u = \widehat{d}_\Gamma$, and $K_M \leq -a \leq 0$. Then*

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = -\int_{\Omega\setminus D} R_{rnrn} \frac{GK}{\kappa_r}\, d\mu \geq a \int_{\Omega\setminus D} \sigma_{n-2}(\kappa)\, d\mu. \tag{28}$$

In particular, if γ is convex, then

$$\mathcal{G}(\Gamma) \geq \mathcal{G}(\gamma). \tag{29}$$

Furthermore, if $\Gamma = \partial B_\rho$, where B_ρ denotes a geodesic ball of radius ρ in M, we have

$$\mathcal{G}(\partial B_\rho) \geq n\omega_n + a \int_{B_\rho} \sigma_{n-2}(\kappa)\, d\mu, \tag{30}$$

with equality only if $R_{rnrn} \equiv -a$ on B_ρ.

Proof. Inequality (28) follows immediately from Theorem 4.9 (with $X = \emptyset$). If γ is convex, then all of its outer parallel hypersurfaces, which are level sets of u fibrating $\Omega \setminus D$, are convex as well by Lemma 3.1. Thus $\sigma_{n-2}(\kappa) \geq 0$ which yields (29). To see (30), let B_ε denote the geodesic ball of radius $\varepsilon < \rho$ with the same center as B_ρ. By (28),

$$\mathcal{G}(\partial B_\rho) - \mathcal{G}(\partial B_\varepsilon) \geq a \int_{B_\rho\setminus B_\varepsilon} \sigma_{n-2}(\kappa)\, d\mu.$$

Now letting $\varepsilon \to 0$, yields (30). Indeed it is well-known that $\mathcal{G}(\partial B_\varepsilon) \to n\omega_n$ as $\varepsilon \to 0$ since $n\omega_n$ is the total curvature of Euclidean spheres (see Lemma 10.19 for an estimate for the rate of convergence). □

Note that (29) also follows from the Gauss-Bonnet theorem when $n = 2$. Indeed the Gauss-Bonnet theorem quickly shows that, when $n = 2$, (29) holds for any convex hypersurface γ nested inside another convex hypersurface Γ. As has been observed by Borbely [26], this monotonicity property of total curvature for nested convex hypersurfaces also holds in hyperbolic spaces of any dimension. More specifically, it follows from Corollary 4.10, since one can construct a function u with Γ and γ as level sets such that all other level sets of u are convex as well in the region between Γ and γ [26, Lem. 1]. In a general Cartan-Hadamard manifold, however, the monotonicity of total curvature does not always hold, as has been demonstrated by Dekster [54]. Finally we record another observation with regard to curvature of geodesic balls:

Corollary 4.12. *Let B_ρ be a geodesic ball of radius ρ in M, and suppose that $K_M \leq -a \leq 0$. Then*

$$\mathcal{G}(\partial B_\rho) \geq \mathcal{G}(\partial B^a_\rho), \tag{31}$$

where B^a_ρ is a geodesic ball of radius ρ in the hyperbolic space $\mathbf{H}^n(-a)$. Furthermore, equality holds only if $R_{rnrn} \equiv -a$ on B_ρ.

Proof. As in the proof of Corollary 4.11, let B_r denote the geodesic ball of radius $r < \rho$ with the same center as B_ρ. By basic Riemannian comparison theory [100, p. 184], principal curvatures of ∂B_r are bounded below by $\sqrt{a}\coth(\sqrt{a}r)$. Hence, on ∂B_r,

$$\sigma_{n-2}(\kappa) \geq (n-1)(\sqrt{a}\coth\sqrt{a}r)^{n-2}.$$

Let $A(r,\theta)d\theta$ denote the volume (surface area) element of ∂B_r, and $H(r,\theta)$ be its (normalized) mean curvature function in geodesic spherical coordinates (generated by the exponential map based at the center of B_r). By [100, (1.5.4)],

$$\frac{d}{dr}A(r,\theta) = (n-1)H(r,\theta)A(r,\theta) \geq (n-1)\sqrt{a}\coth(\sqrt{a}r)\,A(r,\theta),$$

which after an integration yields

$$A(r,\theta) \geq \left(\frac{\sinh(\sqrt{a}r)}{\sqrt{a}}\right)^{n-1}.$$

Thus from (28) we obtain,

$$\begin{aligned}
\mathcal{G}(\partial B_\rho) - \mathcal{G}(\partial B_\varepsilon) &\geq \int_\varepsilon^\rho \int_{\partial B_r} \sigma_{n-2}(\kappa)A(r,\theta)d\theta dr \\
&\geq \int_\varepsilon^\rho \int_{\partial B_r} (n-1)(\sqrt{a}\coth\sqrt{a}r)^{n-2}\left(\frac{\sinh\sqrt{a}r}{\sqrt{a}}\right)^{n-1} d\theta dr \\
&= n\omega_n \int_\varepsilon^\rho (n-1)\sqrt{a}(\cosh\sqrt{a}r)^{n-2}\sinh\sqrt{a}r\; dr \\
&= n\omega_n (\cosh\sqrt{a}r)^{n-1}\Big|_\varepsilon^\rho.
\end{aligned}$$

Letting $\varepsilon \to 0$, $\mathcal{G}(\partial B_\varepsilon) \to n\omega_n$ and we find

$$\mathcal{G}(\partial B_\rho) \geq n\omega_n(\cosh\sqrt{a}\rho)^{n-1} = \mathcal{G}(\partial B^a_\rho),$$

as desired. □

5. Newton Operators and Reilly Type Integral Formulas

In [124, 125] Reilly developed a number of integral formulas for the invariants of the Hessian of functions on a Riemannian manifold M, which have found numerous applications in submanifold geometry. Here we establish some other formulas in this genre which will be used in the proof of Theorem 1.2. More equations of this type will be developed in Sections 10.4 and 10.5. We assume that u is a $\mathcal{C}^{1,1}$ function on a domain $\Omega \subset M$, and the computations below take place at a twice differentiable point of u. Recall that σ_r denotes the r^{th} symmetric elementary function as defined by (27). Let

$$\sigma_r(\nabla^2 u) := \sigma_r\big(\lambda_1(\nabla^2 u), \dots, \lambda_n(\nabla^2 u)\big),$$

where λ_i denote the eigenvalues of $\nabla^2 u$. Note that $\sigma_r(\nabla^2 u)$ generate the coefficients of the characteristic polynomial

$$P(\lambda) := \det(\lambda I^n - \nabla^2 u) = \lambda^n - \sigma_1(\nabla^2 u)\lambda^{n-1} + \cdots + (-1)^n \sigma_n(\nabla^2 u).$$

Thus $\sigma_0(\nabla^2 u) := 1$, and one may also compute that [124, Prop. 1.2(a)], in any given orthonormal frame $E_1, \dots, E_n$,

$$\sigma_r(\nabla^2 u) = \frac{1}{r!}\delta^{i_1,\dots,i_r}_{j_1,\dots,j_r} u_{i_1 j_1} \cdots u_{i_r j_r}, \tag{32}$$

where $\delta^{i_1,\dots,i_m}_{j_1,\dots,j_m}$ is the generalized Kronecker delta function, which is equal to 1 (resp. -1) if $i_1, \dots, i_m$ are distinct and $(j_1, \dots, j_m)$ is an even (resp. odd) permutation of $(i_1, \dots, i_m)$; otherwise, it is equal to 0.

We will use formulas of Reilly [125] to estimate $\int_\Omega \sigma_r(\nabla^2 u) d\mu$ in terms of the quermassintegrals $\int_\Gamma \sigma_r(\kappa) d\sigma$ of $\Gamma = \partial\Omega$ (see Corollary 5.4). To this end we will employ the notion of *Newton operators* [124, 125] associated to $\nabla^2 u$, which may be defined recursively by setting $\mathcal{T}^u_0 := \mathrm{Id}$ and

$$\begin{aligned} \mathcal{T}^u_r &:= \sigma_r(\nabla^2 u)\,\mathrm{Id} - \mathcal{T}^u_{r-1} \cdot \nabla^2 u \\ &= \sigma_r(\nabla^2 u)\,\mathrm{Id} - \sigma_{r-1}(\nabla^2 u)\nabla^2 u + \cdots + (-1)^r (\nabla^2 u)^r. \end{aligned} \tag{33}$$

Thus $\mathcal{T}^u_r$ is the truncation of the matrix polynomial $P(\nabla^2 u)$ obtained by removing the terms of order bigger than r. In particular $\mathcal{T}^u_n = P(\nabla^2 u)$. So, by the Cayley-Hamilton theorem, $\mathcal{T}^u_n = 0$. Consequently, when $\nabla^2 u$ is nondegenerate, (33) yields that

$$\mathcal{T}^u_{n-1} = \sigma_n(\nabla^2 u)(\nabla^2 u)^{-1} = \det(\nabla^2 u)(\nabla^2 u)^{-1} = \mathcal{T}^u, \tag{34}$$

where $\mathcal{T}^u$ is the Hessian cofactor operator discussed in Section 4.1. See [124, Prop. 1.2] for other basic identities which relate σ and T. In particular, by [124, Prop. 1.2(c)], we have $\mathrm{Trace}(\mathcal{T}_r^u \cdot \nabla^2 u) = (r+1)\sigma_{r+1}(\nabla^2 u)$. So, by Euler's identity for homogenous polynomials,

$$(\mathcal{T}_r^u)_{ij} u_{ij} = \mathrm{Trace}(\mathcal{T}_r^u \cdot \nabla^2 u) = (r+1)\sigma_{r+1}(\nabla^2 u) = \frac{\partial \sigma_{r+1}(\nabla^2 u)}{\partial u_{ij}} u_{ij}. \tag{35}$$

Thus it follows from (32) that

$$(\mathcal{T}_r^u)_{ij} = \frac{\partial \sigma_{r+1}(\nabla^2 u)}{\partial u_{ij}} = \frac{1}{r!} \delta^{i,i_1,\dots,i_r}_{j,j_1,\dots,j_r} u_{i_1 j_1} \cdots u_{i_r j_r}. \tag{36}$$

Furthermore, by [125, Prop. 1(11)] (note that the sign of the Riemann tensor R in [125] is opposite to the one in this paper) we have

$$\big(\mathrm{div}(\mathcal{T}_r^u)\big)_j = \frac{1}{(r-1)!} \delta^{i,i_1,\dots,i_r}_{j,j_1,\dots,j_r} u_{i_1 j_1} \cdots u_{i_{r-1} j_{r-1}} R_{i j_r i_r k} u_k. \tag{37}$$

Another useful identity [125, p. 462] is

$$\mathrm{div}\big(\mathcal{T}_r^u(\nabla u)\big) = \big\langle \mathcal{T}_r^u, \nabla^2 u \big\rangle + \big\langle \mathrm{div}(\mathcal{T}_r^u), \nabla u \big\rangle, \tag{38}$$

where $\langle \cdot, \cdot \rangle$ here indicates the Frobenius inner product (i.e., $\langle A, B \rangle := A_{ij} B_{ij}$ for any pair of matrices of the same dimension). Next note that the divergence of $\mathcal{T}_r^u$ may be defined by virtually the same argument used for $\mathcal{T}^u$ in Section 4.1 to yield the following generalization of (15):

$$\big(\mathrm{div}(\mathcal{T}_r^u)\big)_j = \nabla_i (\mathcal{T}_r^u)_{ij}. \tag{39}$$

Here we assume, as was the case earlier, that all local computations take place with respect to the principal curvature frame E_i. Recall that, $\mathcal{T}^u = \mathcal{T}_{n-1}^u$ by (34). Furthermore, $\mathcal{T}_n^u = 0$ as we mentioned earlier. Thus the following observation generalizes Lemma 4.2.

Lemma 5.1.

$$\mathrm{div}\left(\mathcal{T}_{r-1}^u\left(\frac{\nabla u}{|\nabla u|^r}\right)\right) = \left\langle \mathrm{div}(\mathcal{T}_{r-1}^u), \frac{\nabla u}{|\nabla u|^r}\right\rangle + r\frac{\big\langle \mathcal{T}_r^u(\nabla u), \nabla u\big\rangle}{|\nabla u|^{r+2}}.$$

Proof. By Leibnitz rule and (39) we have

$$\begin{aligned}
\mathrm{div}\left(\mathcal{T}_{r-1}^u\left(\frac{\nabla u}{|\nabla u|^r}\right)\right) &= \nabla_i\left((\mathcal{T}_{r-1}^u)_{ij}\frac{u_j}{|\nabla u|^r}\right)\\
&= \left\langle \mathrm{div}(\mathcal{T}_{r-1}^u), \frac{\nabla u}{|\nabla u|^r}\right\rangle + (\mathcal{T}_{r-1}^u)_{ij}\left(\frac{u_{ij}}{|\nabla u|^r} - r\frac{u_j u_\ell u_{\ell i}}{|\nabla u|^{r+2}}\right),
\end{aligned}$$

where the computation to obtain the second term on the right is identical to the one performed earlier in the proof of Lemma 4.2. To develop this term further, note that by (33)

$$(\mathcal{T}_{r-1}^u)_{ij} u_{\ell i} = \sigma_r(\nabla^2 u)\delta_{\ell j} - (\mathcal{T}_r^u)_{\ell j},$$

which in turn yields

$$(\mathcal{T}_{r-1}^u)_{ij}u_{\ell i}\frac{u_j u_\ell}{|\nabla u|^2} = \sigma_r(\nabla^2 u) - (\mathcal{T}_r{}^u)_{ij}\frac{u_i u_j}{|\nabla u|^2}.$$

Hence

$$\begin{aligned}(\mathcal{T}_{r-1}^u)_{ij}\left(\frac{u_{ij}}{|\nabla u|^r} - r\frac{u_j u_\ell u_{\ell i}}{|\nabla u|^{r+2}}\right) &= \frac{r\sigma_r(\nabla^2 u)}{|\nabla u|^r} - \frac{r}{|\nabla u|^r}\left(\sigma_r(\nabla^2 u) - (\mathcal{T}_r{}^u)_{ij}\frac{u_i u_j}{|\nabla u|^2}\right)\\ &= r(\mathcal{T}_r{}^u)_{ij}\frac{u_i u_j}{|\nabla u|^{r+2}},\end{aligned}$$

which completes the proof. □

Recall that $\sigma_{n-1}(\kappa) = GK$. Thus the next observation generalizes Lemma 4.1.

Lemma 5.2. *For $r \leq n-1$,*

$$\frac{\langle \mathcal{T}_r{}^u(\nabla u), \nabla u\rangle}{|\nabla u|^{r+2}} = \sigma_r(\kappa).$$

Proof. Recall that in the principal curvature frame,

$$u_i = 0 \;\text{ for }\; i \neq n, \quad u_n = |\nabla u|, \;\text{ and }\; \frac{u_{ii}}{|\nabla u|} = \kappa_i \;\text{ for }\; i \neq n, \tag{40}$$

where the last equality is by (13). Thus (36) yields that

$$\begin{aligned}(\mathcal{T}_r{}^u)_{ij}\frac{u_i u_j}{|\nabla u|^{r+2}} &= \frac{1}{r!}\delta^{ii_1\cdots i_r}_{jj_1\cdots j_r}u_{i_1 j_1}\cdots u_{i_r j_r}\frac{u_i u_j}{|\nabla u|^{r+2}}\\ &= \frac{1}{r!}\delta^{ni_1\cdots i_r}_{nj_1\cdots j_r}\frac{u_{i_1 j_1}\cdots u_{i_r j_r}}{|\nabla u|^r}\cdot\frac{|\nabla u|^2}{|\nabla u|^2}\\ &= \frac{1}{r!}\delta^{ni_1\cdots i_r}_{nj_1\cdots j_r}\kappa_{i_1}\ldots\kappa_{i_r}\\ &= \sigma_r(\kappa),\end{aligned}$$

where in transition from the second to the third line in the computation above we have used the fact that for $\delta^{ni_1\cdots i_r}_{nj_1\cdots j_r}$ not to vanish, $i_1, \ldots, i_r$, $j_1, \ldots, j_r$ all must be different from n, which in turn implies that $u_{i_m j_m} = \kappa_m$, for $1 \leq m \leq r$, by (40). □

Using the identities established above we can now establish:

Proposition 5.3. *Let Ω be a domain in a Riemannian manifold with $\mathcal{C}^{1,1}$ boundary Γ and u be a $\mathcal{C}^{1,1}$ function on $\mathrm{cl}(\Omega)$ with $|\nabla u| \neq 0$ almost everywhere, and $u \equiv 0$ on Γ. Then, for $1 \leq r \leq n-1$,*

$$\int_\Omega \sigma_{r+1}(\nabla^2 u)d\mu \leq \frac{1}{r+1}\int_\Gamma \sigma_r(\kappa)|\nabla u|^{r+1}d\sigma + C\int_\Omega \sigma_{r-1}(\nabla^2 u)d\mu,$$

where C depends only on the sectional curvature of Ω and the Lipschitz constant of u.

Proof. By (35) and (38) we have

$$\operatorname{div}\big(\mathcal{T}_r{}^u(\nabla u)\big) = (r+1)\sigma_{r+1}(\nabla^2 u) + \big\langle \operatorname{div}(\mathcal{T}_r{}^u), \nabla u\big\rangle.$$

Furthermore, (37) and (32) yield that

$$\big|\big\langle \operatorname{div}(\mathcal{T}_r{}^u), \nabla u\big\rangle\big| \leq \left|\delta^{i,i_1,\dots,i_r}_{j,j_1,\dots,j_r} u_{i_1j_1}\cdots u_{i_{r-1}j_{r-1}} R_{ij_ri_rk} u_k u_j\right| \leq C\sigma_{r-1}(\nabla^2 u).$$

Recall that the outward normal to Γ is given by $\nabla u/|\nabla u|$. Thus by Lemma 5.2 and Stokes' theorem

$$\begin{aligned}
\int_\Gamma \sigma_r(\kappa)|\nabla u|^{r+1} d\sigma &= \int_\Gamma \left\langle \mathcal{T}_r{}^u(\nabla u), \frac{\nabla u}{|\nabla u|}\right\rangle d\sigma \\
&= \int_\Omega \operatorname{div}\big(\mathcal{T}_r{}^u(\nabla u)\big) d\mu \\
&\geq (r+1)\int_\Omega \sigma_{r+1}(\nabla^2 u) d\mu - C\int_\Omega \sigma_{r-1}(\nabla^2 u) d\mu,
\end{aligned}$$

as desired. □

Finally we arrive at the main result of this section:

Corollary 5.4. *Let Ω and u be as in Proposition 5.3. Then*

$$\int_\Omega \sigma_{r+1}(\nabla^2 u) d\mu \leq C\left(\sum_{\ell=0}^{r}\int_\Gamma \sigma_\ell(\kappa) d\sigma + 1\right), \qquad r = 0,\dots,n-1, \tag{41}$$

where C depends only on sectional curvature of Ω and the Lipschitz constant of u.

Proof. The desired inequality holds for $r=0$ since by Stokes' theorem

$$\int_\Omega \sigma_1(\nabla^2 u) d\mu = \int_\Omega \Delta u \, d\mu = \int_\Gamma |\nabla u| d\sigma \leq \sup_\Gamma |\nabla u| \int_\Gamma d\sigma,$$

and $\sigma_0(\kappa) = 1$. Other cases follow by an induction via Proposition 5.3. □

6. Projection into the Cut Locus of Convex Hypersurfaces

Recall that a hypersurface is d-convex if its distance function is convex, as we discussed in Section 3. Here we will study the cut locus of d-convex hypersurfaces and establish the following result:

Theorem 6.1. *Let Γ be a d-convex hypersurface in a Cartan-Hadamard manifold M, and let Ω be the convex domain bounded by Γ. Then for any point $x \in \Omega$ and any of its footprints x° in* $\operatorname{cut}(\Gamma)$,

$$d_\Gamma(x^\circ) \geq d_\Gamma(x).$$

Throughout this section we will assume that M is a Cartan-Hadamard manifold. In particular the exponential map $\exp_p\colon T_pM \to \mathbf{R}^n$ will be a global diffeomorhpism. The proof of the above theorem is based on the notion of tangent cones. For any set $X \subset \mathbf{R}^n$ and $p \in X$, the *tangent cone* T_pX of X at p is the limit of all secant rays which emanate from p and pass through a sequence of points of $X \setminus \{p\}$ converging to p. For a set $X \subset M$ and $p \in X$, the tangent cone is defined as

$$T_pX := T_p\big(\exp_p^{-1}(X)\big) \subset T_pM \simeq \mathbf{R}^n.$$

We say that a tangent cone is *proper* if it does not fill up the entire tangent space. A set $X \subset \mathbf{R}^n$ is a *cone* provided that there exists a point $p \in X$ such that for every $x \in X$ and $\lambda \geq 0$, $\lambda(x-p) \in X$. Then p will be called an *apex* of X. The following observation is proved in [41, Prop. 1.8].

Lemma 6.2 ([41])**.** *For any convex set $X \subset M$, and $p \in \partial X$, T_pX is a proper convex cone in T_pM, and $\exp_p^{-1}(X) \subset T_pX$.*

It will also be useful to record that for a given a set $X \subset \mathbf{R}^n$ and $p \in X$, T_pX is the limit of dilations of X based at p [73, Sec. 2]. More precisely, if we identify p with the origin o of $\mathbf{R}^n$, and for $\lambda \geq 1$ set $\lambda X := \{\lambda x \mid x \in X\}$, then T_oX is the *outer limit* [129] of the sets λX:

$$T_oX = \limsup_{\lambda\to\infty} \lambda X. \tag{42}$$

This means that for every $x \in T_oX \setminus \{o\}$ there exists a sequence of numbers $\lambda_i \to \infty$ such that $\lambda_i X$ eventually intersects any open neighborhood of x. Equivalently, we may record that:

Lemma 6.3 ([73])**.** *Let $X \subset \mathbf{R}^n$ and $o \in X$. Then $x \in T_oX \setminus \{o\}$ if there exists a sequence of points $x_i \in X \setminus \{o\}$ such that $x_i \to o$ and $x_i/|x_i| \to x/|x|$.*

The last lemma yields:

Lemma 6.4. *Let $\Gamma \subset \mathbf{R}^n$ be a closed hypersurface, and $o \in \mathrm{cut}(\Gamma) \cap \Gamma$. Suppose that $T_o\Gamma$ bounds a convex cone containing Γ. Then*

$$\mathrm{cut}(T_o\Gamma) \subset T_o\mathrm{cut}(\Gamma).$$

Proof. By (4) $\mathrm{cut}(T_o\Gamma) = \mathrm{cl}(\mathrm{medial}(T_o\Gamma))$. So it suffices to show that $\mathrm{medial}(T_o\Gamma) \subset T_o\mathrm{cut}(\Gamma)$, since $\mathrm{cut}(T_o\Gamma)$ is closed by definition. Let $x \in \mathrm{medial}(T_o\Gamma)$. Then there exists a sphere S centered at x which is contained in (the cone bounded by) $T_o\Gamma$, and touches $T_o\Gamma$ at multiple points. Suppose that S has radius r. Then, by (42), for each natural number i we may choose a number λ_i so large that the sphere S_i of radius $r-(1/i)$ centered at x is contained in $\lambda_i\Gamma$. Let S_i' be the largest sphere contained in $\lambda_i\Gamma$ centered at x which contains S_i. Then S_i' must intersect $\lambda_i\Gamma$ at some point y. Let S_i'' be the largest sphere contained in $\lambda_i\Gamma$ which passes through y. Then the center c_i of S_i'' lies in skeleton$(\lambda_i\Omega)$, and therefore belongs to $\mathrm{cut}(\lambda_i\Gamma)$, by Lemma 2.4. Now note that $\mathrm{cut}(\lambda_i\Gamma) = \lambda_i\mathrm{cut}(\Gamma)$. So

$$x_i := \frac{c_i}{\lambda_i} \in \frac{\mathrm{cut}(\lambda_i\Gamma)}{\lambda_i} \in \mathrm{cut}(\Gamma).$$

Furthermore, note that $c_i \to x$, since S_i'' and S_i have a point in common, S_i'' is a maximal sphere in $\lambda_i\Gamma$, S_i is a maximal sphere in $T_o\Gamma$, and $\lambda_i\Gamma \to T_o\Gamma$ according to (42). Thus $x_i \to o$, and $x_i/|x_i| \to x/|x|$. So $x \in T_o\mathrm{cut}(\Gamma)$ by Lemma 6.3, which completes the proof. □

For any set $X \subset \mathbf{R}^n$ we define cone(X) as the set of all rays which emanate from the origin o of $\mathbf{R}^n$ and pass through a point of X. Furthermore we set

$$\widehat{X} := X \cap \mathbf{S}^{n-1}.$$

Lemma 6.5. *Let X be the boundary of a proper convex cone with interior points in $\mathbf{R}^n$ and apex at o. Suppose that X is not a hyperplane. Then*

$$\widehat{\mathrm{cut}(X)} = \mathrm{cut}(\widehat{X}),$$

where $\mathrm{cut}(\widehat{X})$ denotes the cut locus of $\widehat{X}$ as a hypersurface in $\mathbf{S}^{n-1}$.

Proof. Let $x \in \widehat{\mathrm{cut}(X)}$. Then, since X is not a hyperplane, there exists a sphere S centered at x which is contained inside the cone bounded by X and touches X at multiple points, or else x is a limit of the centers of such spheres, by (4). Consequently, $\widehat{\mathrm{cone}(S)}$ forms a sphere in $\mathbf{S}^{n-1}$, centered at x, which is contained inside $\widehat{X}$ and touches $\widehat{X}$ at multiple points, or is the limit of such spheres respectively. Thus x belongs to $\mathrm{cut}(\widehat{X})$, which yields that $\widehat{\mathrm{cut}(X)} \subset \mathrm{cut}(\widehat{X})$. The reverse inequality may be established similarly. □

Using the last lemma, we next show:

Lemma 6.6. *Let X be as in Lemma 6.5. Suppose that X is not a hyperplane. Then for every point $x \in X$, there exists a point $s \in \mathrm{cut}(X)$ such that*

$$\langle s, x\rangle > 0.$$

Proof. We may replace x by $x/|x|$. Then, by Lemma 6.5, it is enough to show that $\langle s, x\rangle > 0$ for some $s \in \mathrm{cut}(\widehat{X})$, or equivalently that $\delta_{\mathbf{S}^{n-1}}(s, x) < \pi/2$, where $\delta_{\mathbf{S}^{n-1}}$ denotes the distance in $\mathbf{S}^{n-1}$. To this end let s be a footprint of x on $\mathrm{cut}(\widehat{X})$. Suppose towards a contradiction that $\delta_{\mathbf{S}^{n-1}}(s, x) \geq \pi/2$. Consider the great sphere G in $\mathbf{S}^{n-1}$ which passes through s and is orthogonal to the geodesic segment xs; see Figure 3. Let G^+ be the hemisphere bounded by G which contains x. Then the interior of G^+

is disjoint from $\mathrm{cut}(\widehat{X})$, since $\delta_{\mathbf{S}^{n-1}}(x, G) \leq \delta_{\mathbf{S}^{n-1}}(x, s) = \delta_{\mathbf{S}^{n-1}}(x, \mathrm{cut}(\widehat{X}))$. Next note that the intersection of the convex cone bounded by X with $\mathbf{S}^{n-1}$ is a convex set in $\mathbf{S}^{n-1}$. Thus G divides this convex set into two subregions. Consider the region, say R, which contains x, or lies in G^+, and let S be a sphere of largest radius in R. Then S must touch the boundary of R at least twice. Since S cannot touch G more than once, it follows that S must touch $\widehat{X}$, because the boundary of R consists of a part of G and a part of $\widehat{X}$. First suppose that S touches $\widehat{X}$ multiple times. Then the center of S belongs to $\mathrm{cut}(\widehat{X})$. But this is impossible, since $S \subset R \subset G^+$. We may suppose then that S touches $\widehat{X}$ only once, say at a point y.

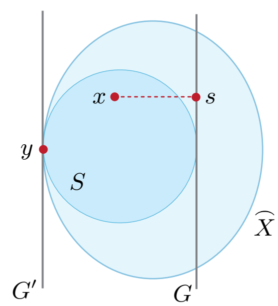

FIGURE 3.

Now we claim that the diameter of S is $\geq \pi/2$. Indeed let G' be the great sphere which passes through y and is tangent to S. Then G' supports $\widehat{X}$, and R is contained entirely between G and G'. The maximum length of a geodesic segment orthogonal to both G and G' is then equal to the diameter of S, since the points where S touches G and G' must be antipodal points of S. In particular the length of the diameter of S must be greater than $\delta_{\mathbf{S}^{n-1}}(x, s)$ as desired.

Finally let S' be the largest sphere contained in $\widehat{X}$ which passes through y. Then the center, say z, of S' belongs to $\mathrm{cut}(\widehat{X})$ by Lemma 2.4. But the diameter of S' is $< \pi$, since X is not a hyperplane by assumption. So, since the diameter of S is $\geq \pi/2$, it follows that z is contained in the interior of S and therefore in the interior R. Hence we reach the desired contradiction since, as we had noted earlier, R does not contain points of $\mathrm{cut}(\widehat{X})$ in its interior. □

For $x \in \Omega$, set

$$\Omega_x := \{y \in \Omega \mid d_\Gamma(y) > d_\Gamma(x)\}, \qquad \text{and} \qquad \Gamma_x := \partial \Omega_x.$$

Lemma 6.7. $\mathrm{cut}(\Gamma_x) \subset \mathrm{cut}(\Gamma)$.

Proof. As we discussed in the proof of Lemma 6.4, it suffices to show that $\mathrm{medial}(\Gamma_x) \subset \mathrm{cut}(\Gamma)$ by (4) . Let $y \in \mathrm{medial}(\Gamma_x)$. Then there exists a sphere $S \subset \mathrm{cl}(\Omega_x)$ centered at

y which intersects Γ_x in multiple points. Let S' be the sphere centered at y with radius equal to the radius of S plus $d(x,\Gamma)$. Then $S' \subset \mathrm{cl}(\Omega)$ and it intersects Γ in multiple points. So, again by (4), $y \in \mathrm{cut}(\Gamma)$ as desired. □

We say that a geodesic segment $\alpha\colon [0,a] \to M$ is *perpendicular* to a convex set X provided that $\alpha(0) \in \partial X$ and $\langle \alpha'(0), x - \alpha(0)\rangle \leq 0$ for all $x \in T_{\alpha(0)}X$. The following observation is well-known, see [22, Lem. 3.2].

Lemma 6.8 ([22])**.** *Let X be a convex set in a Cartan-Hadamard manifold M. Then geodesic segments which are perpendicular to X at distinct points never intersect.*

We need to record one more observation, before proving Theorem 6.1. An example of the phenomenon stated in the following lemma occurs when Γ is the inner parallel curve of a (noncircular) ellipse in $\mathbf{R}^2$ which passes through the foci of the ellipse, and p is one of the foci.

Lemma 6.9. *Let Γ be a d-convex hypersurface in a Cartan-Hadamard manifold M, and $p \in \Gamma \cap \mathrm{cut}(\Gamma)$. Suppose that $T_p\Gamma$ is a hyperplane. Then $T_p\mathrm{cut}(\Gamma)$ contains a ray which is orthogonal to $T_p\Gamma$.*

Proof. Let $\alpha(t)$, $t \geq 0$, be the geodesic ray, with $\alpha(0) = p$, such that $\alpha'(0)$ is orthogonal to $T_p\Gamma$ and points towards Ω. We have to show that $\alpha'(0) \in T_p\mathrm{cut}(\Gamma)$. To this end we divide the argument into two cases as follows.

First suppose that there exists a sphere in $\mathrm{cl}(\Omega)$ which touches Γ only at p. Then the center of that sphere coincides with $\alpha(t_0)$ for some $t_0 > 0$. We claim that then $\alpha(t) \in \mathrm{cut}(\Gamma)$ for all $t \leq t_0$. To see this note that $\alpha(t)$ has a unique footprint on Γ, namely p, for all $t \leq t_0$. For $0 < t \leq t_0$, let $\Gamma^t := (\widehat{d}_\Gamma)^{-1}(-t)$ be the inner parallel hypersurface of Γ at distance t. Suppose, towards a contradiction, that $\alpha(t) \notin \mathrm{cut}(\Gamma)$. Then, by Lemma 2.2, $\widehat{d}_\Gamma$ is $\mathcal{C}^1$ near $\alpha(t)$, which in turn yields that Γ^t is $\mathcal{C}^1$ in a neighborhood U^t of $\alpha(t)$. Furthermore, Γ^t is convex by the d-convexity assumption on Γ. So, by Lemma 6.8, the outward geodesic rays which are perpendicular to U^t never intersect, and thus yield a homeomorphism between U^t and a neighborhood U of p in Γ. Furthermore, since $\widehat{d}_\Gamma$ is $\mathcal{C}^1$ near U^t, each point of U^t has a unique footprint on Γ by Lemma 2.2. Thus there exists a sphere centered at each point of U^t which lies in $\mathrm{cl}(\Omega)$ and passes through a point of U. Furthermore each point of U is covered by such a sphere. So it follows that a ball rolls freely on the convex side of U, and therefore U is $\mathcal{C}^{1,1}$, by the same argument we gave in the proof of Lemma 2.6. But, again by Lemma 2.6, if U is $\mathcal{C}^{1,1}$, then $\widehat{d}_\Gamma$ is $\mathcal{C}^1$ near U, which is not possible since $p \in U$ and $p \in \mathrm{cut}(\Gamma)$. Thus we arrive at the desired contradiction. So we conclude that $\alpha(t) \in \mathrm{cut}(\Gamma)$ as claimed, for $0 < t \leq t_0$, which in turn yields that $\alpha'(0) \in T_p\mathrm{cut}(\Gamma)$ as desired.

So we may assume that there exists no sphere in $\mathrm{cl}(\Omega)$ which touches Γ only at p. Now for small $\varepsilon > 0$ let S_ε be a sphere of radius ε in $\mathrm{cl}(\Omega)$ whose center c_ε is as close to p as possible, among all spheres of radius ε in $\mathrm{cl}(\Omega)$. Then S_ε must intersect Γ in multiple points, since Γ is convex and S_ε cannot intersect Γ only at p. Thus $c_\varepsilon \in \mathrm{cut}(\Gamma)$. Let v be the initial velocity of the geodesic $c_\varepsilon p$, and $\theta(\varepsilon)$ be the supremum of the angles between v and the initial velocities of the geodesics connecting c_ε to each of its footprints on Γ. We claim that $\theta(\varepsilon) \to 0$, as $\varepsilon \to 0$. To see this let $(T_{c_\varepsilon}M)^1$ denote the unit sphere in $T_{c_\varepsilon}M$, centered at c_ε. Furthermore, let $X \subset (T_{c_\varepsilon}M)^1$ denote the convex hull spanned by the initial velocities of the geodesics connecting $c(\varepsilon)$ to its footprints. Then v must lie in X, for otherwise S_ε may be pulled closer to p. Indeed if $v \notin X$, then v is disjoint from a closed hemisphere of $(T_{c_\varepsilon}M)^1$ containing X. Let w be the center of the opposite hemisphere. Then $\langle v, w\rangle > 0$. Thus perturbing $c(\varepsilon)$ in the direction of w will bring S_ε closer to p without leaving $\mathrm{cl}(\Omega)$, which is not possible. So $v \in X$ as claimed. Now note that the footprints of c_ε converge to p, since c_ε converges to p. Furthermore, since $T_p\Gamma$ is a hyperplane, it follows that the angle between every pair of geodesics which connect c_ε to its footprints vanishes. Thus X collapses to a single point, which can only be v. Hence $\theta(\varepsilon) \to 0$ as claimed. Consequently $c_\varepsilon p$ becomes arbitrarily close to meeting Γ orthogonally, or more precisely, the angle between $\alpha'(0)$ and the initial velocity vector of pc_ε vanishes as $\varepsilon \to 0$. Hence, since $c_\varepsilon \in \mathrm{cut}(\Gamma)$, it follows once again that $\alpha'(0) \in T_p\mathrm{cut}(\Gamma)$ which completes the proof. □

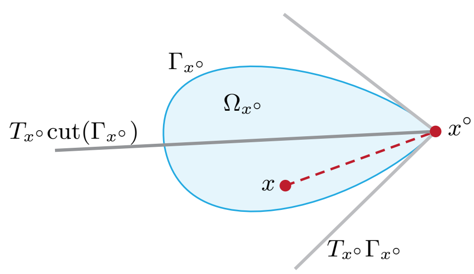

FIGURE 4.

Finally we are ready to prove the main result of this section:

Proof of Theorem 6.1. Suppose, towards a contradiction, that $d(x, \Gamma) > d(x^\circ, \Gamma)$ for some point $x \in \Omega$. Then

$$x \in \Omega_{x^\circ}, \tag{43}$$

see Figure 4. Since x° is a footprint of x on Γ, $\mathrm{cut}(\Gamma)$ lies outside a sphere of radius $d(x^\circ, x)$ centered at x. So if we let v be the initial velocity of the geodesic $x^\circ x$, then

$\langle y, v\rangle \leq 0$, for all $y \in T_{x^\circ}\mathrm{cut}(\Gamma)$, where we identify $T_{x^\circ}\mathrm{cut}(\Gamma)$ with $\mathbf{R}^n$ and x° with the origin of $\mathbf{R}^n$. By Lemma 6.7, $T_{x^\circ}\mathrm{cut}(\Gamma_{x^\circ}) \subset T_{x^\circ}\mathrm{cut}(\Gamma)$. Thus $\langle y, v\rangle \leq 0$, for all $y \in T_{x^\circ}\mathrm{cut}(\Gamma_{x^\circ})$. Furthermore, by Lemma 6.4, $\mathrm{cut}(T_{x^\circ}\Gamma_{x^\circ}) \subset T_{x^\circ}\mathrm{cut}(\Gamma_{x^\circ})$. So

$$\langle s, v\rangle \leq 0, \quad \text{for all} \quad s \in \mathrm{cut}(T_{x^\circ}\Gamma_{x^\circ}). \tag{44}$$

Furthermore, since Γ is d-convex, $T_{x^\circ}\Gamma_{x^\circ}$ bounds a convex cone by Lemma 6.2. Thus, since $T_{x^\circ}\Gamma_{x^\circ}$ contains v, it must be a hyperplane, by Lemma 6.6. Consequently, by Lemma 6.9, $T_{x^\circ}\mathrm{cut}(\Gamma_{x^\circ})$ contains a ray which is orthogonal to $T_{x^\circ}\Gamma_{x^\circ}$. By (44), v must be orthogonal to that ray. So $v \in T_{x^\circ}\Gamma_{x^\circ}$, which in turn yields that $x \in \Gamma_{x^\circ}$. The latter is impossible by (43). Hence we arrive at the desired contradiction. □

Having established Theorem 6.1, we now derive the following consequence of it, which is how Theorem 6.1 will be applied later in this work, in Section 10.4. Set

$$\widehat{r}(\,\cdot\,) := d\big(\,\cdot\,, \mathrm{cut}(\Gamma)\big).$$

Recall that, by Lemma 2.1, $\widehat{r}$ is Lipschitz and thus is differentiable almost everywhere.

Corollary 6.10. *Let Γ be a d-convex hypersurface in a Cartan-Hadamard manifold M, and set $u := \widehat{d}_\Gamma$. Suppose that $\widehat{r}$ is differentiable at a point $x \in M \setminus \mathrm{cut}(\Gamma)$. Then*

$$\big\langle \nabla u(x), \nabla\widehat{r}(x)\big\rangle \geq 0.$$

In particular (since $\widehat{r}$ is Lipschitz), the above inequality holds for almost every $x \in M \setminus \mathrm{cut}(\Gamma)$.

Proof. Since $\widehat{r}$ is differentiable at x, x has a unique footprint x° on $\mathrm{cut}(\Gamma)$, by Lemma 2.2(i). Let α be a geodesic connecting x to x°. Then, by Lemma 2.2(ii), $\alpha'(0) = -\nabla\widehat{r}(x)$. Furthermore, by Theorem 6.1, $u \circ \alpha = -\widehat{d}_\Gamma \circ \alpha$ is nonincreasing. Finally, recall that by Proposition 2.7, u is $\mathcal{C}^1$ on $M \setminus \mathrm{cut}(\Gamma)$, and therefore $u \circ \alpha$ is $\mathcal{C}^1$ as well. Thus

$$0 \geq (u \circ \alpha)'(0) = \big\langle \nabla u(\alpha(0)), \alpha'(0)\big\rangle = \big\langle \nabla u(x), -\nabla\widehat{r}(x)\big\rangle,$$

as desired. □

7. Inf-Convolutions and Proximal Maps

In this section we discuss how to smooth the distance function $\widehat{d}_\Gamma$ of a hypersurface Γ in a Riemannian manifold M via inf-convolution. We also derive some basic estimates for the derivatives of the smoothing via the associated proximal maps. For $t > 0$, the *inf-convolution* (or more precisely *Moreau envelope* or *Moreau-Yosida regularization*) of a function $u\colon M \to \mathbf{R}$ is given by

$$\widetilde{u}^{\,t}(x) := \inf_y \left\{ u(y) + \frac{d^2(x,y)}{2t} \right\}. \tag{45}$$

It is well-known that $\widetilde{u}^t$ is the unique viscosity solution of the Hamilton-Jacobi equation $f_t+(1/2)|\nabla f|^2=0$ for functions $f\colon \mathbf{R}\times M\to\mathbf{R}$ satisfying the initial condition $f(0,x)=u(x)$. Furthermore, when $M=\mathbf{R}^n$, $\widetilde{u}^t$ is characterized by the fact that its epigraph is the Minkowski sum of the epigraphs of u and $|\cdot|^2/(2t)$ [132, Thm. 1.6.17]. The following properties are well-known,

$$\widetilde{(\widetilde{u}^t)}^s=\widetilde{u}^{t+s}, \qquad \text{and} \qquad \widetilde{\lambda u}^t=\lambda\widetilde{u}^{\lambda t}, \tag{46}$$

e.g., see [16, Prop. 12.22]. A simple but highly illustrative example of inf-convolution occurs when it is applied to $\rho(x):=d(x_0,x)$, the distance from a single point $x_0\in M$. Then

$$\widetilde{\rho}^t(x)=\begin{cases}\rho^2(x)/(2t), & \text{if } \rho(x)\leq t,\\ \rho(x)-t/2, & \text{if } \rho(x)>t,\end{cases} \tag{47}$$

which is known as the *Huber function*; see Figure 5 which shows the graph of $\widetilde{\rho}^t$ when $M=\mathbf{R}$ and $x_0=0$. Note that $\widetilde{\rho}^t$ is $\mathcal{C}^{1,1}$ and convex, $\inf(\widetilde{\rho}^t)=\inf(\rho)$, $|\nabla\widetilde{\rho}^t|\leq 1$

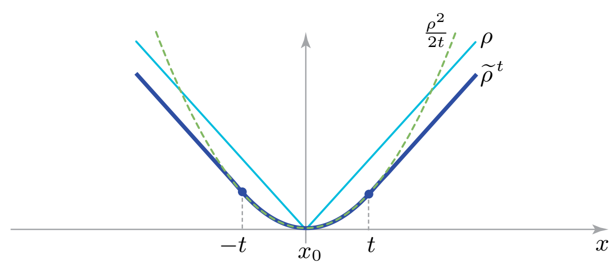

FIGURE 5.

everywhere, $|\nabla\widetilde{\rho}^t|=1$ when $\rho>t$, and $|\nabla^2\widetilde{\rho}^t|\leq C/t$. Remarkably enough, all these properties are shared by the inf-convolution of $\widehat{d}_\Gamma$ when Γ is d-convex, as we demonstrate below.

Some of the following observations are well-known or easy to establish in $\mathbf{R}^n$ or even Hilbert spaces [16, 36]. In the absence of a linear structure, however, finer methods are required to examine the inf-convolution on Riemannian manifolds, especially with regard to its differential properties [9, 10, 17, 21, 63]. First let us record that, by [9, Cor. 4.5]:

Lemma 7.1 ([9]). *Let u be a convex function on a Cartan-Hadamard manifold. Then for all $t>0$ the following properties hold:*

(i) $\widetilde{u}^t$ *is* $\mathcal{C}^1$ *and convex.*
(ii) $t\mapsto\widetilde{u}^t(x)$ *is nonincreasing, and* $\lim_{t\to 0}\widetilde{u}^t(x)=u(x)$.
(iii) $\inf(\widetilde{u}^t)=\inf(u)$, *and minimum points of* $\widetilde{u}^t$ *coincide with those of* u.

See also [17, Ex. 2.8] for part (i) above. Next let us rewrite (45) as

$$\widetilde{u}^t(x) = \inf_y F(y), \qquad F(y) = F(x,y) := u(y) + \frac{d^2(x,y)}{2t}.$$

Since $d^2(x,y)$ is strongly convex and u is convex, $F(y)$ is strongly convex and thus its infimum is achieved at a unique point

$$x^* := \mathrm{prox}_t^u(x),$$

which is called the *proximal point* [16] or *resolvent* [17] of $\widetilde{u}^t$ at x. In other words,

$$\widetilde{u}^t(x) = F\big(x^*\big).$$

The next estimate had been observed earlier [10, Prop. 2.1] for $2tL$.

Lemma 7.2. *Let u be an L-Lipschitz function on a Riemannian manifold. Then*

$$d(x,x^*) \leq tL.$$

Proof. Suppose, towards a contradiction, that $d(x^*,x) > tL$. Then there exists an $\varepsilon > 0$ such that

$$d(x^*,x) \geq (1+\varepsilon)tL.$$

Choose a point x' on the geodesic segment between x and x^* with

$$d(x,x') = d(x,x^*) - \varepsilon tL.$$

Since ε may be chosen arbitrarily small, we may assume that x' is arbitrarily close to x^*. Thus by the local L-Lipschitz assumption, $u(x') - u(x^*) \leq Ld(x^*,x')$. Consequently,

$$\begin{aligned}
F(x') - F(x^*) &= u(x') - u(x^*) + \frac{d^2(x,x') - d^2(x,x^*)}{2t} \\
&\leq Ld(x^*,x') + \frac{(d(x,x') - d(x,x^*))(d(x,x') + d(x,x^*))}{2t} \\
&\leq Ld(x^*,x') - d(x^*,x')\ \frac{d(x,x') + d(x,x^*)}{2t} \\
&= d(x^*,x')\left(L - \frac{2d(x,x^*) - d(x^*,x')}{2t}\right) \\
&\leq d(x^*,x')\left(L - \frac{2(1+\varepsilon)tL - d(x^*,x')}{2t}\right) \\
&= d(x^*,x')\left(\frac{d(x^*,x') - 2\varepsilon tL}{2t}\right) \quad = \quad -\frac{1}{2}\varepsilon^2 tL.
\end{aligned}$$

So $F(x') < F(x^*)$ which contradicts the minimality of x^*, and completes the proof. □

The following properties of proximal maps will be used in perturbation arguments in Section 10.4. Part (i) below, which shows that the proximal map is *nonexpansive*, is well-known [17], and part (ii) follows from [9, Prop. 3.7]. Recall that $d_x(\,\cdot\,) := d(x,\,\cdot\,)$.

Lemma 7.3 ([9,17])**.** *Let u be a convex function on a Cartan-Hadamard manifold. Then*

(i) $d(x_1^*, x_2^*) \le d(x_1, x_2)$,

(ii) If x^* is a regular point of u, then

$$\nabla u(x^*) = -\frac{d(x,x^*)}{t}\nabla d_x(x^*), \qquad \text{and} \qquad \nabla \widetilde{u}^{\,t}(x) = \frac{d(x,x^*)}{t}\nabla d_{x^*}(x).$$

(iii) $\nabla u(x^*)$ *and* $\nabla \widetilde{u}^{\,t}(x)$ *are tangent to the geodesic connecting x^* to x, and*

$$|\nabla u(x^*)| = |\nabla \widetilde{u}^{\,t}(x)|.$$

(iv) If u is L-Lipschitz, then so is $\widetilde{u}^t$.

Proof. For part (i) see [17, Thm. 2.2.22]. For part (ii) note that by definition $F(y) \ge F(x^*)$. Furthermore, x^* is a regular point of F, since by assumption x^* is a regular point of u. Consequently,

$$0 = \nabla F(x^*) = \nabla_y F(x,y)\big|_{y=x^*} = \nabla u(x^*) + \frac{d(x,x^*)}{t}\nabla d_x(x^*),$$

which yields the first equality in (ii). Next we prove the second inequality in (ii) following [9, Prop. 3.7]. To this end note that

$$\begin{aligned}\widetilde{u}^{\,t}(z) &= \inf_y F(z,y) \le F(z,x^*) = u(x^*) + \frac{d^2(z,x^*)}{2t},\\ \widetilde{u}^{\,t}(x) &= \inf_y F(x,y) = F(x,x^*) = u(x^*) + \frac{d^2(x,x^*)}{2t}.\end{aligned}$$

So it follows that

$$\widetilde{u}^{\,t}(z) - \frac{d(z,x^*)^2}{2t} \le u(x^*) = \widetilde{u}^{\,t}(x) - \frac{d^2(x,x^*)}{2t}.$$

Hence $g(\cdot) := \widetilde{u}^{\,t}(\cdot) - d(\cdot,x^*)^2/(2t)$ achieves its maximum at x. Further note that g is $\mathcal{C}^1$ since $\widetilde{u}^{\,t}$ is $\mathcal{C}^1$ by Lemma 7.1. Thus

$$0 = \nabla g(x) = \nabla \widetilde{u}^{\,t}(x) - \frac{d(x,x^*)}{t}\nabla d_{x^*}(x),$$

which yields the second equality in (ii). Next, to establish (iii), let $\alpha\colon [0,s_0] \to M$ be the geodesic with $\alpha(0) = x^*$ and $\alpha(s_0) = x$. Then, by Lemma 2.2,

$$\nabla d_x(x^*) = -\alpha'(0), \qquad \text{and} \qquad \nabla d_{x^*}(x) = \alpha'(s_0).$$

So $\nabla u(x^*)$ and $\nabla \widetilde{u}^t(x)$ are tangent to α and

$$|\nabla u(x^*)| = \left|\frac{d(x,x^*)}{t}\right| = |\nabla \widetilde{u}^t(x)|$$

as desired. Finally, to establish (iv), note that if u is L-Lipschitz, then $|\nabla u| \le L$ almost everywhere. Thus by part (iii), $|\nabla \widetilde{u}^t(x)| = |\nabla u(x^*)| \le L$ for almost every $x \in M$. So $\widetilde{u}^t$ is L-Lipschitz. ∎

Recall that we say a function $u\colon M \to \mathbf{R}$ is *locally* $\mathcal{C}^{1,1}$ provided that it is $\mathcal{C}^{1,1}$ in some choice of local coordinates around each point. There are other notions of $\mathcal{C}^{1,1}$ regularity [10, 63] devised in order to control the Lipschitz constant; however, all these definitions yield the same class of locally $\mathcal{C}^{1,1}$ functions; see [10]. The $\mathcal{C}^{1,1}$ regularity of functions is closely related to the more robust notion of semiconcavity which is defined as follows. We say that u is C*-semiconcave* (or is uniformly semiconcave with a constant C) on a set $\Omega \subset M$ provided that there exists a constant $C > 0$ such that for every $x_0 \in \Omega$, the function

$$x \;\mapsto\; u(x) - Cd^2(x, x_0) \tag{48}$$

is concave on Ω. Furthermore, we say u is C*-semiconvex*, if $-u$ is semiconcave.

Lemma 7.4 ([10,36])**.** *If a function u on a Riemannian manifold is both $C/2$-semiconvex and $C/2$-semiconcave on some bounded domain Ω, then it is locally $\mathcal{C}^{1,1}$ on Ω. Furthermore $|\nabla^2 u| \leq C$ almost everywhere on Ω.*

The above fact is well-known in $\mathbf{R}^n$, see [36, Cor. 3.3.8] (note that the constant C in the book of Cannarsa and Sinestrari [36] corresponds to $2C$ in this work due to a factor of $1/2$ in their definition of semiconcavity.) The Riemannian analogue follows from the Euclidean case via local coordinates to obtain the $\mathcal{C}^{1,1}$ regularity (since semiconcavity is preserved under $\mathcal{C}^2$ diffeomorphisms), and then differentiating along geodesics to estimate the Hessian, see the proof of [36, Cor. 3.3.8], and using Rademacher's theorem. The above lemma has also been established in [10, Thm 1.5]. The next observation, with a different estimate for C, has been known [10, Prop. 7.1(2)]. Here we provide another argument via Lemma 7.2.

Proposition 7.5. *Suppose that u is a convex function on a bounded domain Ω in a Riemannian manifold. Then for all $0 < t \leq t_0$, $\widetilde{u}^t$ is $C/(2t)$-semiconcave on Ω for*

$$C \geq \sqrt{-K_0}\, 3t_0 L \, \coth\left(\sqrt{-K_0}\, 3t_0 L\right), \tag{49}$$

where K_0 is the lower bound for the curvature of $B_{t_0 L}(\Omega)$, and L is the Lipschitz constant of u on Ω. In particular, $\widetilde{u}^t$ is locally $\mathcal{C}^{1,1}$, and

$$|\nabla^2 \widetilde{u}^t| \leq \frac{C}{t} \tag{50}$$

almost everywhere on Ω.

Proof. Since by Lemma 7.1, $\widetilde{u}^t$ is convex, it is $C/(2t)$-semiconvex. Thus as soon as we show that $\widetilde{u}^t$ is $C/(2t)$-semiconcave, $\widetilde{u}^t$ will be $\mathcal{C}^{1,1}$ and (50) will hold by Lemma 7.4, which will finish the proof. To establish the semiconcavity of $\widetilde{u}^t$ note that, by Lemma

7.2,

$$\widetilde{u}^t(x) = \inf_{y\in B_{tL}(x)} \left(u(y) + \frac{1}{2t} d^2(x,y) \right).$$

Let C be as in (49) and, according to (48), set

$$f(x) := \widetilde{u}^t(x) - \frac{C}{2t} d^2(x,x_0) = \inf_{y\in B_{tL}(x)} \left(u(y) - \frac{1}{2t}\Big(C d^2(x,x_0) - d^2(x,y) \Big) \right).$$

We have to show that f is concave on Ω. To this end it suffices to show that f is locally concave on Ω, since a locally concave function is concave. Indeed suppose that f is locally concave on Ω and let $\alpha\colon [a,b]\to\Omega$ be a geodesic. Then $-f\circ\alpha$ is locally convex. Thus, since $-f\circ\alpha$ is $\mathcal{C}^1$, $-(f\circ\alpha)'$ is nondecreasing, which yields that $-f\circ\alpha$ is convex [132, Thm. 1.5.10]. Now, to establish that f is locally concave on Ω, set

$$r := t_0 L.$$

We claim that f is concave on $B_r(p)$, for all $p\in\Omega$. To see this first note that if $x\in B_r(p)$ then $B_{tL}(x)\subset B_r(x)\subset B_{2r}(p)$. So, for $x\in B_r(p)$,

$$f(x) = \inf_{y\in B_{2r}(p)} \left(u(y) - \frac{1}{2t}\Big(C d^2(x,x_0) - d^2(x,y) \Big) \right). \tag{51}$$

Since the infimum of a family of concave functions is concave, it suffices to check that the functions on the right hand side of (51) are concave on $B_{2r}(p)$ for each y. So we need to show that

$$g(x) := \frac{1}{2}\big(C d^2(x,x_0) - d^2(x,y) \big)$$

is convex on $B_{2r}(p)$ for each y. To this end note that the eigenvalues of $\nabla^2 d^2_{x_0}(x)/2$ are bounded below by 1 [98, Thm. 6.6.1]. Furthermore, since $x\in B_r(p)$, and $y\in B_{2r}(p)$, we have $x\in B_{3r}(y)$. Thus the eigenvalues of $\nabla^2 d^2_y(x)/2$ are bounded above by

$$\lambda := \sqrt{-K_0}\,3r\coth\left(\sqrt{-K_0}\,3r\right),$$

by [98, Thm. 6.6.1]. So the eigenvalues of $\nabla^2 g$ on $B_{2r}(p)$ are bounded below by $C-\lambda$. Hence g is convex on $B_{2r}(p)$ if $C\geq\lambda$, which is indeed the case by (49). So f is concave on $B_{2r}(p)$ which completes the proof. □

Proposition 7.6. *Let Γ be a closed hypersurface in a Cartan-Hadamard manifold M and set $u := \widehat{d}_\Gamma$. Then*

(i) $\widetilde{u}^t = u - t/2$ *on* $M\setminus U_t(\mathrm{cut}(\Gamma))$.
(ii) $|\nabla\widetilde{u}^t|\equiv 1$ *on* $M\setminus U_t(\mathrm{cut}(\Gamma))$.
(iii) $|\nabla\widetilde{u}^t|\leq 1$ *on* M *if* Γ *is* d*-convex.*

Proof. Let $x\in M\setminus U_t(\mathrm{cut}(\Gamma))$, and B be the geodesic ball of radius t centered at x. Then all points of $\mathrm{cl}(B)$ are regular points of u. Consider the level set $\{u=c\}$. For c

small, these level sets will be disjoint from B. Let c_0 be the supremum of all constants c such that $\{u = c\}$ is disjoint from $\mathrm{cl}(B)$. Then $\{u = c_0\}$ intersects ∂B at a point x_0. So $c_0 = u(x_0)$. Let $\alpha(s)$ be the geodesic in M with $\alpha(0) = x_0$ and $\alpha'(0) = \nabla u(x_0)$. Then $u(\alpha(s)) = u(x_0) + s$. In particular we have

$$u(x) = u(x_0) + t.$$

By Lemma 7.2, $x^* \in \mathrm{cl}(B)$ so by assumption, x^* is a regular point of u. Thus

$$0 = \nabla F(x^*) = \nabla u(x^*) + \frac{d(x, x^*)}{t} \nabla d_x(x^*),$$

which implies $d(x, x^*) = t$, since $|\nabla u(x^*)| = |\nabla d_x(x^*)| = 1$. So

$$F(x^*) = u(x^*) + \frac{t^2}{2t} = u(x^*) + \frac{t}{2}.$$

On the other hand, since $x^* \in \partial B$, it lies outside the set $\{u < u(x_0)\}$. So $u(x^*) \geq u(x_0)$, which yields

$$u(x^*) + \frac{t}{2} \geq u(x_0) + \frac{t}{2} = F(x_0).$$

So $F(x^*) \geq F(x_0)$, which yields that $F(x^*) = F(x_0)$. Thus

$$\widetilde{u}^t(x) = F(x^*) = F(x_0) = u(x_0) + \frac{t}{2},$$

which yields

$$\widetilde{u}^t(x) = u(x) - \frac{t}{2}.$$

So we have established part (i) of the proposition. To see part (ii) note that $|\nabla u| \equiv 1$ on $M \setminus U_t(\mathrm{cut}(\Gamma))$. Thus by (i) $|\nabla \widetilde{u}^t| \equiv |\nabla u| \equiv 1$ on $M \setminus U_t(\mathrm{cut}(\Gamma))$. To see part (iii), let x be a point where $|\nabla \widetilde{u}^t(x)| \neq 0$, and let $\alpha(s)$ be the geodesic in M with $\alpha(0) = x$ and $\alpha'(0)$ parallel to $\nabla \widetilde{u}^t(x)$. Then

$$f(s) := \widetilde{u}^t \circ \alpha(s)$$

is a convex function. So f' is nondecreasing. If s_1 is sufficiently large, then $\alpha(s_1) \in M \setminus U_t(\mathrm{cut}(\Gamma))$. Thus, by (ii),

$$|\nabla \widetilde{u}^t(x)| = f'(0) \leq f'(s_1) = \big\langle \nabla \widetilde{u}^t(\alpha(s_1)), \alpha'(s_1) \big\rangle \leq |\nabla \widetilde{u}^t(\alpha(s_1))| = 1,$$

which establishes (iii), and completes the proof. □

8. Continuity of Total Curvature for $\mathcal{C}^{1,1}$ Hypersurfaces

In this section we establish the continuity of the total curvature function on the space of $\mathcal{C}^{1,1}$ hypersurfaces in a Cartan-Hadamard manifold M. As a consequence we then show that if the total curvature inequality (2) holds for $\mathcal{C}^2$ hypersurfaces, it holds for $\mathcal{C}^{1,1}$ hypersurfaces as well (Corollary 8.4). For $r > 0$, let $\mathrm{Reach}_r(M)$ denote the space of all closed embedded hypersurfaces Γ in M with $\mathrm{reach}(\Gamma) \geq r$ (as defined in Section

2). We assume that $\mathrm{Reach}_r(M)$ is endowed with the topology induced by the Hausdorff distance on bounded subsets of M, which is given by

$$d^H(X,Y) := \inf\big\{\delta > 0 \mid X \subset U_\delta(Y) \text{ and } Y \subset U_\delta(X)\big\};$$

see [32, Sec. 7.3.1] for basic properties of d^H. Recall that elements of $\mathrm{Reach}_r(M)$ are $\mathcal{C}^{1,1}$ by Lemma 2.6, so their total curvature is well defined. The principal result of this section is:

Theorem 8.1. *The total curvature mapping*

$$\mathrm{Reach}_r(M) \ni \Gamma \overset{\mathcal{G}}{\longmapsto} \mathcal{G}(\Gamma) \in \mathbf{R}$$

is continuous, for any $r > 0$.

First we need to record the following observation:

Lemma 8.2. *Let $\Gamma \in \mathrm{Reach}_r(M)$, and $\Gamma^m \in \mathrm{Reach}_r(M)$ be a sequence of hypersurfaces converging to Γ with respect to Hausdorff distance. Set $u := \widehat{d}_\Gamma$, $u^m := \widehat{d}_{\Gamma^m}$, and let $\delta := r/8$. Then on $U := U_{2\delta}(\Gamma)$ and for m sufficiently large:*

$$\|u^m\|_{\mathcal{C}^{1,1}(U)}, \quad \|u\|_{\mathcal{C}^{1,1}(U)} \leq C,$$

where C depends only on r and the lower bound for the sectional curvatures of M on $U_r(\Gamma)$. Moreover a subsequence (still called u^m) converges weakly to u in $\mathcal{W}^{2,p}(U)$ for for $p > n$, and strongly in $\mathcal{C}^{1,\alpha}(U)$ for $\alpha \in (0, 1 - n/p)$. In particular,

$$\|u^m - u\|_{\mathcal{C}^1(U)} \to 0,$$

as $m \to \infty$.

Proof. The Hausdorff convergence of Γ^m to Γ implies that u^m converges to u uniformly on $U_r(\Gamma)$, since $\mathrm{cl}(U_r(\Gamma))$ is compact. In particular for m large, we have that $U_{r/8}(\Gamma^m) \subset U_{r/4}(\Gamma)$. The uniform $\mathcal{C}^{1,1}$ estimates for u^m, u on U now follow from Proposition 2.8. Thus by the Rellich-Kondrachov compactness theorem [74, Thm. 7.22], a subsequence of u^m converges weakly to u in $\mathcal{W}^{2,p}(U)$. Furthermore, since for $p > n$, $\mathcal{W}^{2,p}(U)$ compactly embeds into $\mathcal{C}^{1,\alpha}(U)$, $\alpha \in (0, 1 - n/p)$ [74, Thm. 7.26], which yields strong convergence of u^m to u in $\mathcal{C}^{1,\alpha}(U)$. □

Now we are ready to establish the main result of this section:

Proof of Theorem 8.1. Let Γ^m, u, u^m, and U be as in Lemma 8.2, and Ω, Ω_m be the domains bounded by Γ and Γ^m respectively. Let η be a smooth nonnegative cutoff function on M given by $\eta \equiv 0$ on $\{u \leq -2\delta\}$ and $\eta \equiv 1$ on $\{u > -\delta\}$. By Lemmas 4.2 and 4.3,

$$\mathcal{G}(\Gamma^m) \;=\; \int_{\Omega_m} \mathrm{div}\big(\eta\,\mathcal{T}^m(\nabla u^m)\big)\,d\mu,$$

where we have set $\mathcal{T}^m := \mathcal{T}^{u^m}$. Furthermore, by (38), we have

$$\begin{aligned} \operatorname{div}\big(\eta\,\mathcal{T}^m(\nabla u^m)\big) &= \eta\operatorname{div}\big(\mathcal{T}^m(\nabla u^m)\big) + \langle \mathcal{T}^m(\nabla u^m), \nabla\eta\rangle \\ &= \eta\,\langle \operatorname{div}(\mathcal{T}^m), \nabla u^m\rangle + \eta\,\langle \mathcal{T}^m, \nabla^2 u^m\rangle + \langle \mathcal{T}^m(\nabla u^m), \nabla\eta\rangle. \end{aligned} \tag{52}$$

Recall that $\mathcal{T}^m = \mathcal{T}^m_{n-1}$ by (34). Thus, by (35) when (u^m_{ij}) is diagonal,

$$\langle \mathcal{T}^m, \nabla^2 u^m\rangle = (\mathcal{T}^m)_{ij} u^m_{ij} = n\sigma_n(u^m) = n u^m_{11}\dots u^m_{nn} = 0,$$

since $u^m_{nn} = 0$ (because u^m is the distance function of Γ^m). So we conclude that

$$\mathcal{G}(\Gamma^m) = \int_{\Omega_m} \eta\,\langle \operatorname{div}(\mathcal{T}^m), \nabla u^m\rangle d\mu + \int_\Lambda \langle \mathcal{T}^m(\nabla u^m), \nabla\eta\rangle d\mu,$$

where, $\Lambda := \{-2\delta < u < -\delta\}$. Thus by (36) and (37),

$$\begin{aligned} \mathcal{G}(\Gamma^m) = \frac{1}{(n-2)!}\int_{\Omega_m} \eta\,\delta^{i,i_1,\dots,i_{n-1}}_{j,j_1,\dots,j_{n-1}} R_{ij_{n-1}i_{n-1}k}\, u^m_{i_1j_1}\cdots u^m_{i_{n-2}j_{n-2}} u^m_j u^m_k d\mu \\ + \frac{1}{(n-1)!}\int_\Lambda \eta_\ell\, \delta^{k,i_1,\dots,i_{n-1}}_{\ell,j_1,\dots,j_{n-1}} u^m_{i_1j_1}\cdots u^m_{i_{n-1}j_{n-1}} u^m_k d\mu. \end{aligned}$$

Replacing u^m by u and Ω_m by Ω on the right hand side of the last expression, we also obtain a similar expression for $\mathcal{G}(\Gamma)$. To compare these two expressions note that, for m sufficiently large, $\Omega\triangle\Omega_m := (\Omega_m\setminus\Omega)\cup(\Omega\setminus\Omega_m)\subset U$. Furthermore, by Lemma 8.2, the second derivatives of u and u^m are bounded almost everywhere on U. Thus, since $\Omega\triangle\Omega_m \to 0$, as $m\to\infty$, it follows that

$$\begin{aligned} &\mathcal{G}(\Gamma^m) - \mathcal{G}(\Gamma) = \\ &\frac{1}{(n-2)!}\int_\Omega \eta\delta^{i,i_1,\dots,i_{n-1}}_{j,j_1,\dots,j_{n-1}} R_{ij_{n-1}i_{n-1}k}\big(u^m_{i_1j_1}\cdots u^m_{i_{n-2}j_{n-2}} u^m_j u^m_k - u_{i_1j_1}\cdots u_{i_{n-2}j_{n-2}} u_j u_k\big) d\mu \\ &+\frac{1}{(n-1)!}\int_\Lambda \eta_\ell\,\delta^{k,i_1,\dots,i_{n-1}}_{\ell,j_1,\dots,j_{n-1}}\big(u^m_{i_1j_1}\cdots u^m_{i_{n-1}j_{n-1}} u^m_k - u_{i_1j_1}\cdots u_{i_{n-1}j_{n-1}} u_k\big) d\mu + o(1), \end{aligned} \tag{53}$$

where $o(1)\to 0$ as $m\to\infty$. It remains to show then that the integrals in the last two lines of (53) vanish as $m\to\infty$. We verify this for the more complicated integral in the second line of (53) (the argument for the other integral will be similar). Note that the integral in the second line of (53) is the sum of the following two integrals

$$\begin{aligned} \mathcal{A} &:= \int_\Omega \eta\,\delta^{i,i_1,\dots,i_{n-1}}_{j,j_1,\dots,j_{n-1}} R_{ij_{n-1}i_{n-1}k}\big(u^m_{i_1j_1}\cdots u^m_{i_{n-2}j_{n-2}}((u^m-u)_j u^m_k (u^m-u)_k u_j)\big) d\mu \\ \mathcal{B} &:= \int_\Omega \eta\,\delta^{i,i_1,\dots,i_{n-1}}_{j,j_1,\dots,j_{n-1}} R_{ij_{n-1}i_{n-1}k}\big(u^m_{i_1j_1}\cdots u^m_{i_{n-2}j_{n-2}} - u_{i_1j_1}\cdots u_{i_{n-2}j_{n-2}}\big) u_j u_k d\mu. \end{aligned}$$

As $m\to\infty$, $\mathcal{A}$ vanishes, since by Lemma 8.2, $u^m\to u$ with respect to the $\mathcal{C}^1$ norm on U. So it remains to check that B vanishes as well. To see this set

$$w(s) := s\,u^m + (1-s)u, \qquad \overline{\eta} = \eta\delta^{i,i_{n-1}}_{j,j_{n-1}} R_{ij_{n-1}i_{n-1}k} u_j u_k.$$

Then, using the chain rule and (35), we obtain

$$\mathcal{B} = C\int_\Omega \overline{\eta}\big(\sigma_{n-2}(u^m) - \sigma_{n-2}(\nabla^2 u)\big)\, d\mu$$
$$= C\int_\Omega \overline{\eta}\left(\int_0^1 \frac{d}{ds}\sigma_{n-2}(w)\, ds\right) d\mu = C\int_0^1\int_\Omega \overline{\eta}(\mathcal{T}^w_{n-3})_{ij}(u^m-u)_{ij}\, d\mu ds$$
$$= C\int_0^1\int_\Omega \overline{\eta}\big\langle \mathcal{T}^w_{n-3}, \nabla^2(u^m-u)\big\rangle\, d\mu ds,$$

for a dimensional constant $C = C(n)$. Finally, by the analogue of (52) for $\mathcal{T}^w_{n-3}$, we have

$$\overline{\eta}\left\langle \mathcal{T}^w_{n-3}, \nabla^2(u^m-u)\right\rangle$$
$$= \operatorname{div}\big(\overline{\eta}\,\mathcal{T}^w_{n-3}(\nabla(u^m-u))\big) - \overline{\eta}\left\langle \operatorname{div}(\mathcal{T}^w_{n-3}), \nabla(u^m-u)\right\rangle - \big\langle \mathcal{T}^w_{n-3}(\nabla(u^m-u)), \nabla\overline{\eta}\big\rangle,$$

which by Lemma 8.2 vanishes as $m\to\infty$, after integration over Ω, and applying Stokes' theorem to the first term. Thus $\mathcal{B}$ vanishes as desired. □

Recall that, by Proposition 4.8, for every $\varepsilon > 0$ there exists $\lambda(\varepsilon) > 0$ such that the Greene-Wu convolution $\widehat{u}^\varepsilon_{\lambda(\varepsilon)}$ is $\mathcal{C}^\infty$. Set

$$\widehat{u}^{\,\varepsilon} := \widehat{u}^\varepsilon_{\lambda(\varepsilon)} = \overline{u}^{\,\varepsilon} \circ_{\lambda(\varepsilon)} \phi.$$

Next, for each convex hypersurface Γ with distance function u, let $\widehat{\Gamma}^\varepsilon$ be the hypersurface given by $\{\widehat{u}^{\,\varepsilon} = 0\}$. The next observation also follows from [120, Prop. 6], where it is shown that the second fundamental form of $\widehat{\Gamma}^\varepsilon$ is uniformly bounded above (although [120, Prop. 6] is stated in manifolds with strictly negative curvature, the proof works in the nonpositively curved case as well).

Lemma 8.3. *Let Γ be a $\mathcal{C}^{1,1}$ convex hypersurface in a Riemannian manifold. Then, for ε sufficiently small,* $\operatorname{reach}(\widehat{\Gamma}^\varepsilon) > \delta$ *for some $\delta > 0$, depending on* $\operatorname{reach}(\Gamma)$ *and ε.*

Proof. By Lemma 2.6, $r := \operatorname{reach}(\Gamma) > 0$. Furthermore, by Lemma 2.7, $u := \widehat{d_\Gamma}$ is $\mathcal{C}^{1,1}$ on $U_r(\Gamma)$. Thus it follows that $|\nabla^2 u|$ is uniformly bounded above on $U_r(\Gamma)$ almost everywhere, by Rademacher's theorem. This in turn yields that $|\nabla^2\widehat{u}^\varepsilon|$ is also uniformly bounded above on $U_r(\Gamma)$ for ε sufficiently small, as we will discuss below. Furthermore, since by Proposition 4.8, $\widehat{u}^\varepsilon$ converges to u with respect to the $\mathcal{C}^1$-norm on $U_r(\Gamma)$, where $|\nabla u| \equiv 1$, we may assume that $|\nabla\widehat{u}^\varepsilon| \geq 1/2$ on $U_r(\Gamma)$. Then, since for ε sufficiently small, $\widehat{\Gamma}^\varepsilon \subset U_r(\Gamma)$, it follows by (13) that the principal curvatures of $\widehat{\Gamma}^\varepsilon$ are uniformly bounded above. Consequently, for ε small, $\widehat{\Gamma}^\varepsilon$ have uniformly bounded reach by the Riemannian version of Blaschke's rolling theorem [94].

It remains only to show that $|\nabla^2\widehat{u}^\varepsilon|$ is bounded above on $U_r(\Gamma)$. The argument here follows the same general approach as in the proof of [80, Lem. 6]. See also [120, p. 630] for a more explicit formulation to control $\nabla^2\widehat{u}^\varepsilon$. To start, let $p \in U_r(\Gamma)$ be a point where $\nabla^2\overline{u}^\varepsilon$ exists, and let $c\colon(-\delta,\delta)\to M$ be a geodesic with $c(0)=p$. By (23),

$$\widehat{u}^\varepsilon\big(c(t)\big) = \frac{1}{\lambda^n}\int_{v\in T_{c(t)}M}\phi\left(\frac{|v|}{\lambda}\right)\overline{u}^\varepsilon\big(\exp_{c(t)}(v)\big)\,d\mu_{c(t)}.$$

Let $\mathcal{P}_{0,t}\colon T_pM\to T_{c(t)}M$ denote parallel translation along c. Then, as in [80, p. 280], we may rewrite the last expression as

$$\widehat{u}^\varepsilon\big(c(t)\big) = \frac{1}{\lambda^n}\int_{v\in T_pM}\phi\left(\frac{|v|}{\lambda}\right)\overline{u}^\varepsilon\big(\exp_{c(t)}(\mathcal{P}_{0,t}(v))\big)\,d\mu_p,$$

so that the integration takes place over a single tangent plane. Let $B_{\lambda(\varepsilon)}(p)$ denote the ball of radius $\lambda(\varepsilon)$ centered at p in T_pM, and recall that ϕ vanishes outside $B_{\lambda(\varepsilon)}(p)$. This, together with differentiation under the integral, yields that

$$\frac{d^2}{dt^2}\widehat{u}^\varepsilon\big(c(t)\big)\Big|_{t=0} = \frac{1}{\lambda^n}\int_{v\in B_{\lambda(\varepsilon)}(p)}\phi\left(\frac{|v|}{\lambda}\right)\frac{d^2}{dt^2}\overline{u}^\varepsilon\big(c_v(t)\big)\Big|_{t=0}\,d\mu_p, \tag{54}$$

where $c_v\colon(-\delta,\delta)\to M$ is the curve generated by parallel translation of v along c:

$$c_v(t) := \exp_{c(t)}\circ\mathcal{P}_{0,t}(v).$$

(In particular, $c_0=c$). Note that the integrand in (54) exists, because by assumption $\nabla^2\overline{u}^\varepsilon$ exists at p. Next recall that $\overline{u}^\varepsilon := u+\frac{\varepsilon}{2}\rho^2$. Thus since $|\nabla^2 u|$ is bounded above on $U_r(\Gamma)$, due to the $\mathcal{C}^{1,1}$ regularity assumption on u, it follows that $|\nabla^2\overline{u}^\varepsilon|$ is uniformly bounded above on $U_r(\Gamma)$ as well for small ε. Furthermore, note that $c_v'(0)$ and $c_v''(0)$ depend continuously on v, since the exponential map and parallel translation are $\mathcal{C}^\infty$ diffeomorphisms. Now since $v\in B_{\lambda(\varepsilon)}(p)$, which has compact closure, it follows that the integrand in (54) is bounded, and this bound may be chosen uniformly for almost every $p\in U_r(\Gamma)$. So, since $|\nabla^2\widehat{u}^\varepsilon|$ is continuous, it is bounded on $U_r(\Gamma)$ as claimed. □

Lemma 8.3 together with Theorem 8.1 now quickly yields:

Corollary 8.4. *Let Γ be a convex $\mathcal{C}^{1,1}$ hypersurface in a Cartan-Hadamard manifold M. There exists a sequence of $\mathcal{C}^\infty$ convex hypersurfaces Γ^i in M, converging to Γ with respect to $\mathcal{C}^1$ norm, such that*

$$\mathcal{G}(\Gamma^i)\to\mathcal{G}(\Gamma).$$

Proof. By Lemma 2.6, $r := \mathrm{reach}(\Gamma) > 0$. Furthermore, by Lemma 8.3, $\Gamma^i := \widehat{\Gamma}^{1/i} \in \mathrm{Reach}_C(M)$, for some $C>0$. So Γ and Γ^i belong to $\mathrm{Reach}_{C'}(M)$ for $C' := \min\{r, C\}$. Furthermore, by Proposition 4.8, $\Gamma^i\to\Gamma$ with respect to $\mathcal{C}^1$ topology, as $i\to\infty$. Thus, by Theorem 8.1, $\mathcal{G}(\Gamma^i)\to\mathcal{G}(\Gamma)$. □

Note 8.5. Corollary 8.4 shows that the total curvature inequality (2) for $\mathcal{C}^{1,1}$ convex hypersurfaces follows once it is established for $\mathcal{C}^2$ convex hypersurfaces. Furthermore, since the total curvature function is continuous on the space $K^2(M)$ of $\mathcal{C}^2$ convex hypersurfaces with its $\mathcal{C}^2$ topology, it suffices to establish the inequality for any dense subclass of $K^2(M)$. In particular, one may assume that Γ has strictly positive curvature and is even analytic. To see that these form a dense subset of $K^2(M)$, let $u := \widehat{d}_\Gamma$ be the distance function of $\Gamma \in K^2(M)$ and x_0 be any point in the interior of the convex set bounded by Γ. Then $u^\varepsilon(\cdot) := u(\cdot) + \varepsilon d^2(\cdot, x_0)$ is a convex $\mathcal{C}^2$ function with positive definite Hessian. Thus the level sets $\Gamma^\varepsilon := (u^\varepsilon)^{-1}(0)$ yield a family of positively curved hypersurfaces in $K^2(M)$ which converge to Γ with respect to the $\mathcal{C}^2$ topology as $\varepsilon \to 0$. So the space of positively curved hypersurfaces $K^2_+(M)$ is dense in $K^2(M)$. Next note $K^2(M) \subset \mathrm{Emb}^2(\mathbf{S}^{n-1}, M)$, the space of $\mathcal{C}^2$ embedded spheres in M. But real analytic submanifolds are dense in the space of $\mathcal{C}^2$ embedded submanifolds (e.g., this follows quickly from [92, Thm. 5.1, p. 65]). In particular, any hypersurface $\Gamma \in K^2_+(M)$, may be approximated by a family of real analytic hypersurfaces Γ_i which converge to it with respect to the $\mathcal{C}^2$ topology. Since Γ has positive curvature, Γ_i will eventually have positive curvature as well, and thus belong to $K^2_+(M)$. So real analytic hypersurfaces are dense in $K^2_+(M)$, which in turn is dense in $K^2(M)$.

9. Regularity and Curvature of the Convex Hull

For any convex hypersurface Γ in a Cartan-Hadamard manifold M and $\varepsilon > 0$, the outer parallel hypersurface $\Gamma^\varepsilon := (\widehat{d}_\Gamma)^{-1}(\varepsilon)$ is $\mathcal{C}^{1,1}$, by Lemma 2.6, and therefore its total curvature $\mathcal{G}(\Gamma^\varepsilon)$ is well-defined by Rademacher's theorem. We set

$$\mathcal{G}(\Gamma) := \lim_{\varepsilon \to 0} \mathcal{G}(\Gamma^\varepsilon). \tag{55}$$

Recall that $\varepsilon \mapsto \mathcal{G}(\Gamma^\varepsilon)$ is a decreasing function by Corollary 4.11. Thus, as $\mathcal{G}(\Gamma^\varepsilon) \geq 0$, it follows that $\mathcal{G}(\Gamma)$ is well-defined. The convex hull of a set $X \subset M$, denoted by $\mathrm{conv}(X)$, is the intersection of all closed convex sets in M which contain X. We set

$$X_0 := \partial \,\mathrm{conv}(X).$$

Note that if $\mathrm{conv}(X)$ has nonempty interior, then X_0 is a convex hypersurface. In this section we show that the total positive curvature of a closed embedded $\mathcal{C}^{1,1}$ hypersurface Γ in a Cartan-Hadamard manifold cannot be smaller than that of Γ_0 (Corollary 9.6), following the same general approach indicated in [101]. First we record a basic observation, which follows from Lemma 6.2 and a local characterization of convex sets in Riemannian manifolds [1, 99]:

Lemma 9.1. *Let X be a compact set in a Cartan-Hadamard manifold M, and $p \in X_0 \setminus X$. Then there exits a geodesic segment of M on X_0 which connects p to a point of X.*

Proof. Let $\widehat{\mathrm{conv}(X)} := \exp_p^{-1}(\mathrm{conv}(X))$. By Lemma 6.2, $\widehat{\mathrm{conv}(X)} \subset T_p\,\mathrm{conv}(X)$, and $T_p\,\mathrm{conv}(X)$ is a proper convex cone in T_pM. Thus there exists a hyperplane H in T_pM which passes through p and with respect to which $\widehat{\mathrm{conv}(X)}$ lies on one side. Next note that $H \cap \widehat{\mathrm{conv}(X)}$ is star-shaped about p. Indeed if $q \in H \cap \widehat{\mathrm{conv}(X)}$, then the line segment pq in H is mapped by $\exp_p$ to a geodesic segment in M which has to lie in $\mathrm{conv}(X)$, since $\mathrm{conv}(X)$ is convex. Consequently pq lies in $\widehat{\mathrm{conv}(X)}$ as desired. Now suppose, towards a contradiction, that there exists no geodesic segment in X_0 which connects p to a point of X. Then, since $H \cap \widehat{\mathrm{conv}(X)}$ is star-shaped about p, it follows that H is disjoint from $\widehat{X} := \exp_p^{-1}(X)$. So there exists a sphere $\widehat{S}$ in T_pM which passes through p and contains $\widehat{X}$ in the interior of the ball that it bounds. Let $S := \exp_p(\widehat{S})$. Then X lies in the interior of the compact region bounded by S in M. Furthermore S has positive curvature on the closure of a neighborhood U of p, since $\widehat{S}$ has positive curvature at p. Let S_ε denote the inner parallel hypersurface of S at distance ε, and U' be the image of U in S_ε. Then p will not be contained in S_ε, but we may choose $\varepsilon > 0$ so small that S_ε still contains X, U' has positive curvature, and S_ε intersects $\mathrm{conv}(X)$ only at U'. Let Y be the intersection of the compact region bounded by S_ε with $\mathrm{conv}(X)$. Then interior of Y is a locally convex set in M, as defined in [1]. Consequently Y is a convex set by a result of Karcher [99], see [1, Prop. 1]. So we have constructed a closed convex set in M which contains X but not p, which yields the desired contradiction, because $p \in \mathrm{conv}(X)$. □

Lemma 9.1 shows that the curvature of $X_0 \setminus X$ vanishes at every twice differentiable point. To further investigate the regularity of the convex hull, we may invoke the theory of semi-concave functions as follows. The last sentence in the next lemma is due to a theorem of Alexandrov [3], which states that semi-convex functions are twice differentiable almost everywhere [36, Prop. 2.3.1]. Many different proofs of this result are available, e.g. [15, 61, 70, 86]; see [132, p. 31] for a survey.

Lemma 9.2. *Let Γ be a convex hypersurface in a Riemannian manifold M. Then for each point p of Γ there exists a local coordinate chart (U, ϕ) of M around p such that $\phi(U \cap \Gamma)$ forms the graph of a semi-convex function $f\colon V \to \mathbf{R}$ for some open set $V \subset \mathbf{R}^{n-1}$. In particular Γ is twice differentiable almost everywhere.*

Proof. Let U be a small normal neighborhood of p in M, and set $\phi := \exp_p^{-1}$. By Lemma 6.2, we may identify T_pM with $\mathbf{R}^n$ such that $\phi(\Gamma \cap U)$ forms the graph of some function $f\colon V \subset \mathbf{R}^{n-1} \to \mathbf{R}$ with $f \geq 0$. We claim that f is semiconvex. Indeed, since Γ is convex, through each point $q \in \Gamma \cap U$ there passes a sphere S_q of radius r, for some fixed $r > 0$, which lies outside the domain Ω bounded by Γ. The images of small open neighborhoods of q in S_q under f yield $\mathcal{C}^2$ functions $g_q\colon V_q \to \mathbf{R}$ which support the graph of f from below in a neighborhood V_q of $x_q := f^{-1}(q) \in V$. Note that the Hessian of g_q at x_q depends continuously on q. So it follows that the second symmetric derivatives of f are uniformly bounded below, i.e.,

$$f(x+h) + f(x-h) - 2f(x) \geq C|h|^2,$$

for all x in V, where $C := \sup_q |\nabla^2 g_q(x_q)|$. Thus f is semiconvex [36, Prop. 1.1.3]. □

Lemmas 9.1 and 9.2 now indicate that the curvature of $X_0 \setminus X$ vanishes almost everywhere; however, in the absence of $\mathcal{C}^{1,1}$ regularity for X_0, this information is of little use, see [133] for a survey of curvature properties of convex hypersurfaces with low regularity. To further explore the regularity of X_0, we use the last two lemmas to obtain the following observation. We say that a set $X \subset M$ is *differentiable* at a point $p \in X$, if the tangent cone T_pX is a hyperplane in T_pM.

Lemma 9.3. *Let X be a compact set in a Riemannian manifold M. Suppose that* $\mathrm{conv}(X)$ *has nonempty interior, and X is differentiable at all point of $X \cap X_0$. Then X_0 is $\mathcal{C}^1$.*

Proof. A function f defined on an open subset of $\mathbf{R}^n$ is differentiable at a point x if and only if the tangent cone to the graph of f at $f(x)$ is a hyperplane (see the proof of [73, Lem. 3.1]). Furthermore, it is well-known that a semiconvex function is $\mathcal{C}^1$ if it is differentiable at each point [36, Prop. 3.3.4]. Thus since, by Lemma 9.2, any convex hypersurface in M may be represented locally as the graph of a semiconvex function, it suffices to show that X_0 is differentiable at each point. More specifically, we need to show that the tangent cone T_pX_0 is a hyperplane for each point $p \in X_0 \setminus X$. Suppose, towards a contradiction, that T_pX_0 is not a hyperplane for some point $p \in X_0 \setminus X$. Then since, by Lemma 6.2, T_pX_0 is a convex hypersurface in T_pM, there passes a pair of different support hyperplanes $\widehat{H}$, $\widehat{H}'$ of T_pX_0 through p in T_pM. Let H, H' be the images of these hyperplanes under $\exp_p$. Then H, H' are complete hypersurfaces in M which support X_0 at p. Furthermore, H, H' pass through a point q of X by Lemma 9.1. Note that $\widehat{H}$ and $\widehat{H}'$ are transversal along the line passing through p and $\exp^{-1}(q)$. Thus, since $\exp_p$ is a diffeomorphism, it follows that H and H' are transversal along the geodesic which passes through p and q in M. Thus T_qX cannot be a hyperplane, which contradicts the differentiability assumption on $X \cap X_0$. □

If Γ is $\mathcal{C}^1$, then its outward unit normal vector field ν is well defined. Then for every point $p \in \Gamma$ we set

$$p^\varepsilon := \exp_p\big(\varepsilon\nu(p)\big),$$

and let Γ^ε denote the outer parallel hypersurface of Γ at distance ε. In the next lemma we use a 2-jet approximation result from viscosity theory [67] together with basic comparison theory for Riemannian submanifolds [144].

Lemma 9.4. *Let Γ be a $\mathcal{C}^1$ convex hypersurface in a Cartan-Hadamard manifold, and p be a twice differentiable point of Γ. Then p^ε is a twice differentiable point of Γ^ε for all $\varepsilon \geq 0$. Furthermore, $GK_{\Gamma^\varepsilon}(p^\varepsilon)$ depends continuously on ε.*

Proof. Since p is a twice differentiable point of Γ, we may construct via normal coordinates and [67, Lem. 4.1, p. 211], a pair of $\mathcal{C}^2$ hypersurfaces $S_\pm$ in M which pass through p, lie on either side of Γ, and have the same shape operator as Γ at p,

$$\mathcal{S}_{S_+}(p) = \mathcal{S}_\Gamma(p) = \mathcal{S}_{S_-}(p). \tag{56}$$

Since $S_\pm$ are $\mathcal{C}^2$, their distance functions are $\mathcal{C}^2$ in an open neighborhood of p, by Lemma 2.5. So the outer parallel hypersurfaces $S^\varepsilon_\pm$, which are obtained by moving points of $S_\pm$ by a small distance ε along geodesics tangent to their outward normals, are $\mathcal{C}^2$; where by outward normals we mean the normals which point to the same side of $S_\pm$ as the outward normal ν of Γ points at p. Furthermore, by Riccati's equation [78, Cor. 3.3], $\mathcal{S}_{S^\varepsilon_\pm}(p^\varepsilon)$ are determined by the initial conditions $\mathcal{S}_{S_\pm}(p)$. Thus (56) implies that

$$\mathcal{S}_{S^\varepsilon_-}(p^\varepsilon) = \mathcal{S}_{S^\varepsilon_+}(p^\varepsilon).$$

This yields that p^ε is a twice differentiable point of Γ^ε for ε sufficiently small, since $S^\varepsilon_\pm$ support Γ^ε on either side of p^ε. To estimate ε independently of p, note that, since Γ is convex, the principal curvatures of $S_\pm$ at p are all nonnegative, with respect to $\nu(p)$. Thus, by replacing $S_\pm$ by smaller neighborhoods of p in $S_\pm$, we may assume that all principal curvatures of $S_\pm$, in the outward direction, are uniformly bounded below by $-\delta$ for some $\delta \geq 0$ independent of p, i.e.,

$$\mathcal{S}_{S_\pm}(q) \geq -\delta I$$

for all $q \in S_\pm$. Consequently, since $\mathcal{S}_{S_\pm}$ are solutions to Riccati's equation, it follows from standard ODE theory that the principal curvatures of $S^\varepsilon_\pm$ remain bounded for $0 \leq \varepsilon < \overline{\varepsilon}$, where $\overline{\varepsilon} > 0$ depends only on δ, which controls the initial conditions of the equation, and the curvature of M, which controls the coefficients of the equation. If $\delta = 0$, which occurs when $GK_\Gamma(p) > 0$, we may set $\overline{\varepsilon} = \infty$ by Lemmas 6.8 and 2.5. Since δ can be chosen arbitrarily small, by shrinking $S_\pm$ to p, it can be shown that we may always set $\overline{\varepsilon} = \infty$. Alternatively, this also follows from an explicit geometric estimate for the focal distance of Riemannian submanifolds. Indeed principal curvatures of $S^\varepsilon_\pm$ blow up precisely at a focal point of $S_\pm$, i.e., when outward geodesics orthogonal to $S_\pm$ reach $\mathrm{cut}(S_\pm)$. Thus finiteness of principal curvatures of $S^\varepsilon_\pm$, for $0 \leq \varepsilon < \overline{\varepsilon}$, implies that each point of $S^\varepsilon_\pm$ has a unique footprint on $S_\pm$. So $S^\varepsilon_\pm$ will be $\mathcal{C}^2$ for $0 \le \varepsilon < \overline{\varepsilon}$, by Lemma 2.5. Since δ may be chosen arbitrarily small, $\overline{\varepsilon}$ may be arbitrarily large by [144, Cor. 42(a)]. So, as discussed above, p^ε will be a twice differentiable point of Γ^ε for $\varepsilon \ge 0$. In particular $GK_{\Gamma^\varepsilon}(p^\varepsilon)$ will be well-defined for $\varepsilon \ge 0$, and since the shape operator is a solution to Riccati's equation, it follows that $\varepsilon \mapsto GK_{\Gamma^\varepsilon}(p^\varepsilon)$ is continuous for $\varepsilon \ge 0$, which completes the proof. □

The next observation is contained essentially in Kleiner's work, see [101, p. 42 (++)]. Here we employ the above lemmas to give a more detailed treatment as follows:

Proposition 9.5 ([101])**.** *Let X be a compact set in a Cartan-Hadamard manifold M. Suppose that* $\mathrm{conv}(X)$ *has nonempty interior, and there exists an open neighborhood U of X_0 in M such that $X \cap U$ is a $\mathcal{C}^{1,1}$ hypersurface. Then*

$$\mathcal{G}(X \cap X_0) = \mathcal{G}(X_0).$$

Proof. By Lemma 9.3, X_0 is $\mathcal{C}^1$. In particular its outward unit normal vector field ν is well defined, and its outer parallel hypersurfaces X_0^ε are generated by points $p^\varepsilon := \exp_p(\varepsilon\nu)$ where $p \in X_0$. For any set $A \subset X$, we define A^ε as the collection of all point p^ε with $p \in A$. Then we have

$$\mathcal{G}(X_0^\varepsilon) = \mathcal{G}\big((X_0 \setminus X)^\varepsilon\big) + \mathcal{G}\big((X_0 \cap X)^\varepsilon\big).$$

Note that as $\varepsilon \to 0$, $\mathcal{G}(X_0^\varepsilon) \to \mathcal{G}(X_0)$ by definition (55). So to complete the proof it suffices to show that

$$\mathcal{G}((X_0 \setminus X)^\varepsilon) \to 0, \qquad \text{and} \qquad \mathcal{G}\big((X_0 \cap X)^\varepsilon\big) \to \mathcal{G}(X_0 \cap X).$$

First we check that $\mathcal{G}((X_0 \setminus X)^\varepsilon) \to 0$. To this end note that if p is a twice differentiable point of $X_0 \setminus X$, which is almost every point by Lemma 9.2, then by Lemma 9.4, p^ε is a twice differentiable point of Γ^ε for all $0 \le \varepsilon \le \overline{\varepsilon}$. Furthermore, by Lemma 9.1, $GK(p) := GK_\Gamma(p) = 0$. So Lemma 9.4 yields that

$$GK(p^\varepsilon) \to 0 \tag{57}$$

for almost every $p \in X \setminus X_0$, where $GK(p^\varepsilon) := GK_{\Gamma^\varepsilon}(p^\varepsilon)$. Now, following Kleiner [101, p. 42], we set $\overline{p} := p^{\overline{\varepsilon}}$, and for all $\varepsilon \in [0, \overline{\varepsilon}]$ let

$$r^\varepsilon \colon X_0^{\overline{\varepsilon}} \to X_0^\varepsilon$$

be the projection $\overline{p} \mapsto p^\varepsilon$. In particular note that $p^\varepsilon = r^\varepsilon(\overline{p})$. Set $J(p^\varepsilon) := \mathrm{Jac}_{\overline{p}}(r^\varepsilon)$. Then, for all $\varepsilon \in [0, \overline{\varepsilon}]$, we have

$$\mathcal{G}\big((X_0 \setminus X)^\varepsilon\big) = \int_{(X_0 \cap X)^{\overline{\varepsilon}}} GK(p^\varepsilon) J(p^\varepsilon) d\sigma. \tag{58}$$

By Lemma 9.4, for almost all $\overline{p} \in (X_0 \backslash X)^{\overline{\varepsilon}}$ we may assume that p^ε is a twice differentiable point of X_0^ε for all $\varepsilon \in [0, \overline{\varepsilon}]$, and $GK(p^\varepsilon)$ is continuous on $\varepsilon \in [0, \overline{\varepsilon}]$. Next note that, for all $\varepsilon \in [0, \overline{\varepsilon}]$ and twice differentiable points $\overline{p}$,

$$J(p^\varepsilon) \leq 1, \tag{59}$$

since in a Hadamard space projection into convex sets is nonexpansive [29, Cor. 2.5]. Now by (57) and (59) we have

$$GK(p^\varepsilon)J(p^\varepsilon) \to 0,$$

for almost all $\overline{p} \in (X_0 \setminus X)^{\overline{\varepsilon}}$. Next note that, at every twice differentiable point $\overline{p}$, the second fundamental form of $X_0^{\overline{\varepsilon}}$ is bounded above, since $X_0^{\overline{\varepsilon}}$ is supported from below by balls of radius $\overline{\varepsilon}$ at each point. As discussed in [101, p. 42–43], it follows that there exists a constant C such that for all $\varepsilon \in [0, \overline{\varepsilon}]$ and twice differentiable points p^ε,

$$GK(p^\varepsilon)J(p^\varepsilon) \leq C. \tag{60}$$

Hence, by the dominated convergence theorem, the right hand side of (58) vanishes as $\varepsilon \to 0$, as desired. For the convenience of the reader, we provide an alternative self-contained proof of (60) as follows. To see this note that if p^ε is a twice differentiable point, then the Riccati equation holds on the interval $[\varepsilon, \overline{\varepsilon}]$ by Lemma 9.4. So on this interval, by [78, Thm. 3.11], we have

$$J'(\varepsilon) = (n-1)H(\varepsilon)J(\varepsilon),$$

where $J(\varepsilon) := J(p^\varepsilon)$, and $H(\varepsilon) := H_{X_0^\varepsilon}(p^\varepsilon) \geq 0$ is the mean curvature of X_0^ε at p^ε (recall that, as we pointed out in Section 3, the sign of our mean curvature is opposite to that in [78]). Furthermore, by Riccati's equation for principal curvatures of parallel hypersurfaces [78, Cor. 3.5], and Lemma 9.4, we have

$$GK'(\varepsilon) = -GK(\varepsilon)\left((n-1)H(\varepsilon) + \mathrm{Ric}(\varepsilon)\sum_{i=1}^{n-1}\frac{1}{\kappa_i(\varepsilon)}\right) \geq -(n-1)H(\varepsilon)GK(\varepsilon),$$

$GK(\varepsilon) := GK(p^\varepsilon)$ and $\mathrm{Ric}(\varepsilon)$ denotes the Ricci curvature of M at p^ε with respect to a normal to X_0^ε. So we have

$$\big(GK(\varepsilon)J(\varepsilon)\big)' \geq -(n-1)H(\varepsilon)GK(\varepsilon)J(\varepsilon) + GK(\varepsilon)(n-1)H(\varepsilon)J(\varepsilon) = 0.$$

Note that $J(\overline{\varepsilon}) = 1$, since $r^{\overline{\varepsilon}}$ is the identity map. Hence, for $\varepsilon \leq \overline{\varepsilon}$,

$$GK(\varepsilon)J(\varepsilon) \leq GK(\overline{\varepsilon})J(\overline{\varepsilon}) = GK(\overline{\varepsilon}).$$

But $GK(\overline{\varepsilon})$ is uniformly bounded above, since as we had mentioned earlier, a ball of radius $\overline{\varepsilon}$ rolls freely inside $X_0^{\overline{\varepsilon}}$. So we obtain (60), which completes the argument for $\mathcal{G}((X_0 \backslash X)^\varepsilon) \to 0$. It remains to show then that $\mathcal{G}\big((X_0 \cap X)^\varepsilon\big) \to \mathcal{G}(X_0 \cap X)$. To see this note that $GK(p^\varepsilon)J(p^\varepsilon) \to GK(p)J(p)$ by Lemma 9.4. Then the dominated convergence theorem, as we argued above, completes the proof. □

Finally we arrive at the main result of this section:

Corollary 9.6. *Let Γ be a closed $\mathcal{C}^{1,1}$ hypersurface embedded in a Cartan-Hadamard manifold. Then*

$$\mathcal{G}_+(\Gamma) \geq \mathcal{G}(\Gamma_0).$$

Proof. Note that $\mathcal{G}_+(\Gamma) \geq \mathcal{G}_+(\Gamma \cap \Gamma_0)$. Furthermore, since Γ is supported by Γ_0 from above, $GK_\Gamma(p) \geq GK_{\Gamma_0}(p) \geq 0$ for all twice differentiable points $p \in \Gamma \cap \Gamma_0$. Hence $\mathcal{G}_+(\Gamma \cap \Gamma_0) = \mathcal{G}(\Gamma \cap \Gamma_0)$. Finally, $\mathcal{G}(\Gamma \cap \Gamma_0) = \mathcal{G}(\Gamma_0)$ by Proposition 9.5, which completes the proof. □

Note 9.7. Proposition 9.5 would follow immediately from Lemma 9.1, if we could establish a refinement of Lemma 9.3 which would ensure that X_0 is $\mathcal{C}^{1,1}$, whenever X is $\mathcal{C}^{1,1}$ near X_0. Here we show that this refinement holds for closed $\mathcal{C}^{1,1}$ hypersurfaces Γ bounding a domain Ω in nonnegatively curved complete manifolds. Indeed, when Γ is $\mathcal{C}^{1,1}$, there exists $\varepsilon > 0$ such that the inner parallel hypersurface $\Gamma^{-\varepsilon}$, obtained by moving a distance ε along inward normals is embedded, by Lemma 2.6. Let $D \subset \Omega$ be the domain bounded by $\Gamma^{-\varepsilon}$. We claim that

$$(\operatorname{conv}(D))^\varepsilon = \operatorname{conv}(D^\varepsilon), \tag{61}$$

where $(\cdot)^\varepsilon$ denotes the outer parallel hypersurface. This shows that a ball (of radius ε) rolls freely inside $\operatorname{conv}(D^\varepsilon)$. Thus $(D^\varepsilon)_0$ is $\mathcal{C}^{1,1}$ by Lemma 2.6 which completes the proof, since $D^\varepsilon = \Omega$. So $(D^\varepsilon)_0 = \Omega_0 = \Gamma_0$. To prove (61) note that, since $D \subset \operatorname{conv}(D)$, we have $D^\varepsilon \subset (\operatorname{conv}(D))^\varepsilon$, which in turn yields

$$\operatorname{conv}(D^\varepsilon) \subset \operatorname{conv}((\operatorname{conv}(D))^\varepsilon) = (\operatorname{conv}(D))^\varepsilon.$$

To establish the reverse inclusion, suppose that $p \notin \operatorname{conv}(D^\varepsilon)$. Then there exists a convex set Y which contains D^ε but not p. Consequently the inner parallel hypersurface $Y^{-\varepsilon}$ contains D and is disjoint from $B^\varepsilon(p)$, the ball of radius ε centered at p. But $Y^{-\varepsilon}$ is convex, since the signed distance function is convex inside convex sets in nonnegatively curved manifolds [131, Lem. 3.3 p. 211]. So $\operatorname{conv}(D)$ is disjoint from $B^\varepsilon(p)$, which in turn yields that $p \notin (\operatorname{conv}(D))^\varepsilon$. So we have established that

$$(\operatorname{conv}(D))^\varepsilon \subset \operatorname{conv}(D^\varepsilon)$$

as desired.

10. Proof of the Total Curvature Inequality

Here we combine the results of previous sections to obtain Theorem 1.2. First note that, by Corollary 9.6 and definition (55), it suffices to establish the total curvature inequality (2) for $\mathcal{C}^{1,1}$ convex hypersurfaces Γ. To this end, by Corollary 8.4, we may further assume that Γ is $\mathcal{C}^\infty$. Then, by Proposition 3.3, we may replace Γ by a $\mathcal{C}^{1,1}$ d-convex hypersurface in $M \times \mathbf{R}$, which we will still call M. More specifically, according to Proposition 3.3, we may assume that Γ is an outer parallel hypersurface of a compact convex set $X \subset M$ without interior points. When a hypersurface Γ of M satisfies this property we say that it is *parallel-convex* or *p-convex*, and call X the *core* of Ω. Note that any p-convex hypersurface Γ in M is automatically d-convex, due to Lemma 3.1, and if X is the core of Ω, then $\mathrm{cut}(\Gamma) = X$. Thus the cut locus of a p-convex hypersurface has nice structure. Indeed any compact convex subset of M is an embedded submanifold with smooth totally geodesic relative interior by a result of Cheeger and Gromoll [41, Thm. 1.6]. To prove Theorem 1.2 it now suffices to show:

Proposition 10.1. *Let Γ be a $\mathcal{C}^{1,1}$ p-convex hypersurface in a Cartan-Hadamard manifold M. Then Γ satisfies the total curvature inequality* (2).

The outline for proving Proposition 10.1 is as follows. First, in Section 10.1, we will approximate the distance function of Γ by a family v^r of $\mathcal{C}^{1,1}$ convex functions with a single minimum point x_0 in the core of Ω, and also estimate the derivatives of v^r. Next in Section 10.2 we show that the total curvature of the zero level set of v^r converges to that of Γ. Then in Section 10.3 we apply the comparison formula to the level sets of v^r to obtain a formula for the total curvature of Γ in terms of a number of integrals. Finally, in Sections 10.4, 10.5, and 10.6 we will estimate each of these integrals, with the aid of results in previous sections, to complete the proof. We should point out that the p-convexity assumption on Γ is invoked only to obtain the estimates in Section 10.4. In all other sections it will be enough to assume that Γ is d-convex.

10.1. **Smoothing the distance function.** As outlined above, to prove Proposition 10.1, we start by approximating the distance function

$$u := \widehat{d}_\Gamma,$$

by a family v^r of $\mathcal{C}^{1,1}$ convex functions which converges to u as $r \to 0$. In particular $\Gamma_r := \{v^r = 0\}$ converges to Γ. The derivatives of v^r will satisfy the uniform bounds described in Proposition 10.3, Corollary 10.4, and Lemma 10.8 below. Furthermore, v^r will have a single minimum point x_0, where x_0 may be chosen to be any point on the core of Ω, or the minimum set of u. We will call x_0 the *center* of Ω. We will construct v^r so that it will be *radial* near x_0, i.e., it will depend only on the distance from x_0 on a neighborhood of x_0, while away from x_0, it will coincide with the inf-convolution of a perturbation of u; see Figure 6. To start, we fix $0 < \delta < 2/3$, and for $0 < r < 1$ set

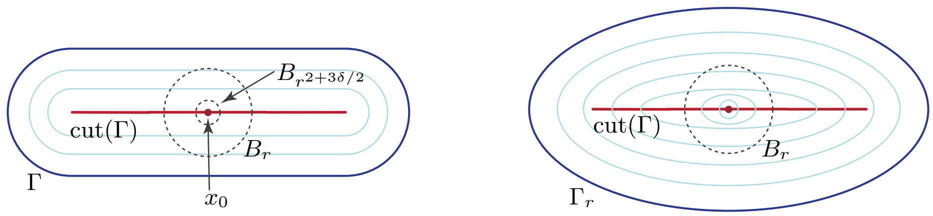

FIGURE 6.

$$t = t(r) := r^{1+\delta/2}.$$

Note that $t/r \to 0$ as $r \to 0$. Next choose a point $x_0 \in \mathrm{cut}(\Gamma)$ or a minimum point of u, and set

$$\rho(x) := d(x_0, x).$$

Recall that ρ is convex by Lemma 3.1. Now set

$$\psi(x) = \psi_r(x) := r^{1+\delta}\rho(x).$$

We perturb u to another convex function which has a unique minimum point at the center x_0 by setting

$$w = w_r := u + 3r^{1+\delta}\rho = u + 3\psi_r.$$

Next let $m := u(x_0)$ and set

$$h = h_r := \max\left\{\widetilde{w_r}^{\,t(r)},\ \psi_r + m\right\},$$

where recall that $\widetilde{f}^t$ denotes the inf-convolution of f. Note that $\widetilde{w_r}^{\,t}$ is convex by Lemma 7.1, and thus h_r is convex as well. Let

$$B_r := B_r(x_0).$$

Lemma 10.2.

(i) $h_r \equiv \psi_r + m$ *in* $B_{r^{2+3\delta/2}}$.
(ii) $h_r \equiv \widetilde{w}_r^t$ *in* $M \setminus B_r$.

Proof. Note that $w(x) - m = w(x) - u(x_0) \leq (1 + 3r^{1+\delta})\rho(x)$. Thus by (46) and (47), when $\rho(x) \leq t$,

$$\begin{aligned}
\widetilde{w}^{\,t}(x) - m &\leq \big(\widetilde{(1 + 3r^{1+\delta})\rho}\big)^{\,t}(x) \\
&= (1 + 3r^{1+\delta})\widetilde{\rho}^{\,(1+3r^{1+\delta})t}(x) \\
&\leq \frac{1 + 3r^{1+\delta}}{2(1 + 3r^{1+\delta})t}\rho^2(x) \\
&= \frac{\rho^2(x)}{2t}.
\end{aligned}$$

So it follows that, when $\rho(x) \leq r^{2+3\delta/2}$,

$$\widetilde{w}^{\,t}(x) - m \leq \frac{1}{2}\frac{\rho(x)}{t}\rho(x) \leq \frac{1}{2}r^{1+\delta}\rho(x) < \psi(x),$$

which yields (i). To obtain (ii) note that w is L-Lipschitz for $L = 1+3r^{1+\delta}$. Furthermore by definition,

$$\widetilde{w}^{\,t}(x) - m = \inf_y \left\{ u(y) + 3r^{1+\delta}\rho(y) - m + \frac{d^2(x,y)}{2t} \right\},$$

and by Lemma 7.2, the infimum is achieved for $x^* := \mathrm{prox}_t^{w-m}(x) = \mathrm{prox}_t^{w}(x)$ with $d(x, x^*) \leq tL$. Since by Lemma 2.1, ρ is 1-Lipschitz, $|\rho(x) - \rho(x^*)| \leq d(x, x^*) \leq tL$. So $\rho(x^*) \geq \rho(x) - tL$. Hence, when $\rho(x) \geq r$,

$$\widetilde{w}^t(x) - m \geq 3r^{1+\delta}\rho(x^*) \geq 3r^{1+\delta}\big(\rho(x) - tL\big) \geq 2r^{1+\delta}\rho(x) > \psi(x),$$

which completes the proof. □

Now we set

$$v = v^r := \widetilde{h_r}^{\,t(r)}. \tag{62}$$

Proposition 10.3. *As $r \to 0$, $v^r \to u$ uniformly on any given compact subset of M. Furthermore, v^r is convex, $\mathcal{C}^{1,1}$, has a unique minimum point at x_0, coincides with $\widetilde{\psi_r}^{\,t(r)}$ on $B_{r^{2+3\delta/2}}$, coincides with $\widetilde{w_r}^{\,2t(r)}$ outside B_r, and satisfies the following inequalities*

$$\begin{aligned}
|\nabla v^r| &\leq 1 + 3r^{1+\delta} && \text{on} \quad M, \\
|\nabla v^r| &\geq r^{1+\delta} && \text{on} \quad M \setminus B_{r^{2+3\delta/2}}, \\
|\nabla v^r| &= \frac{\rho}{r^{1+\delta/2}} && \text{on} \quad B_{r^{2+3\delta/2}}.
\end{aligned}$$

Proof. Convergence of v^r to u, its convexity, and that it has a unique minimum point at x_0 all follow from Lemma 7.1. The $\mathcal{C}^{1,1}$ regularity of v^r is due to Proposition 7.5. Lemma 10.2 together with (46) yields that $v^r = \widetilde{w}^{\,2t}$ outside B_r. Furthermore, via (46) and (47), Lemma 10.2 yields that

$$v^r - m = \widetilde{\psi}^t = \big(\widetilde{r^{1+\delta}\rho}\big)^t = r^{1+\delta}\widetilde{\rho}^{r^{1+\delta}t} = r^{1+\delta}\frac{\rho^2}{2r^{1+\delta}t} = \frac{\rho^2}{2t} = \frac{\rho^2}{2r^{1+\delta/2}},$$

when $\rho \leq r^{1+\delta}t = r^{2+3\delta/2}$. In particular v^r is radial on $B_{r^{2+3\delta/2}}$ and satisfies

$$\big\langle \nabla v^r, \nabla\rho\big\rangle = |\nabla v^r| = \frac{\rho}{r^{1+\delta/2}} = \frac{r^{2+3\delta/2}}{r^{1+\delta/2}} = r^{1+\delta},$$

when $\rho = r^{2+3\delta/2}$. Now note that $|\nabla v^r| \geq \langle \nabla v^r, \nabla\rho\rangle$, and since v^r is convex, $\langle \nabla v^r, \nabla\rho\rangle$ is nondecreasing. Hence it follows that $|\nabla v^r| \geq r^{1+\delta}$, if $\rho \geq r^{2+3\delta/2}$, as claimed. Next note that, since by Lemma 2.1, u and ρ are 1-Lipschitz, then

$$|\nabla w_r| = \big|\nabla u + 3r^{1+\delta}\nabla\rho\big| \leq |\nabla u| + 3r^{1+\delta}|\nabla\rho| \leq 1 + 3r^{1+\delta}, \tag{63}$$

almost everywhere. So w_r is L-Lipschitz, for $L := 1 + 3r^{1+\delta}$. Therefore $\widetilde{w_r}^t$ will also be L-Lipschitz by Lemma 7.3(iv). Consequently h_r will be L-Lipschitz, since ψ_r is $r^{1+\delta}$-Lipschitz. So, again by Lemma 7.3(iv), $v^r = \widetilde{h}^t$ is L-Lipschitz. Thus $|\nabla v^r| = |\nabla \widetilde{h}^t| \leq L$, as desired. □

Recall that, by Proposition 7.5, $|\nabla^2 v^r| \leq C/t$ almost everywhere on Ω. So

$$|\nabla^2 v^r| \leq \frac{C}{r^{1+\delta/2}} \qquad \text{a.e. on} \quad B_r. \tag{64}$$

Throughout this section C will denote a constant independent of r whose value may change from line to line. Proposition 10.3 together with (64) yields:

Corollary 10.4.

$$\frac{|\nabla^2 v^r|}{|\nabla v^r|} \leq \frac{C}{r^{2+3\delta/2}} \qquad \textit{a.e. on} \quad B_r \setminus B_{r^{2+3\delta/2}}, \tag{65}$$

$$\frac{|\nabla^2 v^r|}{|\nabla v^r|} \leq \frac{C}{\rho} \qquad \textit{a.e. on} \quad B_{r^{2+3\delta/2}}. \tag{66}$$

Recall that by (13) the principal curvatures of the level sets of v^r are given by

$$\kappa_\ell^r = \frac{v_{\ell\ell}^r}{|\nabla v^r|} \leq \frac{|\nabla^2 v^r|}{|\nabla v^r|},$$

Thus Corollary 10.4 estimates how fast these curvatures blow up near x_0 as $r \to 0$, or $v^r \to u$. A few more important estimates for derivatives of v^r will be established in Lemma 10.8 below.

10.2. **Convergence of total curvature.** Recall that $\Gamma_r := \{v^r = 0\}$ denotes the zero level set of v^r. Here we show that the total curvature of Γ_r converges to that of Γ. By Proposition 10.3, Γ_r is a $\mathcal{C}^{1,1}$ convex hypersurfaces which converges to Γ as $r \to 0$. We need to check that:

Lemma 10.5. *As $r \to 0$,*

$$\mathcal{G}(\Gamma_r) \to \mathcal{G}(\Gamma).$$

Proof. By Theorem 8.1, it suffices to check that $\mathrm{reach}(\Gamma_r)$ is bounded away from 0 for r small. To this end, as we discussed in the proof of Lemma 8.3, it is enough to check that $|\nabla^2 v^r|$ is uniformly bounded above on an open neighborhood U of Γ. By Proposition 2.7, u is $\mathcal{C}^{1,1}$ on U, assuming U is sufficiently small. So $|\nabla^2 u|$ is bounded above almost everywhere on U. It suffices then to show that, as $r \to 0$, $|\nabla^2(v^r - u)| \to 0$ uniformly almost everywhere on U. Note that

$$|\nabla^2(v^r - u)| \leq |\nabla^2(v^r - w_r)| + |\nabla^2(w_r - u)|.$$

Furthermore, $|\nabla^2(w_r - u)| \to 0$ uniformly almost everywhere on U, since $\nabla(w_r - u) = \nabla\psi_r$, which vanishes on U, as $r \to 0$, with respect to the $\mathcal{C}^2$ topology. So it remains only to show that

$$|\nabla^2(v^r - w_r)| \to 0$$

uniformly almost everywhere on U. To establish this claim, note that $v^r = \widetilde{w_r}^{2t(r)}$ on U by Proposition 10.3, assuming r is small. Next let $\mathcal{P}_{p,q}\colon T_pM \to T_qM$ denote parallel translation along the (unique) geodesic connecting points p and q of M. We will show that, if U is sufficiently small, then for almost every point $p \in U$ and unit vector $E \in T_pM$,

$$\nabla^2 v^r(p)E = \mathcal{P}_{p^*,p}\nabla^2 w_r(p^*)E^* + \mathcal{O}(r),$$

where $p^* := \mathrm{prox}_{2t}^{w_r}(p)$, $E^* := dp^*(E)$, and $|\mathcal{O}(r)| \to 0$ as $r \to 0$. Establishing the above equation will complete the proof. To this end first note that, choosing U sufficiently small, we can make sure that p^* is arbitrarily close to Γ for every $p \in U$. Thus, for almost every point $p \in U$, we may assume that p^* is a twice differentiable point of w. Now, for such a choice of p, let $\alpha_s = \alpha(s)$ be a geodesic with $\alpha(0) = p$ and $\alpha'(0) = E$. Then

$$\nabla^2 v(p)E = \lim_{s\to 0}\frac{\mathcal{P}_{\alpha_s,p}\nabla v\big(\alpha_s\big) - \nabla v\big(p\big)}{s}.$$

Let $\alpha_s^* := (\alpha_s)^*$ and note that $\alpha_0^* = p^*$. Then, by Lemma 7.3,

$$\nabla v\big(\alpha_s\big) = \mathcal{P}_{\alpha_s^*,\alpha_s}\nabla w\big(\alpha_s^*\big).$$

Thus if we set

$$\begin{aligned} A(s,t) &:= \mathcal{P}_{p^*,p}\left(\mathcal{P}_{\alpha_s^*,p^*}\nabla w(\alpha_s^*) - \nabla w\big(p^*\big)\right), \\ B(s,t) &:= \big(\mathcal{P}_{\alpha_s,p}\mathcal{P}_{\alpha_s^*,\alpha_s} - \mathcal{P}_{p^*,p}\mathcal{P}_{\alpha_s^*,p^*}\big)\nabla w\big(\alpha_s^*\big), \end{aligned}$$

then we have

$$\mathcal{P}_{\alpha_s,p}\nabla v\big(\alpha_s\big) - \nabla v\big(p\big) = A(s,t) + B(s,t).$$

Furthermore note that

$$\lim_{s\to 0}\frac{A(s,t)}{s}=\mathcal{P}_{p^*,p}\left(\lim_{s\to 0}\frac{\mathcal{P}_{\alpha_s^*,p^*}\nabla w(\alpha_s^*)-\nabla w(p^*)}{s}\right)=\mathcal{P}_{p^*,p}\nabla^2 w(p^*)E^*.$$

Thus

$$\nabla^2 v(p)E=\mathcal{P}_{p^*,p}\nabla^2 w(p^*)E^*+\frac{B(s,t)}{s}+\mathcal{O}(s).$$

So to complete the proof it now suffices to show that

$$|B(s,t)|\leq C s\, t. \tag{67}$$

To see this let $z:=\mathcal{P}_{\alpha_s,p}\mathcal{P}_{\alpha_s^*,\alpha_s}\nabla w\big(\alpha_s^*\big)=\mathcal{P}_{\alpha_s,p}\nabla v\big(\alpha_s\big)$. Then $\nabla w\big(\alpha_s^*\big)=\mathcal{P}_{\alpha_s,\alpha_s^*}\mathcal{P}_{p,\alpha_s}(z)$. So it follows that

$$|B(s,t)|=|z-\mathcal{P}_{p^*,p}\mathcal{P}_{\alpha_s^*,p^*}\mathcal{P}_{\alpha_s,\alpha_s^*}\mathcal{P}_{p,\alpha_s}(z)|=|z-\mathcal{P}_c(z)|,$$

where $\mathcal{P}_c$ indicates parallel translation around the closed curve c composed of geodesic segments $p\alpha_s$, $\alpha_s\alpha_s^*$, $\alpha_s^*p^*$, and p^*p. Thus $|B(s,t)|$ measures the holonomy of z around c, which depends continuously on z and c. But z and c, in turn, depend continuously on s and t. Hence we obtain the desired estimate (67). Indeed,

$$B(s,t)=C\cdot R\big(E,\nabla v(p)\big)z\cdot s\,t+\mathcal{O}(s\,t),$$

since the velocity at p of the segment $p\alpha_s$ of c is $\alpha'(0)=E$ by assumption, and the velocity at p of the segment pp^* is $\nabla\widetilde{w}^{\,2t}(p)=\nabla v(p)$ by Lemma 7.3. □

10.3. **Applying the comparison formula.** Here we apply the comparison formula of Theorem 4.9 to the level sets of v^r. We start by setting

$$\mathrm{cut}_r(\Gamma):=\mathrm{cut}(\Gamma)\setminus B_{r/2};$$

see Figure 7. Furthermore, let

$$\theta=\theta(r):=Ct(r)=Cr^{1+\delta/2}.$$

Note that for our purposes in this section, C may be chosen to be any positive constant

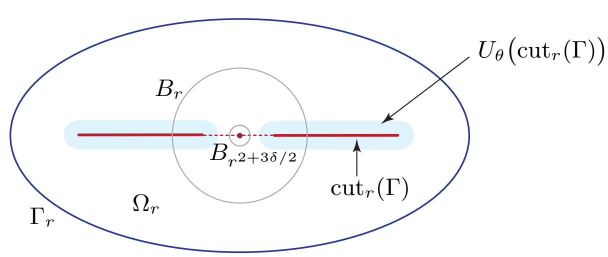

FIGURE 7.

≥ 1 (due to Proposition 7.6(ii)). Furthermore, $\theta/r\to 0$ as $r\to 0$, which is an essential feature of our argument. Let $\eta = \eta^r$ be a cutoff function for $U_\theta(\mathrm{cut}_r(\Gamma))$, as defined by (24). Next let γ be a level set of v^r in $B_{r^{2+3\delta/2}}$, D be the domain bounded by γ, and set $\Omega_r := \{v^r < 0\}$. Note that if r is sufficiently small, then $D \subset \Omega_r$, and $U_\theta(\mathrm{cut}_r(\Gamma)) \subset \Omega_r \setminus D$. Now applying Theorem 4.9 to v^r over $\Omega_r \setminus D$, with $X = \mathrm{cut}_r(\Gamma)$, yields

$$\mathcal{G}(\Gamma_r) - \mathcal{G}(\gamma) = \mathrm{I}(\Omega_r \setminus D) + \mathrm{II}(\Omega_r \setminus D) + \mathrm{III}(\Omega_r \setminus D),$$

where

$$\mathrm{I}(\,\cdot\,) := \int_{(\,\cdot\,)} \left(\eta_k \frac{GK}{\kappa_k}\frac{v_{nk}}{|\nabla v|} - \eta_n GK\right) d\mu, \qquad \mathrm{II}(\,\cdot\,) := \int_{(\,\cdot\,)} \eta R_{\ell k \ell n}\frac{GK}{\kappa_\ell \kappa_k}\frac{v_{nk}}{|\nabla v|} d\mu,$$

and

$$\mathrm{III}(\,\cdot\,) := -\int_{(\,\cdot\,)} \eta R_{\ell n \ell n}\frac{GK}{\kappa_\ell} d\mu.$$

Here $GK = GK^r$ and $\kappa_i = \kappa_i^r$ denote the Gauss-Kronecker curvature and principal curvatures of the level sets of v^r respectively (recall also our curvature notation (26), and that we assume $k \leq n-1$ in the summations above). Further note that since γ lies in $B_{r^{2+3\delta/2}}$ where v^r is radial, γ is a geodesic sphere. As is well-known, when the radius of a geodesic sphere goes to zero, its total curvature converges to $n\omega_n$ (see Lemma 10.19 for an estimate for the rate of convergence). Thus, letting γ shrink to x_0, we obtain

$$\mathcal{G}(\Gamma_r) = n\omega_n + \mathrm{I}(\Omega_r) + \mathrm{II}(\Omega_r) + \mathrm{III}(\Omega_r).$$

Letting $r \to 0$ in the above expression, and using Lemma 10.5, we find that

$$\mathcal{G}(\Gamma) = n\omega_n + \lim_{r\to 0}\mathrm{I}(\Omega_r) + \lim_{r\to 0}\mathrm{II}(\Omega_r) + \lim_{r\to 0}\mathrm{III}(\Omega_r).$$

It remains then to estimate the above limits. To this end we estimate each of the corresponding integrals on $\Omega_r \setminus B_r$ and on B_r by a number of different methods. First in Section 10.4 we will show that:

$$\mathrm{I}^0(\Omega) := \lim_{r\to 0}\mathrm{I}(\Omega_r \setminus B_r) \geq 0, \qquad \text{and} \qquad \mathrm{II}^0(\Omega) := \lim_{r\to 0}\mathrm{II}(\Omega_r \setminus B_r) = 0. \tag{68}$$

Note also that, since v^r is convex, the principal curvatures of its level sets are nonnegative. Thus

$$\mathrm{III}^0(\Omega) := \lim_{r\to 0}\mathrm{III}(\Omega_r) \geq a \lim_{r\to 0}\int_{\Omega_r} \eta^r \sigma_{n-2}(\kappa^r) d\mu \geq 0,$$

where $-a$ is the supremum of the sectional curvatures of Ω. Next in Sections 10.6 and 10.5 we will show that

$$\lim_{r\to 0}\mathrm{I}(B_r) = 0, \qquad \text{and} \qquad \lim_{r\to 0}\mathrm{II}(B_r) = 0, \tag{69}$$

respectively. Hence we obtain

$$
\begin{aligned}
\mathcal{G}(\Gamma) &= n\omega_n + \mathrm{I}^0(\Omega) + \mathrm{III}^0(\Omega) \\
&\geq n\omega_n + \mathrm{III}^0(\Omega) \\
&\geq n\omega_n + a\lim_{r\to 0}\int_\Omega \eta^r\sigma_{n-2}(\kappa^r)d\mu,
\end{aligned}
\tag{70}
$$

which completes the proof of Proposition 10.1 and consequently that of Theorem 1.2. Note that the last inequality in (70) is sharp since it holds for geodesic spheres in hyperbolic space (see Corollaries 4.11 and 4.12). Thus it remains to establish (68) and (69) to complete the proof of Proposition 10.1.

10.4. **Estimates away from the center.** Here we use the estimates for the inf-convolution and associated proximal maps developed in Section 7, together with the Reilly type formulas in Section 5, to establish the claims made in (68). To this end it suffices to show that:

Proposition 10.6. *There exists a constant $C > 0$ such that for r sufficiently small:*
(i) $\mathrm{I}(\Omega_r \setminus B_r) \geq -Cr^{\delta/2}$,
(ii) $|\mathrm{II}(\Omega_r \setminus B_r)| \leq Cr^{\delta/2}$.

To establish the above proposition recall that $\mathrm{cut}_r(\Gamma) := \mathrm{cut}_r(\Gamma) \setminus B_{r/2}$, $\theta := Ct = Cr^{1+\delta/2}$, and set

$$\mathcal{U}_r := B_r \cup U_\theta\big(\mathrm{cut}_r(\Gamma)\big),$$

see Figure 8. Note that, since the cutoff function η vanishes on $U_\theta(\mathrm{cut}_r(\Gamma))$,

$$\mathrm{I}(\Omega_r \setminus B_r) = \mathrm{I}(\Omega_r \setminus \mathcal{U}_r), \qquad \text{and} \qquad \mathrm{II}(\Omega_r \setminus B_r) = \mathrm{II}(\Omega_r \setminus \mathcal{U}_r), \tag{71}$$

by the definitions of I and II above. Hence it suffices to develop the estimates that we

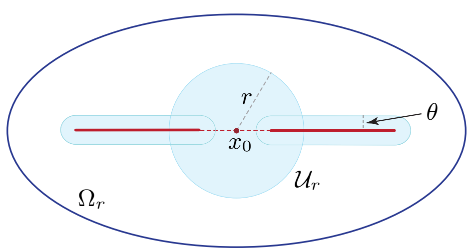

FIGURE 8.

need outside $\mathcal{U}_r$. To this end let us first record the following simple observation:

Lemma 10.7. *For r sufficiently small,*

$$\mathcal{U}_r = B_r \cup U_\theta(\mathrm{cut}(\Gamma)).$$

Proof. It is obvious that $\mathcal{U}_r \subset B_r \cup U_\theta(\mathrm{cut}(\Gamma))$. To obtain the reverse inclusion it suffices to check that if $x \notin \mathcal{U}_r$, then $x \notin U_\theta(\mathrm{cut}(\Gamma))$ or $d(x, \mathrm{cut}(\Gamma)) \geq \theta$. To this end let x° be a footprint of x on $\mathrm{cut}(\Gamma)$. If $x^\circ \in \mathrm{cut}_r(\Gamma)$, then $d(x, \mathrm{cut}(\Gamma)) = d(x, \mathrm{cut}_r(\Gamma)) \geq \theta$ and we are done. Suppose then that $x^\circ \in B_{r/2}$. Then $d(x, \mathrm{cut}(\Gamma)) \geq r/2$. Since $\theta/r \to 0$, we may choose r sufficiently small so that $r/2 \geq \theta$, which completes the proof. □

Using the last lemma together with results of Section 7 we next show:

Lemma 10.8. *Let $\theta = C_0 t = C_0 r^{1+\delta/2}$ where $C_0 \geq 6$. Then, for r sufficiently small, the following inequalities hold on $\Omega_r \setminus \mathcal{U}_r$*

(i) $|\nabla v^r| \geq 1$,
(ii) $|\nabla(v^r - u)| \leq 3r^{1+\delta}$,
(iii) $\big|\nabla_k |\nabla v^r|^2\big| \leq C r^{\delta/2}$,

where ∇_k refers to differentiation along the principal directions of the level sets of v^r.

Proof. Let $x \in \Omega_r \setminus \mathcal{U}_r$. By Lemma 10.7, we may assume that r is so small that $d(x, \mathrm{cut}(\Gamma)) \geq \theta = C_0 t$. Set

$$x^* := \mathrm{prox}_{2t}^{w_r}(x).$$

Recall that, by (63), w_r is L-Lipschitz for $L := 1 + 3r^{1+\delta}$. Thus $d(x, x^*) \leq 2tL$ by Proposition 7.2. So if $r \leq 1/6$,

$$d\big(x^*, \mathrm{cut}(\Gamma)\big) \geq d\big(x, \mathrm{cut}(\Gamma)\big) - d(x, x^*) \geq (6 - 2L)t \geq (4 - 6r^{1+\delta})t > 3t.$$

Hence x^* will be a regular point of u and so of w. Let $\alpha(s)$, $0 \leq s \leq s_0 := d(x, x^*)$, be a geodesic with $\alpha(0) = x^*$ and $\alpha(s_0) = x$. Recall that, by Proposition 10.3,

$$v = \widetilde{w}^{2t}$$

outside B_r. Thus, since x^* is a regular point of w, it follows from Lemma 7.3 that, outside B_r

$$\begin{aligned} \nabla w(x^*) &= -\frac{s_0}{2t}\nabla d_x(x^*) &= \frac{s_0}{2t}\,\alpha'(0), \\ \nabla v(x) &= \frac{s_0}{2t}\nabla d_{x^*}(x) &= \frac{s_0}{2t}\,\alpha'\big(s_0\big). \end{aligned}$$

Furthermore, since u is nondecreasing in radial directions from x_0, $\langle \nabla u, \nabla \psi \rangle \geq 0$. So

$$|\nabla v(x)|^2 = |\nabla w(x^*)|^2 = |\nabla u(x^*)|^2 + 6\langle \nabla u(x^*), \nabla \psi(x^*)\rangle + 9|\nabla \psi(x^*)|^2 \geq 1,$$

since $|\nabla u(x^*)| = 1$, which establishes (i). Next let $f(s) := \langle \nabla u(\alpha(s)), \alpha'(s)\rangle$ and note that, since u is convex,

$$f'(s) = \big\langle \nabla^2 u(\alpha'(s)), \alpha'(s)\big\rangle \geq 0.$$

Hence

$$\big\langle \nabla u(x), \nabla v(x)\big\rangle = \frac{s_0}{2t} f(s_0) \geq \frac{s_0}{2t} f(0) = \big\langle \nabla u(x^*), \nabla w(x^*)\big\rangle.$$

So it follows that

$$\begin{aligned}|\nabla v(x)-\nabla u(x)|^2 &= |\nabla w(x^*)|^2 - 2\langle \nabla v(x), \nabla u(x)\rangle + |\nabla u(x^*)|^2\\ &\leq |\nabla w(x^*)|^2 - 2\langle \nabla w(x^*), \nabla u(x^*)\rangle + |\nabla u(x^*)|^2\\ &= |\nabla w(x^*) - \nabla u(x^*)|^2\\ &= |\nabla u(x^*) + 3r^{1+\delta}\nabla\rho(x^*) - \nabla u(x^*)|^2\\ &= 9r^{2+2\delta},\end{aligned}$$

and we obtain (ii). Finally, to obtain (iii) note that we may write

$$|\nabla v(x)|^2 = g(x^*),$$

where

$$g(\cdot) := |\nabla w(\cdot)|^2 = \big|\nabla u(\cdot) + 3r^{1+\delta}\nabla\rho(\cdot)\big|^2 = 1 + 6r^{1+\delta}\langle\nabla u(\cdot), \nabla\rho(\cdot)\rangle + 9r^{2+2\delta}.$$

Let E_k be a principal direction at x for the level set $\{v = v(x)\}$, and choose a geodesic $x = x(s)$ starting at x in the direction E_k. Then, by the chain rule,

$$\begin{aligned}\big|\nabla_k|\nabla v|^2\big| &= \left|\frac{d}{ds}g\big(x^*(s)\big)\big|_{s=0}\right|\\ &= \big|dg(x^*)\, dx^*(E_k)\big|\\ &= 6r^{1+\delta}\Big|\big\langle\nabla^2 u(x^*)dx^*(E_k), \nabla\rho(x^*)\big\rangle + \big\langle\nabla u(x^*), \nabla^2\rho(x^*)dx^*(E_k)\big\rangle\Big|\\ &\leq Cr^{1+\delta}\Big(|\nabla^2 u(x^*)||\nabla\rho(x^*)| + |\nabla u(x^*)||\nabla^2\rho(x^*)|\Big)\big|dx^*(E_k)\big|\\ &\leq Cr^{1+\delta}\Big(|\nabla^2 u(x^*)| + |\nabla^2\rho(x^*)|\Big) \qquad (72)\\ &\leq Cr^{1+\delta}\left(\frac{1}{t}+\frac{1}{r}\right) \qquad (73)\\ &\leq Cr^{\delta/2},\end{aligned}$$

for r sufficiently small, where dg denotes the differential map of g, and dx^* is the differential of the proximal map $x \mapsto x^*$. The inequality (72) above uses the facts that $|\nabla\rho(x^*)| = |\nabla u(x^*)| = 1$, since ρ and u are both distance functions, and that $|dx^*(E_k)| \leq 1$, since proximal maps are nonexpansive by Lemma 7.3(i). To obtain (73) we have used the fact that

$$|\nabla^2\rho(x^*)| \leq \frac{C}{\rho(x^*)},$$

which holds since the nonzero eigenvalues of $\nabla^2\rho(x^*)$ are principal curvatures of the geodesic sphere which passes through x^* and is centered at x_0. Since $d(x, x_0) = \rho(x) > r$ and $d(x, x^*) \leq 2tL \leq 8t$, the radius of this sphere is given by

$$\rho(x^*) = d(x^*, x_0) \geq d(x, x_0) - d(x^*, x) \geq r - 8t \geq \frac{r}{2},$$

for r sufficiently small. So $|\nabla^2\rho(x^*)| \leq C/r$ as indicated in (73). Furthermore, (73) also uses the fact that u and $\widetilde{u}^t$ have parallel level sets by Proposition 7.6(i). More specifically, for $k < n$, $u_{nn} = u_{nk} = 0$ at x^* and the principal curvatures of the level sets of u, namely $u_{kk} = \kappa_k$ at x^* are related to those of $\widetilde{u}^t$ at

$$y := \mathrm{prox}_t^u(x^*)$$

by Riccati's equation for principal curvatures of parallel hypersurfaces [78, Cor. 3.5]. This yields that

$$\kappa_k(x^*) = \widetilde{\kappa}_k(y) + \mathcal{O}(t),$$

since by Lemma 7.3(iii) the geodesic connecting x^* and y is orthogonal to the level sets of u. Furthermore, Proposition 7.6(i) also ensures that $\widetilde{u}^t$ is linear along this geodesic which yields that $\widetilde{u}^t_{nn} = \widetilde{u}^t_{nk} = 0$ at y. So it follows, via Proposition 7.5, that

$$|\nabla^2 u(x^*)| = |\nabla^2 \widetilde{u}^t(y)| + \mathcal{O}(t) \leq \frac{C}{t},$$

as indicated in (73), which completes the proof. □

Using the estimates from the last lemma, we now derive a Reilly type formula relating the generalized mean curvatures of the level sets of v to the symmetric functions of $\nabla^2 v$.

Lemma 10.9. *On $\Omega_r \setminus \mathcal{U}_r$ and for $2 \leq m \leq n-1$,*

$$\sigma_m(\kappa) \leq \sigma_m(\nabla^2 v) + Cr^\delta \sigma_{m-2}(\kappa).$$

Proof. Recall that, by (13), in a principal curvature frame for level sets of v we have

$$\nabla^2 v = \begin{pmatrix} \kappa_1|\nabla v| & & 0 & v_{1n} \\ & \ddots & & \vdots \\ 0 & & \kappa_{n-1}|\nabla v| & v_{(n-1)n} \\ v_{n1} & \cdots & v_{(n-1)n} & v_{nn} \end{pmatrix}.$$

Using Lemma 4.6(iv) we obtain

$$\begin{aligned} \sum_{i=0}^{n} \sigma_i(\nabla^2 v) t^i &= \det(I^n + t\nabla^2 v) \\ &= (1 + tv_{nn}) \prod_{k<n} (1 + t\kappa_k|\nabla v|) - \sum_{k<n} (tv_{kn})^2 \prod_{\ell \neq k} (1 + t\kappa_\ell|\nabla v|) \\ &= (1 + tv_{nn}) \sum_{j=0}^{n-1} \sigma_j(\kappa|\nabla v|) t^j - \sum_{k<n} (tv_{kn})^2 \prod_{\ell \neq k} (1 + t\kappa_\ell|\nabla v|). \end{aligned}$$

Note that $v_{nn} \geq 0$ since v is convex. So differentiating both sides of the last expression m times with respect to t, evaluating the resulting expression at $t = 0$, and discarding the terms containing v_{nn} yields

$$\begin{aligned}
\sigma_m(\nabla^2 v) &\geq \sigma_m\left(\kappa|\nabla v|\right) - \frac{1}{m!}\sum_{k<n}\frac{d^m}{dt^m}\left((tv_{kn})^2\prod_{\ell\neq k}(1+t\kappa_\ell|\nabla v|)\right)\Bigg|_{t=0} \\
&= \sigma_m\left(\kappa|\nabla v|\right) - \frac{1}{m!}\sum_{k<n}\left(\frac{d^2}{dt^2}(tv_{kn})^2\frac{d^{m-2}}{dt^{m-2}}\prod_{\ell\neq k}(1+t\kappa_\ell|\nabla v|)\right)\Bigg|_{t=0} \\
&= \sigma_m\left(\kappa|\nabla v|\right) - \frac{1}{m!}\sum_{k<n}2v_{kn}^2\frac{d^{m-2}}{dt^{m-2}}\left(\prod_{\ell\neq k}(1+t\kappa_\ell|\nabla v|)\right)\Bigg|_{t=0} \\
&\geq \sigma_m\left(\kappa|\nabla v|\right) - Cr^\delta\sigma_{m-2}\left(\kappa|\nabla v|\right) \qquad (74)\\
&\geq \sigma_m(\kappa) - Cr^\delta\sigma_{m-2}(\kappa), \qquad (75)
\end{aligned}$$

by Lemma 10.8 and Proposition 10.3. More specifically, since $|\nabla v| = -v_n$, by parts (i) and (iii) of Lemma 10.8 we have

$$|v_{nk}| = \big|\nabla_k|\nabla v|\big| = \left|\frac{\nabla_k|\nabla v|^2}{2|\nabla v|}\right| \leq \big|\nabla_k|\nabla v|^2\big| \leq Cr^{\delta/2}, \tag{76}$$

which yields (74). Furthermore Lemma 10.8(i) and Proposition 10.3 provide lower and upper bounds for $|\nabla v|$ which yield (75) and complete the proof. □

Applying the last lemma recursively yields the following uniform bound for the total generalized mean curvatures of level sets of v:

Lemma 10.10. *For $2 \leq m \leq n-1$,*

$$\int_{\Omega_r\setminus\mathcal{U}_r}\sigma_m(\kappa)d\mu \leq C.$$

Proof. From Lemma 10.9 and Corollary 5.4 it follows that,

$$\int_{\Omega_r\setminus\mathcal{U}_r}\sigma_m(\kappa)d\mu \leq C + Cr^\delta\int_{\Omega_r\setminus\mathcal{U}_r}\sigma_{m-2}(\kappa)d\mu. \tag{77}$$

Note that the constant C here is indeed independent of r, since according to Corollary 5.4 it is controlled by the Lipschitz constant of v^r which is well bounded (i.e., close to 1). So, recalling that $\sigma_0 := 1$ as we had pointed out in Section 5, we obtain

$$\int_{\Omega_r\setminus\mathcal{U}_r}\sigma_2(\kappa)d\mu \leq C.$$

Next note that by Lemma 10.8(i) and since v is convex,

$$\sigma_1(\kappa) = \sum\kappa_\ell \leq \sum\big(\kappa_\ell|\nabla v|\big) + v_{nn} = \Delta v,$$

on $\Omega_r \setminus \mathcal{U}_r$. Thus by Stokes theorem, Proposition 10.3, and since Γ_r are convex hypersurfaces with diameter bounded above,

$$\int_{\Omega_r\setminus\mathcal{U}_r} \sigma_1(\kappa) d\mu \leq \int_{\Omega_r\setminus\mathcal{U}_r} \Delta v d\mu \leq \int_{\Omega_r} \Delta v d\mu = \int_{\Gamma_r} |\nabla v| d\sigma \leq 2\mathcal{H}^{n-1}(\Gamma_r) \leq C,$$

where $\mathcal{H}$ denotes the Hausdorff measure. So (77) yields that

$$\int_{\Omega_r\setminus\mathcal{U}_r} \sigma_3(\kappa) d\mu \leq C.$$

Hence, by induction, $\int_{\Omega_r\setminus\mathcal{U}_r} \sigma_m(\kappa) \leq C$, for $2 \leq m \leq n-1$. □

We need only one more observation before we can prove Proposition 10.6. Set

$$\widehat{r}(\cdot) := d(\cdot, \mathrm{cut}_r(\Gamma)).$$

Recall that by the p-convexity assumption, Γ has constant distance from $\mathrm{cut}(\Gamma)$. So if we set $\varepsilon := \mathrm{reach}(\Gamma) = d(\Gamma, \mathrm{cut}(\Gamma))$, then we have

$$d(\cdot, \mathrm{cut}(\Gamma)) = u(\cdot) + \varepsilon. \tag{78}$$

Lemma 10.11. *For r sufficiently small,*

$$\nabla \widehat{r} = \nabla u$$

on $U_{2\theta}(\mathrm{cut}_r(\Gamma)) \setminus \mathcal{U}_r$.

Proof. If $x \in U_{2\theta}(\mathrm{cut}_r(\Gamma))$, then $d(x, \mathrm{cut}_r(\Gamma)) \leq 2\theta$. On the other hand, if $x \notin \mathcal{U}_r$, then $x \notin B_r$, and so $d(x, B_{r/2}) \geq r/2$. Since $\theta/r \to 0$ as $r \to 0$, we may choose r so small that $r/2 > 2\theta$. Then any footprint of x on $\mathrm{cut}_r(\Gamma)$ lies outside $B_{r/2}$. Thus, recalling that $\mathrm{cut}_r(\Gamma) := \mathrm{cut}(\Gamma) \setminus B_{r/2}$, we obtain $\widehat{r}(\cdot) = d(\cdot, \mathrm{cut}(\Gamma))$ on $U_{2\theta}(\mathrm{cut}_r(\Gamma)) \setminus B_r$, which yields the desired equality by (78). □

Now we are ready to establish the main result of this section:

Proof of Proposition 10.6. To establish (i) first let us recall that by (71) and definition of I we have

$$\mathrm{I}(\Omega_r \setminus B_r) = \mathrm{I}(\Omega_r \setminus \mathcal{U}_r) = \int_{\Omega_r\setminus\mathcal{U}_r} \eta_k \frac{GK}{\kappa_k} \frac{v_{nk}}{|\nabla v|} d\mu - \int_{\Omega_r\setminus\mathcal{U}_r} \eta_n GK d\mu. \tag{79}$$

We estimate each of the above integrals as follows. By Corollary 6.10 and parts (i), (ii) of Lemma 10.8,

$$\begin{aligned}\left\langle \nabla\widehat{r}(x), \frac{\nabla v(x)}{|\nabla v(x)|}\right\rangle &= \left\langle \nabla\widehat{r}(x), \frac{\nabla u(x)}{|\nabla v(x)|}\right\rangle + \left\langle \nabla\widehat{r}(x), \frac{\nabla (v-u)(x)}{|\nabla v(x)|}\right\rangle \\ &\geq \left\langle \nabla\widehat{r}(x), \frac{\nabla (v-u)(x)}{|\nabla v(x)|}\right\rangle \\ &\geq -\frac{|\nabla(v-u)(x)|}{|\nabla v(x)|} \\ &\geq -Cr^{1+\delta}.\end{aligned}$$

Furthermore, recalling the definition of η (24), we see that $\eta = \phi\circ\widehat{r}$ where ϕ is a nondecreasing function with support on $[\theta, 2\theta]$. Thus

$$0 \leq \phi' \leq \frac{1}{\theta} = \frac{1}{Ct} = \frac{1}{Cr^{1+\delta/2}}. \tag{80}$$

So we have

$$-\eta_n = -\langle\nabla\eta, E_n\rangle = \phi'\left\langle \nabla\widehat{r}, \frac{\nabla v(x)}{|\nabla v(x)|}\right\rangle \geq -C\phi' r^{1+\delta} \geq -Cr^{\delta/2}.$$

Thus we obtain the desired estimate for the second integral in (79),

$$-\int_{\Omega_r\setminus\mathcal{U}_r} \eta_n GK\, d\mu \geq -Cr^{\delta/2}. \tag{81}$$

Next, to estimate the first integral in (79), note that by Lemma 10.11, we may choose r so small that

$$\eta_k = \langle\nabla\eta, E_k\rangle = \phi'\langle\nabla\widehat{r}, E_k\rangle = \phi'\langle\nabla u, E_k\rangle = \phi' u_k$$

on $U_{2\theta}(\mathrm{cut}_r(\Gamma))\setminus\mathcal{U}_r$. Furthermore, note that η_k vanishes outside $U_{2\theta}(\mathrm{cut}_r(\Gamma))$. Thus, since $v_k = 0$,

$$\begin{aligned}\int_{\Omega_r\setminus\mathcal{U}_r} \eta_k \frac{GK}{\kappa_k}\frac{v_{nk}}{|\nabla v|}d\mu &= \int_{U_{2\theta}(\mathrm{cut}_r(\Gamma))\setminus\mathcal{U}_r} \eta_k \frac{GK}{\kappa_k}\frac{v_{nk}}{|\nabla v|}d\mu \\ &= \int_{U_{2\theta}(\mathrm{cut}_r(\Gamma))\setminus\mathcal{U}_r} \phi' \frac{GK}{\kappa_k}\frac{(u_k - v_k)v_{nk}}{|\nabla v|}d\mu.\end{aligned}$$

Now using (80) together with Lemma 10.8(i), (76), and Lemma 10.10 we obtain

$$\begin{aligned}\left|\int_{\Omega_r\setminus\mathcal{U}_r} \eta_k \frac{GK}{\kappa_k}\frac{v_{nk}}{|\nabla v|}d\mu\right| &\leq \int_{\Omega_r\setminus\mathcal{U}_r} \phi'\frac{GK}{\kappa_k}\frac{|(u_k-v_k)v_{nk}|}{|\nabla v|}d\mu \\ &\leq \frac{C}{r^{1+\delta/2}}\cdot r^{1+\delta}\cdot r^{\delta/2}\int_{\Omega_r\setminus\mathcal{U}_r}\sigma_{n-2}(\kappa)d\mu \\ &\leq Cr^{\delta},\end{aligned}$$

which together with (81) completes the proof of (i). Finally, to obtain (ii) we again use Lemma 10.8(i), (76), and Lemma 10.10 to obtain

$$\begin{aligned}|\mathrm{II}(\Omega_r \setminus B_r)| = |\mathrm{II}(\Omega_r \setminus \mathcal{U}_r)| &\leq C\int_{\Omega_r\setminus\mathcal{U}_r} \frac{GK}{\kappa_\ell\kappa_k}\frac{|v_{nk}|}{|\nabla v|}d\mu \\ &\leq Cr^{\delta/2}\int_{\Omega_r\setminus\mathcal{U}_r} \sigma_{n-3}(\kappa)d\mu \\ &\leq Cr^{\delta/2},\end{aligned}$$

which completes the proof. □

Note 10.12. In the proof of Proposition 10.6 we finally made use of Theorem 6.1 via Corollary 6.10. Recall however that, by the p-convexity assumption, Γ is an outer parallel hypersurface of a convex set X without interior points. In this case $\mathrm{cut}(\Gamma) = X$, and X is precisely the minimum set of $\widehat{d}_\Gamma$. Thus it follows that Theorem 6.1, and consequently Corollary 6.10, hold automatically. In short Theorem 6.1 is not strictly necessary for proving the total curvature inequality in Theorem 1.2; however, we have kept the results of Section 6 intact since they are concerned with fundamental properties of cut locus which may be of independent interest. In particular they might be useful for studying the generalized form of the Cartan-Hadamard conjecture, where the p-convexity assumption might not be warranted.

10.5. **Estimates near the center: Part one.** Here we again use Reilly type formulas of Section 5 to show that $\mathrm{II}(B_r)$ vanishes, as claimed in (69). Set

$$A_r := B_r \setminus B_{r^{2+3\delta/2}}.$$

Note that, since $v = v^r$ is radial on $B_{r^{2+3\delta/2}}$, $v_{nk} \equiv 0$ on $B_{r^{2+3\delta/2}}$, for $k < n$. Thus

$$\mathrm{II}(B_r) = \mathrm{II}(A_r).$$

Now for any $\mathcal{C}^{1,1}$ function $f\colon A_r \to \mathbf{R}$ we set

$$\mathcal{K}_r(f) := \operatorname*{ess\,sup}_{A_r} \frac{|\nabla^2 f|}{|\nabla f|}.$$

So $\mathcal{K}_r(f)$ is an essential upper bound for principal curvatures of level sets of f by (13). In particular recall that $v_{kk}/|\nabla v| = \kappa_k$. Furthermore since v is convex, (v_{ij}) is positive semidefinite at every twice differentiable point. So $|v_{kn}| \leq \sqrt{v_{kk}v_{nn}}$. Consequently

$$\frac{|v_{kn}|}{|\nabla v|} \leq \frac{\sqrt{v_{kk}v_{nn}}}{|\nabla v|} \leq \sqrt{\kappa_k\mathcal{K}_r(v)} \quad \text{a.e. on} \quad A_r. \tag{82}$$

Recalling that $|\eta^r| \leq 1$ and using the Cauchy-Schwartz inequality, we now obtain

$$
\begin{aligned}
\big(\mathrm{II}(B_r)\big)^2 &\leq \left(\int_{A_r} \sum_{\ell\neq k} \left| R_{\ell k \ell n} \frac{GK}{\kappa_\ell \kappa_k} \frac{v_{kn}}{|\nabla v|} \right| d\mu \right)^2 \\
&\leq C\mathcal{K}_r(v) \left(\int_{A_r} \sum_{\ell\neq k} \frac{GK}{\kappa_\ell \kappa_k} \sqrt{\kappa_k} d\mu \right)^2 \\
&\leq C\mathcal{K}_r(v) \left(\sum_{\ell\neq k} \int_{A_r} \sqrt{\frac{GK}{\kappa_\ell}} \sqrt{\frac{GK}{\kappa_\ell \kappa_k}} d\mu \right)^2 \\
&\leq C\mathcal{K}_r(v) \left(\sum_{\ell\neq k} \sqrt{\int_{A_r} \frac{GK}{\kappa_\ell} d\mu \int_{A_r} \frac{GK}{\kappa_\ell \kappa_k} d\mu} \right)^2 \qquad (83) \\
&\leq C\mathcal{K}_r(v) \sum_{\ell} \int_{A_r} \frac{GK}{\kappa_\ell} d\mu \sum_{\ell\neq k} \int_{A_r} \frac{GK}{\kappa_\ell \kappa_k} d\mu \\
&= C\mathcal{K}_r(v) \int_{A_r} \sigma_{n-2}(\kappa) d\mu \int_{A_r} \sigma_{n-3}(\kappa) d\mu \\
&\leq C\mathcal{K}_r(v) \int_{B_r} \sigma_{n-2}(\kappa) d\mu \int_{B_r} \sigma_{n-3}(\kappa) d\mu
\end{aligned}
$$

As we already have an estimate for $\mathcal{K}_r(v)$ by (65), it remains only to estimate the integrals in the last line of (83).

Theorem 10.13.

$$\int_{B_r} \sigma_\ell(\kappa) d\mu \leq C^\ell r^{n-\ell},$$

for $1 \leq \ell \leq n-1$.

The above theorem together with (83) and (65) immediately yields

$$\mathrm{II}(B_r)^2 \leq C r^{-(2+3\delta/2)} r^2 r^3 = C r^{3-3\delta/2}, \qquad (84)$$

as desired.

Note 10.14. Theorem 10.13 together with (65) quickly yields the weaker estimate

$$|\mathrm{II}(B_r)| \leq C\mathcal{K}_r(v) \int_{A_r} \sigma_{n-3}(\kappa) d\mu \leq C r^{1-3\delta/2},$$

which is sufficient for our purposes here.

So it remains to prove Theorem 10.13. Recall that $\sigma_0 := 1$, as we pointed out in Section 5. Furthermore, for $\ell < 0$, we set $\sigma_\ell := 0$. In particular note that the terms involving σ_{k-3} below vanish for $k < 3$.

Lemma 10.15. *Let $\widetilde{z}$ be a $\mathcal{C}^{1,1}$ convex function on B_r with $|\nabla\widetilde{z}| \neq 0$ on A_r, and $k \geq 2$. Then at every twice differentiable point of $\widetilde{z}$ on A_r,*

$$\left|\left\langle \operatorname{div}(\mathcal{T}^{\widetilde{z}}_{k-1}), \frac{\nabla\widetilde{z}}{|\nabla\widetilde{z}|^k}\right\rangle\right| \leq C\big(\sigma_{k-2}(\widetilde{\kappa}) + \mathcal{K}_r(\widetilde{z})\sigma_{k-3}(\widetilde{\kappa})\big),$$

where $\widetilde{\kappa} = (\widetilde{\kappa}_1, \ldots, \widetilde{\kappa}_{n-1})$ refers to the principal curvatures of level sets of $\widetilde{z}$.

Proof. In the principal curvature frame, (37) and (40) yield that at every twice differentiable point of $\widetilde{z}$,

$$\begin{aligned}
\left|\left\langle \operatorname{div}(\mathcal{T}^{\widetilde{z}}_{k-1}), \frac{\nabla\widetilde{z}}{|\nabla\widetilde{z}|^k}\right\rangle\right| &= \left|\frac{1}{(k-2)!}\delta^{ii_1\cdots i_{k-1}}_{jj_1\cdots j_{k-1}}\frac{\widetilde{z}_{i_1j_1}\cdots\widetilde{z}_{i_{k-2}j_{k-2}}\widetilde{z}_\ell\widetilde{z}_j}{|\nabla\widetilde{z}|^k}R_{ij_{k-1}i_{k-1}\ell}\right| \\
&= \left|\frac{1}{(k-2)!}\delta^{ii_1\cdots i_{k-1}}_{nj_1\cdots j_{k-1}}\frac{\widetilde{z}_{i_1j_1}\cdots\widetilde{z}_{i_{k-2}j_{k-2}}|\nabla\widetilde{z}|^2}{|\nabla\widetilde{z}|^k}R_{ij_{k-1}i_{k-1}n}\right| \\
&\leq C\left|\delta^{ii_1\cdots i_{k-2}}_{nj_1\cdots j_{k-2}}\frac{\widetilde{z}_{i_1j_1}\cdots\widetilde{z}_{i_{k-2}j_{k-2}}}{|\nabla\widetilde{z}|^{k-2}}\right|,
\end{aligned} \tag{85}$$

where in the last line above we assume that the summation takes place outside of the absolute value sign (by the triangle inequality). Next recall that if $i = n$ in the last line of (85), then for $\delta^{ni_1\cdots i_{k-2}}_{nj_1\cdots j_{k-2}}$ not to vanish, $i_1, \ldots, i_{k-2}$, $j_1, \ldots, j_{k-2}$ all must be different from n, as we had mentioned in the proof of Lemma 5.2. Thus by (40),

$$\left|\delta^{ni_1\cdots i_{k-2}}_{nj_1\cdots j_{k-2}}\frac{\widetilde{z}_{i_1j_1}\cdots\widetilde{z}_{i_{k-2}j_{k-2}}}{|\nabla\widetilde{z}|^{k-2}}\right| \leq \delta^{i_1\cdots i_{k-2}}_{i_1\cdots i_{k-2}}\widetilde{\kappa}_{i_1}\ldots\widetilde{\kappa}_{i_{k-2}} = (k-2)!\sigma_{k-2}(\widetilde{\kappa}). \tag{86}$$

If on the other hand $i \neq n$, then for $\delta^{ii_1\cdots i_{k-2}}_{nj_1\cdots j_{k-1}}$ not to vanish, exactly one of the terms $i_1, \ldots i_{k-2}$, say i_m, must be equal to n. In that case we have

$$\left|\delta^{ii_1\cdots i_{k-2}}_{nj_1\cdots j_{k-2}}\frac{\widetilde{z}_{i_1j_1}\cdots\widetilde{z}_{i_{k-2}j_{k-2}}}{|\nabla\widetilde{z}|^{k-2}}\right| \leq \frac{|\widetilde{z}_{i_mn}|}{|\nabla\widetilde{z}|}\widetilde{\kappa}_{i_1}\cdots\widetilde{\kappa}_{i_{m-1}}\widetilde{\kappa}_{i_{m+1}}\ldots\widetilde{\kappa}_{i_{k-2}}$$

Furthermore, since $\widetilde{z}$ is convex, by (82) we have

$$\frac{|\widetilde{z}_{i_mn}|}{|\nabla\widetilde{z}|} \leq \sqrt{\widetilde{\kappa}_{i_m}\mathcal{K}_r(\widetilde{z})} \leq \frac{\widetilde{\kappa}_{i_m} + \mathcal{K}_r(\widetilde{z})}{2}.$$

So it follows that for $i \neq n$,

$$\left|\delta^{ii_1\cdots i_{k-2}}_{nj_1\cdots j_{k-2}}\frac{\widetilde{z}_{i_1j_1}\cdots\widetilde{z}_{i_{k-2}j_{k-2}}}{|\nabla\widetilde{z}|^{k-2}}\right| \leq C\big(\sigma_{k-2}(\widetilde{\kappa}) + \mathcal{K}_r(\widetilde{z})\sigma_{k-3}(\widetilde{\kappa})\big). \tag{87}$$

Thus (85), (86), and (87) yield the desired inequality at every twice differentiable point of A_r. ∎

Lemma 10.15, together with Lemmas 5.1 and 5.2, now quickly yields:

Lemma 10.16. *Let $\widetilde{z}$ be a $\mathcal{C}^{1,1}$ convex function on B_r with a single minimum point at x_0. Suppose that $\widetilde{z}$ is radial on $B_{r^{2+3\delta/2}}$ and satisfies (66), i.e., $|\nabla^2\widetilde{z}|/|\nabla\widetilde{z}| \leq C/\rho$ almost everywhere on $B_{r^{2+3\delta/2}}$. Then for $k \geq 2$,*

$$\begin{aligned}(88)\quad & k\int_{B_r}\sigma_k(\widetilde{\kappa})d\mu \\ & \leq \int_{B_r}\operatorname{div}\left(\mathcal{T}_{k-1}^{\widetilde{z}}\left(\frac{\nabla\widetilde{z}}{|\nabla\widetilde{z}|^k}\right)\right)d\mu + C\left(\int_{B_r}\sigma_{k-2}(\widetilde{\kappa})d\mu + \mathcal{K}_r(\widetilde{z})\int_{B_r}\sigma_{k-3}(\widetilde{\kappa})d\mu\right).\end{aligned}$$

Lemma 10.16 together with (65) and Stokes' theorem yields:

Lemma 10.17. *For $2 \leq \ell \leq n-1$,*

$$(89)\qquad \int_{B_r}\sigma_\ell(\kappa)d\mu \leq C\left(r^{n-\ell} + \int_{B_r}\sigma_{\ell-2}(\kappa)d\mu + \frac{1}{r^{2+3\delta/2}}\int_{B_r}\sigma_{\ell-3}(\kappa)d\mu\right).$$

Proof. We will glue a radial function ϕ to v near ∂B_r to obtain a convex function $\widetilde{z}$ on B_r which agrees with v on almost all of B_r, but vanishes on ∂B_r. Then we will apply Lemma 10.16 to $\widetilde{z}$, together with Stokes' theorem, to obtain (89). To construct $\widetilde{z}$ recall that x_0 is the center of B_r, $\rho(x) := d(x_0, x)$, and $m := \min_{B_r} v = v(x_0) < 0$. For $x \in B_r$ and $\lambda \in (1/2, 1)$, let

$$\phi_\lambda(x) := m + \frac{|m|}{(1-\lambda)r}\max\{\rho(x) - \lambda r, 0\},$$

and set

$$(90)\qquad s = s(\lambda) := \frac{(1-\lambda)}{2|m|}r^2.$$

Recall that $v = v^r = \widetilde{h}^t$ with $t = r^{1+\delta/2}$, and define $\overline{v} = \overline{v}^r := \widetilde{h}^{t-s}$. Note that $\overline{v}$ shares all the principal properties of v. In particular $\overline{v}$ has a single minimum point at x_0 with $\overline{v}(x_0) = m$. Now in $B_{\lambda r}$, we have $\phi_\lambda = m \leq \overline{v}$. Furthermore on ∂B_r, $\phi_\lambda = m + |m| = 0$, which yields that $\phi_\lambda > \overline{v}$ near ∂B_r. So if we set

$$z = z_\lambda := \max\{\overline{v}, \phi_\lambda\},$$

then z will be convex in B_r, radial near ∂B_r, and coincide with $\overline{v}^r$ on $B_{\lambda r}$. Consequently, the inf-convolution $\widetilde{z}^s$ will be $\mathcal{C}^{1,1}$, convex in B_r and radial near ∂B_r. Moreover, by the semigroup property of inf-convolution (46),

$$(91)\qquad \widetilde{z}^s = \widetilde{(\overline{v}^r)}^s = \widetilde{(\widetilde{h}^{t-s})}^s = \widetilde{h}^t = v^r, \qquad \text{on} \quad B_{\lambda r}.$$

We claim that, for λ close to 1,

$$(92)\qquad \mathcal{K}_r(\widetilde{z}^s) \leq \frac{C}{r^{2+3\delta/2}}.$$

To see this, first note that by (92) and (65),

$$\operatorname*{ess\,sup}_{B_{\lambda r}\setminus B_{r^{2+3\delta/2}}} \frac{|\nabla^2 \widetilde{z}^s|}{|\nabla \widetilde{z}^s|} = \operatorname*{ess\,sup}_{B_{\lambda r}\setminus B_{r^{2+3\delta/2}}} \frac{|\nabla^2 v^r|}{|\nabla v^r|} \leq \mathcal{K}_r(v^r) \leq \frac{C}{r^{2+3\delta/2}}. \tag{93}$$

So (92) holds on $B_{\lambda r}\setminus B_{r^{2+3\delta/2}}$. To establish (92) on $B_r \setminus B_{\lambda r}$, note that in this region we have

$$\phi_\lambda = m + \frac{|m|}{(1-\lambda)r}(\rho - \lambda r) = \frac{m}{1-\lambda} + \frac{|m|}{r(1-\lambda)}\rho.$$

Furthermore, by(46), (90), and (47),

$$\left(\widetilde{\frac{|m|}{r(1-\lambda)}\rho}\right)^s = \frac{|m|}{r(1-\lambda)}\widetilde{\rho}^{\frac{|m|}{r(1-\lambda)}s} = \frac{|m|}{r(1-\lambda)}\widetilde{\rho}^{\frac{r}{2}} = \frac{|m|}{r(1-\lambda)}\left(\rho - \frac{r}{4}\right).$$

Thus it follows that, on $B_r \setminus B_{\lambda r}$,

$$\widetilde{\phi}^s_\lambda = \frac{m}{1-\lambda} + \frac{|m|}{r(1-\lambda)}\left(\rho - \frac{r}{4}\right).$$

In particular $|\nabla \widetilde{z}^s| = |\nabla \widetilde{\phi}^s_\lambda| = |m|/(r(1-\lambda))$ on $B_r \setminus B_{\lambda r}$. Furthermore, recall that by Proposition 7.5, $|\nabla^2 \widetilde{z}^s| \leq C/s$. Thus, applying the mean value theorem along radial geodesics connecting $\partial B_{\lambda r}$ to ∂B_r yields that

$$|\nabla \widetilde{z}^s| \geq \langle \nabla \widetilde{z}^s, \nabla \rho\rangle \geq \frac{|m|}{r(1-\lambda)} - \frac{C}{s}(r - r\lambda) = \frac{|m|}{r(1-\lambda)} - \frac{C}{r} \geq \frac{|m|}{2r(1-\lambda)}$$

in $B_r \setminus B_{\lambda r}$ for λ sufficiently close to 1. Therefore,

$$\operatorname*{ess\,sup}_{B_r\setminus B_{\lambda r}} \frac{|\nabla^2 \widetilde{z}^s|}{|\nabla \widetilde{z}^s|} \leq C\frac{2r(1-\lambda)}{|m|s} \leq \frac{C}{r} < \frac{C}{r^{2+3\delta/2}}, \tag{94}$$

for s small compared to t, or λ sufficiently close to 1. Now (93) and (94) yield that (92) holds as claimed. So, applying Lemma 10.16 to $\widetilde{z}^s$ gives

$$\int_{B_r} \sigma_\ell(\widetilde{\kappa}^s)d\mu \leq C\left(r^{n-\ell} + \int_{B_r}\sigma_{\ell-2}(\widetilde{\kappa}^s)d\mu + \frac{1}{r^{2+3\delta/2}}\int_{B_r}\sigma_{\ell-3}(\widetilde{\kappa}^s)d\mu\right), \tag{95}$$

where $\widetilde{\kappa}^s$ refers to principal curvatures of level sets of $\widetilde{z}^s$. Note that by (91), $\widetilde{z}^s = \widetilde{z}^s_\lambda = v^r$ in $B_{\lambda r}$. Furthermore, $\int_{B_r\setminus B_{\lambda r}} \sigma_\ell(\widetilde{\kappa}^s)d\mu$ vanishes as $\lambda \to 1$ since due to (13), the principal curvatures of $\widetilde{z}^s$ are essentially bounded on $B_r \setminus B_{\lambda r}$ by (94). Thus letting $\lambda \to 1$ in (95), or $s \to 0$, we obtain (89). □

Now we will inductively apply Lemma 10.17 to establish Theorem 10.13. First note that by Stokes' theorem

$$\int_{B_r} \sigma_1(\kappa)d\mu = \int_{B_r} \operatorname{div}\left(\frac{\nabla v^r}{|\nabla v^r|}\right)d\mu \leq \mathcal{H}^{n-1}(\partial B_r) \leq Cr^{n-1},$$

where recall that $\mathcal{H}$ denotes the Hausdorff measure. So Theorem 10.13 holds for $\ell = 1$. Next suppose that there exists a constant C such that

$$\int_{B_r} \sigma_\ell(\kappa) d\mu \leq C^\ell r^{n-\ell}, \tag{96}$$

for $\ell \leq k-1$. Then by Lemma 10.17, and since $r < 1$, and $\delta < 2/3$ by assumption, we have

$$\begin{aligned}\int_{B_r} \sigma_k(\kappa) d\mu &\leq C\left(r^{n-k} + C^{k-2} r^{n+2-k} + C r^{-(2+\frac{3}{2}\delta)} C^{k-3} r^{n+3-k}\right) \\ &= C r^{n-k}\left(1 + C^{k-2} r^2 + C^{k-2} r^{1-\frac{3}{2}\delta}\right) \\ &\leq 3C^{k-1} r^{n-k} \\ &\leq C^k r^{n-k},\end{aligned}$$

for $C \geq 3$. So (96) holds for $\ell \leq k$, which completes the proof of Theorem 10.13.

10.6. **Estimates near the center: Part two.** In this section we show that $\mathrm{I}(B_r)$ vanishes, as claimed in (69). Set

$$\widehat{\mathrm{cut}(\Gamma)} := \exp_{x_0}^{-1}(\mathrm{cut}(\Gamma)),$$

and let $r\mathbf{S}^{n-1}$ denote the sphere of radius r centered at x_0 in $T_{x_0}M \simeq \mathbf{R}^n$. Also let B^n denote the unit ball in $T_{x_0}M$ centered at x_0. Recall that $\mathcal{H}$ stands for Hausdorff measure.

Lemma 10.18.

$$\mathcal{H}^{n-2}\left(r\mathbf{S}^{n-1} \cap \widehat{\mathrm{cut}(\Gamma)}\right) \leq C r^{n-2}.$$

Proof. Recall that, due to the p-convexity assumption on Γ, $\mathrm{cut}(\Gamma)$ is convex. In particular the relative interior of $\mathrm{cut}(\Gamma)$ is a totally geodesic proper submanifold of M [41, Thm. 1.6]. Thus it follows that $\widehat{\mathrm{cut}(\Gamma)}$ lies in a hyperplane of T_pM. So $r\mathbf{S}^{n-1} \cap \widehat{\mathrm{cut}(\Gamma)}$ lies in a great sphere of $r\mathbf{S}^{n-1}$, which yields the desired inequality. □

Note that the proof of the above lemma marks only the second instance where we invoke the p-convexity assumption on Γ (the first instance was in Section 10.4); however, the above lemma holds for cut loci of more general hypersurfaces, as we describe in Note 10.22 below.

Lemma 10.19.

$$0 \leq \mathcal{G}(\partial B_r) - n\omega_n \leq C r^2.$$

Proof. By Corollary 4.11, $\mathcal{G}(\partial B_r) \geq n\omega_n$. Since ∂B_r is a geodesic sphere, a power series expansion of its second fundamental form in normal coordinates, see [42, Thm. 3.1], shows that

$$0 \le GK \le \frac{1}{r^{n-1}}(1 + Cr^2), \tag{97}$$

where GK denotes the Gauss-Kronecker curvature of ∂B_r. Furthermore another power series expansion [77, Thm. 3.1] shows that

$$\left|\mathrm{vol}(\partial B_r) - n\omega_n r^{n-1}\right| \le Cr^{n+1}.$$

Using these inequalities we obtain

$$\begin{aligned} 0 \le \mathcal{G}(\partial B_r) - n\omega_n &\le \frac{1}{r^{n-1}}(1 + Cr^2)\mathrm{vol}(\partial B_r) - n\omega_n \\ &\le r^{1-n}(1 + Cr^2) \cdot n\omega_n r^{n-1}(1 + Cr^2) - n\omega_n \\ &\le n\omega_n(1 + Cr^2 - 1) \\ &\le Cr^2, \end{aligned}$$

as desired. □

Recall that $\mathcal{G}_{\eta^r}(\partial B_r) := \int_{\partial B_r} \eta^r \, GK \, d\sigma$. Using the last lemma, we next show:

Lemma 10.20.

$$\left|\mathcal{G}_{\eta^r}(\partial B_r) - n\omega_n\right| \le C\frac{\theta}{r} = Cr^{\delta/2}.$$

Proof. By the triangle inequality and Lemma 10.19,

$$\begin{aligned} |\mathcal{G}_{\eta^r}(\partial B_r) - n\omega_n| &\le \left|\int_{\partial B_r} GK d\sigma - n\omega_n\right| + \left|\int_{\partial B_r} (1 - \eta^r) GK d\sigma\right| \\ &\le Cr^2 + \int_{\partial B_r \cap U_{2\theta}(\mathrm{cut}(\Gamma))} GK d\sigma. \end{aligned} \tag{98}$$

So it remains to estimate the integral in the last line of (98). To this end we may assume by (97) that r is so small that the Gauss-Kronecker curvature of ∂B_r is less than $2/r^{n-1}$. Then

$$\int_{\partial B_r \cap U_{2\theta}(\mathrm{cut}(\Gamma))} GK d\sigma \le \frac{2}{r^{n-1}} \mathcal{H}^{n-1}\Big(\partial B_r \cap U_{2\theta}\big(\mathrm{cut}(\Gamma)\big)\Big).$$

Next, to estimate $\mathcal{H}^{n-1}(\partial B_r \cap U_{2\theta}(\mathrm{cut}(\Gamma)))$, note that $\exp_{x_0}^{-1}(\partial B_r) = r\mathbf{S}^{n-1}$. Thus,

$$\begin{aligned} \mathcal{H}^{n-1}\Big(\partial B_r \cap U_{2\theta}\big(\mathrm{cut}(\Gamma)\big)\Big) &\le 2\mathcal{H}^{n-1}\left(r\mathbf{S}^{n-1} \cap U_{2\theta}\left(\widehat{\mathrm{cut}(\Gamma)}\right)\right) \\ &= 2r^{n-1}\mathcal{H}^{n-1}\left(\mathbf{S}^{n-1} \cap \frac{1}{r}U_{2\theta}\left(\widehat{\mathrm{cut}(\Gamma)}\right)\right), \end{aligned}$$

where $U_{2\theta}(\widehat{\mathrm{cut}(\Gamma)})$ denotes the tubular neighborhood of radius 2θ about $\widehat{\mathrm{cut}(\Gamma)}$ in T_pM. Furthermore, for r small

$$\begin{aligned}\mathcal{H}^{n-1}\left(\mathbf{S}^{n-1}\cap\frac{1}{r}U_{2\theta}\left(\widehat{\mathrm{cut}(\Gamma)}\right)\right) &= \mathcal{H}^{n-1}\left(\mathbf{S}^{n-1}\cap U_{\frac{2\theta}{r}}\left(\frac{\widehat{\mathrm{cut}(\Gamma)}}{r}\right)\right)\\ &\leq \frac{4\theta}{r}\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1}\cap\frac{\widehat{\mathrm{cut}(\Gamma)}}{r}\right).\end{aligned}$$

Finally note that, by Lemma 10.18,

$$\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1}\cap\frac{\widehat{\mathrm{cut}(\Gamma)}}{r}\right)=\frac{1}{r^{n-2}}\mathcal{H}^{n-2}\left(r\mathbf{S}^{n-1}\cap\widehat{\mathrm{cut}(\Gamma)}\right)\leq C.$$

The last four displayed expressions yield that, for small r,

$$\int_{\partial B_r\cap U_{2\theta}(\mathrm{cut}(\Gamma))}GK\,d\sigma\leq\frac{2}{r^{n-1}}\cdot 2r^{n-1}\cdot\frac{4\theta}{r}\cdot C=16C\frac{\theta}{r}=16Cr^{\delta/2},$$

which together with (98) completes the argument. □

We are now ready to establish the main result of this section:

Proposition 10.21. $|\mathrm{I}(B_r)|\leq Cr^{\delta/2}$.

Proof. Let $\widetilde{z}^s$ be as in the proof of Lemma 10.17. Applying Theorem 4.9 to $\widetilde{z}^s$ in B_r gives

$$\mathcal{G}_{\eta^r}(\partial B_r)-n\omega_n=\widetilde{\mathrm{I}}^s(B_r)+\widetilde{\mathrm{II}}^s(B_r)+\widetilde{\mathrm{III}}^s(B_r),$$

where

$$\widetilde{\mathrm{I}}^s(B_r):=\int_{B_r}\left(\frac{\widetilde{z}^s_{nk}\eta^r_k}{|\nabla\widetilde{z}^s|}\frac{\widetilde{GK}^s}{\widetilde{\kappa}^s_k}-\eta^r_n\widetilde{GK}^s\right)d\mu,\qquad \widetilde{\mathrm{II}}^s(B_r):=\int_{B_r}\eta^rR_{\ell k\ell n}\frac{\widetilde{GK}^s}{\widetilde{\kappa}^s_\ell\widetilde{\kappa}^s_k}\frac{(\widetilde{z}^s)_{kn}}{|\nabla\widetilde{z}^s|}d\mu,$$

and

$$\widetilde{\mathrm{III}}^s(B_r):=-\int_{B_r}\eta^rR_{\ell n\ell n}\frac{\widetilde{GK}^s}{\widetilde{\kappa}^s_\ell}d\mu.$$

Here $\widetilde{GK}^s$ and $\widetilde{\kappa}^s_k$ denote the Gauss-Kronecker curvature and principal curvatures of the level sets of $\widetilde{z}^s$ respectively. Virtually the same proof used in establishing (84) shows that

$$|\widetilde{\mathrm{II}}^s(B_r)|\leq Cr^{3/2-3\delta/4}\leq Cr^{\delta/2}.$$

Furthermore, by Theorem 10.13 and since $|\eta^r|\leq 1$,

$$|\widetilde{\mathrm{III}}^s(B_r)|\leq\int_{B_r}\frac{\widetilde{GK}^s}{\widetilde{\kappa}^s_\ell}d\mu=\int_{B_r}\sigma_{n-2}(\widetilde{\kappa}^s)d\mu\leq Cr^2\leq Cr^{\delta/2}.$$

These estimates together with Lemma 10.20 yield

$$|\widetilde{\mathrm{I}}^s(B_r)|\leq|\mathcal{G}_{\eta^r}(\partial B_r)-n\omega_n|+\left|\widetilde{\mathrm{II}}^s(B_r)\right|+\left|\widetilde{\mathrm{III}}^s(B_r)\right|\leq Cr^{\delta/2}.$$

By (91), $\widetilde{z}^s = v^r$ on $B_{r'}$ for any $r' < r$, assuming s is sufficiently small. Thus

$$|\mathrm{I}(B_{r'})| \leq Cr^{\delta/2}.$$

Finally, since $\mathrm{I}(B_r)$ depends continuously on r, we obtain

$$|\mathrm{I}(B_r)| = \lim_{r' \to r} |\mathrm{I}(B_{r'})| \leq Cr^{\delta/2},$$

as desired. □

Note 10.22. With an eye towards possible applications to the generalized form of the Cartan-Hadamard conjecture, here we describe how Lemma 10.18 may be established for a broader class of convex hypersurfaces. Indeed it is enough to assume that $\mathcal{H}^{n-1}(\mathrm{cut}(\Gamma)) < \infty$, and $\mathrm{cut}(\Gamma)$ is $\mathcal{C}^1$-rectifiable near x_0. These assumptions mean that there are a finite number of $\mathcal{C}^1$ embeddings $\phi_i \colon \Delta_i \to M$ which cover a neighborhood of x_0 in $\mathrm{cut}(\Gamma)$, where Δ_i are simplices of dimension at most $n-1$. These conditions hold, for instance, when Γ is analytic. Indeed in this case $\mathrm{cut}(\Gamma)$ is subanalytic [31, 40], which ensures that $\mathrm{cut}(\Gamma)$ admits a $\mathcal{C}^1$ triangulation [52, 118]. To prove Lemma 10.18 in this setting we proceed as follows. Note that $\mathcal{H}^{n-2}\left(r\mathbf{S}^{n-1} \cap \widehat{\mathrm{cut}(\Gamma)}\right)/r^{n-2} = \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \mathrm{cut}(\Gamma)/r\right)$. Thus it suffices to show that

$$\lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \frac{\widehat{\mathrm{cut}(\Gamma)}}{r}\right) < \infty.$$

To this end we employ the basic property of tangent cones that, as discussed in Section 6, within any bounded subset of $T_{x_0}M$, $\widehat{\mathrm{cut}(\Gamma)}/r \to T_{x_0}\widehat{\mathrm{cut}(\Gamma)}$, with respect to Hausdorff distance (as $r \to 0$) . We will show that

$$\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap T_{x_0}\widehat{\mathrm{cut}(\Gamma)}\right) < \infty, \tag{99}$$

and

$$\lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \frac{\widehat{\mathrm{cut}(\Gamma)}}{r}\right) \leq \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap T_{x_0}\widehat{\mathrm{cut}(\Gamma)}\right), \tag{100}$$

which will complete the proof. Set $\overline{\Delta}_i := \phi_i(\Delta_i)$. Note that $T_{x_0}\widehat{\mathrm{cut}(\Gamma)} = \cup_i T_{x_0}\overline{\Delta}_i$ and, assuming r is sufficiently small, $\mathbf{S}^{n-1} \cap \mathrm{cut}(\Gamma)/r = \cup_i \left(\mathbf{S}^{n-1} \cap \overline{\Delta}_i/r\right)$. Thus to establish (99) and (100) it suffices to check that

$$\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap T_{x_0}\overline{\Delta}_i\right) < \infty, \tag{101}$$

and

$$\lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \frac{\overline{\Delta}_i}{r}\right) \leq \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap T_{x_0}\overline{\Delta}_i\right), \tag{102}$$

respectively. To establish (101) note that $T_{x_0}\overline{\Delta}_i = d_{x_0}\phi_i\left(T_{\phi_i^{-1}(x_0)}\Delta_i\right)$, where d is the differential map. Furthermore, $T_{\phi_i^{-1}(x_0)}\Delta_i$ is a convex cone of dimension at most $n-1$ since $\dim(\Delta_i) \leq n-1$. Therefore $T_{x_0}\overline{\Delta}_i$ will be a convex cone of dimension at most $n-1$ as well. Hence $\mathbf{S}^{n-1} \cap T_{x_0}\overline{\Delta}_i$ will be a convex set of dimension at most $n-2$ in $\mathbf{S}^{n-1}$. In particular it lies in a proper great subsphere of $\mathbf{S}^{n-1}$. So $\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap T_{x_0}\overline{\Delta}_i\right) \leq \mathrm{vol}(\mathbf{S}^{n-2})$ which yields (101) as desired.

It remains to establish (102). To this end we utilize the natural stratifications near x_0 that $\mathbf{S}^{n-1} \cap \overline{\Delta}_i/r$ and $\mathbf{S}^{n-1} \cap T_{x_0}\overline{\Delta}_i$ inherit from Δ_i. Let F_{ij} denote the facets of Δ_i which are adjacent to $\phi_i^{-1}(x_0)$. Note that, since it is $\mathcal{C}^1$ and injective, ϕ_i has full rank in the interior of each F_{ij}. Thus F_{ij} will have the same dimension as $\overline{F}_{ij} := \phi_i(F_{ij})$, and so $\overline{\Delta}_i$ inherits a stratification by $\mathcal{C}^1$ manifolds near x_0 which mirrors that of Δ_i near $\phi_i^{-1}(x_0)$. On the other hand,

$$T_{x_0}\overline{\Delta}_i = \bigcup_j T_{x_0}\overline{F}_{ij}.$$

Thus we obtain a stratification of $T_{x_0}\overline{\Delta}_i$ as well, by $T_{x_0}\overline{F}_{ij}$. Note that the dimension of $T_{x_0}\overline{F}_{ij}$ may be lower than that of $\overline{F}_{ij}$, because the rank of $d_{x_0}\phi_i$ may be lower than $n-1$; however, for every $\mathcal{C}^1$ curve $c\colon [0,\varepsilon) \to \overline{F}_{ij}$ with $c(0) = x_0$, c/r converges, with respect to the $\mathcal{C}^1$-norm, to the ray of $T_{x_0}\overline{F}_{ij}$ generated by the left derivative $c'_{-}(0)$. It follows then that, for r sufficiently small, $\overline{F}_{ij}/r$ will be transversal to $\mathbf{S}^{n-1}$. So $\mathbf{S}^{n-1} \cap \overline{\Delta}_i/r$ will be stratified by $\mathcal{C}^1$ manifolds $\mathbf{S}^{n-1} \cap \overline{F}_{ij}/r$. To establish (102) it now suffices to check that

$$\lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \frac{\overline{F}_{ij}}{r}\right) \leq \mathcal{H}^{n-2}\big(\mathbf{S}^{n-1} \cap T_{x_0}\overline{F}_{ij}\big). \tag{103}$$

To establish (103) note that we may assume $\overline{F}_{ij}$ has the maximal dimension $n-1$, for otherwise both sides of (103) would vanish by transversality. So $\overline{F}_{ij}$ has a well defined field of tangent hyperplanes (of dimension $n-1$) in its interior, which may be extended continuously to the boundary of $\overline{F}_{ij}$, since ϕ_i is $\mathcal{C}^1$ up to the boundary of F_{ij} by assumption. In particular $\overline{F}_{ij}$ has a well-defined tangent hyperplane H at x_0. Note that $T_{x_0}\overline{F}_{ij} \subset H$, and let $\widetilde{F}_{ij}$ denote the projection of $\overline{F}_{ij}$ into H. Since $\overline{F}_{ij}$ and $\widetilde{F}_{ij}$ are tangent at x_0, then, as $r \to 0$, $B^n \cap \overline{F}_{ij}/r$ and $B^n \cap \widetilde{F}_{ij}/r$ become arbitrarily $\mathcal{C}^1$-close. Thus it follows, due to transversality of $\overline{F}_{ij}/r$ and $\widetilde{F}_{ij}/r$ with $\mathbf{S}^{n-1}$, that $\mathbf{S}^{n-1} \cap \overline{F}_{ij}/r$ and $\mathbf{S}^{n-1} \cap \widetilde{F}_{ij}/r$ become arbitrarily $\mathcal{C}^1$-close. So

$$\lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \frac{\overline{F}_{ij}}{r}\right) = \lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1} \cap \frac{\widetilde{F}_{ij}}{r}\right).$$

Consequently, to establish (103) it suffices to check that

$$\lim_{r\to 0} \mathcal{H}^{n-2}\left(\mathbf{S}^{n-1}\cap\frac{\widetilde{F}_{ij}}{r}\right)\le \mathcal{H}^{n-2}(\mathbf{S}^{n-1}\cap T_{x_0}\overline{F}_{ij}). \tag{104}$$

To see this note that $\mathbf{S}^{n-1}\cap\widetilde{F}_{ij}/r$ lies in the $n-2$ dimensional sphere $H\cap\mathbf{S}^{n-1}$ which we may identify with $\mathbf{S}^{n-2}$. Thus

$$\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1}\cap\frac{\widetilde{F}_{ij}}{r}\right)=\mathcal{H}^{n-2}\left(\mathbf{S}^{n-2}\cap\frac{\widetilde{F}_{ij}}{r}\right)=\mathcal{L}\left(\mathbf{S}^{n-2}\cap\frac{\widetilde{F}_{ij}}{r}\right), \tag{105}$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbf{S}^{n-2}$. Furthermore $B^n\cap\widetilde{F}_{ij}/r\to B^n\cap T_{x_0}\overline{F}_{ij}$ with respect to Hausdorff distance, since $\widetilde{F}_{ij}/r$ is simply the projection of $\overline{F}_{ij}/r$ into H, and $B^n\cap\overline{F}_{ij}/r\to B^n\cap T_{x_0}\overline{F}_{ij}$. Thus

$$\mathbf{S}^{n-2}\cap\frac{\widetilde{F}_{ij}}{r}\to\mathbf{S}^{n-2}\cap T_{x_0}\overline{F}_{ij}$$

with respect to Hausdorff distance. But Lebesgue measure is upper semi-continuous with respect to Hausdorff distance. Thus, by (105),

$$\begin{aligned}
\lim_{r\to 0}\mathcal{H}^{n-2}\left(\mathbf{S}^{n-1}\cap\frac{\widetilde{F}_{ij}}{r}\right) &= \lim_{r\to 0}\mathcal{L}\left(\mathbf{S}^{n-2}\cap\frac{\widetilde{F}_{ij}}{r}\right)\\
&\le \mathcal{L}(\mathbf{S}^{n-2}\cap T_{x_0}\overline{F}_{ij})\\
&= \mathcal{H}^{n-2}(\mathbf{S}^{n-2}\cap T_{x_0}\overline{F}_{ij})\\
&= \mathcal{H}^{n-2}(\mathbf{S}^{n-1}\cap T_{x_0}\overline{F}_{ij}).
\end{aligned}$$

which establishes (104) and completes the proof.

11. Proof of the Isoperimetric Inequality

In this section we will use the total curvature inequality, established in Theorem 1.2, to obtain the isoperimetric inequality and prove Theorem 1.1, via the well-known isoperimetric profile argument [128, 130] along the same general lines indicated by Kleiner [101]. The *isoperimetric profile* [19, 20] of any open subset U of a Riemannian manifold M is the function $\mathcal{I}_U\colon [0,\mathrm{vol}(U))\to\mathbf{R}$ given by

$$\mathcal{I}_U(v):=\inf\big\{\mathrm{per}(\Omega)\mid \Omega\subset U,\ \mathrm{vol}(\Omega)=v,\ \mathrm{diam}(\Omega)<\infty\big\},$$

where *diam* is the diameter, *vol* denotes the Lebesgue measure, and *per* stands for perimeter; see [39, 75] for the general definition of perimeter (when $\partial\Omega$ is piecewise $\mathcal{C}^1$, for instance, $\mathrm{per}(\Omega)$ is just the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$). Proving the isoperimetric inequality in Theorem 1.1 is equivalent to showing that

$$\mathcal{I}_M\ge\mathcal{I}_{\mathbf{R}^n},$$

for any Cartan-Hadamard manifold M. To this end it suffices to show that $\mathcal{I}_B \geq \mathcal{I}_{\mathbf{R}^n}$ for a family of (open) geodesic balls $B \subset M$ whose radii grows arbitrarily large and eventually covers any bounded set $\Omega \subset M$. So we fix a geodesic ball B in M and consider its *isoperimetric regions*, i.e., sets $\Omega \subset B$ which have the least perimeter for a given volume, or satisfy $\mathrm{per}(\Omega) = \mathcal{I}_B(\mathrm{vol}(\Omega))$. The existence of these regions are well-known, and they have the following regularity properties:

Lemma 11.1 ([76, 139])**.** *For any $v \in (0, \mathrm{vol}(B))$ there exists an isoperimetric region $\Omega \subset B$ with $\mathrm{vol}(\Omega) = v$. Let $\Gamma := \partial\Omega$, H be the normalized mean curvature of Γ (wherever it is defined), and $\Gamma_0 := \partial\,\mathrm{conv}(\Gamma)$. Then*

- *(i) $\Gamma \cap B$ is $\mathcal{C}^\infty$ except for a closed set $\mathrm{sing}(\Gamma)$ of Hausdorff dimension at most $n-8$. Furthermore, $H \equiv H_0 = H_0(v)$ a constant on $\Gamma \cap B \setminus \mathrm{sing}(\Gamma)$.*
- *(ii) Γ is $\mathcal{C}^{1,1}$ within an open neighborhood U of ∂B in M. Furthermore, $H \leq H_0$ almost everywhere on $U \cap \Gamma$.*
- *(iii) $d\big(\mathrm{sing}(\Gamma), \Gamma_0\big) \geq \varepsilon_0 > 0$.*

In particular Γ is $\mathcal{C}^{1,1}$ within an open neighborhood of Γ_0 in M.

Proof. Part (i) follows from Gonzalez, Massari, and Tamanini [76], and (ii) follows from Stredulinsky and Ziemer [139, Thm 3.6], who studied the identical variational problem in $\mathbf{R}^n$. Indeed the $\mathcal{C}^{1,1}$ regularity near ∂B is based on the classical obstacle problem for graphs which extends in a straightforward way to Riemannian manifolds; see also Morgan [114]. To see (iii) note that by (i), $\mathrm{sing}(\Gamma)$ is closed, and by (ii), $\mathrm{sing}(\Gamma)$ lies in B. So it suffices to check that points $p \in \Gamma \cap \Gamma_0 \cap B$ are not singular. This is the case since $T_p\Gamma \subset T_p\,\mathrm{conv}(\Gamma)$ which is a convex subset of T_pM. Therefore $T_p\Gamma$ is contained in a half-space of T_pM generated by any support hyperplane of $T_p\,\mathrm{conv}(\Gamma)$ at p. This forces $T_p\Gamma$ to be a hyperplane [137, Cor. 37.6]. Consequently Γ will be $\mathcal{C}^\infty$ in a neighborhood of p [65, Thm. 5.4.6], [114, Prop. 3.5]. □

Now let $\Omega \subset B$ be an isoperimetric region with volume v, as provided by Lemma 11.1. By Proposition 9.5, $\mathcal{G}(\Gamma_0) = \mathcal{G}(\Gamma \cap \Gamma_0)$. Furthermore recall that, by Theorem 1.2 and definition (55), $\mathcal{G}(\Gamma_0) \geq n\omega_n$. Thus we have

$$n\omega_n \leq \mathcal{G}(\Gamma_0) = \mathcal{G}(\Gamma \cap \Gamma_0) = \int_{\Gamma\cap\Gamma_0} GK d\sigma, \tag{106}$$

where GK denotes the Gauss-Kronecker curvature of Γ. Note that $GK \geq 0$ on $\Gamma \cap \Gamma_0$, since at these points Γ is locally convex. So the arithmetic versus geometric means inequality yields that $GK \leq H^{n-1}$ on $\Gamma \cap \Gamma_0$. Thus, by (106),

$$
\begin{aligned}
n\omega_n &\leq \int_{\Gamma\cap\Gamma_0} GK d\sigma \\
&\leq \int_{\Gamma\cap\Gamma_0} H^{n-1} d\sigma \\
&= \int_{\Gamma\cap\partial B} H^{n-1} d\sigma + \int_{\Gamma\cap\Gamma_0\cap B} H_0^{n-1} d\sigma \\
&\leq \int_{\Gamma\cap\partial B} H_0^{n-1} d\sigma + \int_{\Gamma\cap B} H_0^{n-1} d\sigma \\
&= H_0^{n-1}\,\mathrm{per}(\Omega).
\end{aligned}
\tag{107}
$$

Hence it follows that

$$
H_0\big(\mathrm{vol}(\Omega)\big) \;\geq\; \left(\frac{n\omega_n}{\mathrm{per}(\Omega)}\right)^{\frac{1}{n-1}} \;=\; \overline{H}_0\big(\mathrm{per}(\Omega)\big),
\tag{108}
$$

where $\overline{H}_0(a)$ is the mean curvature of a ball of perimeter a in $\mathbf{R}^n$. It is well-known that $\mathcal{I}_B$ is continuous and increasing [127], and thus is differentiable almost everywhere. Furthermore, $\mathcal{I}_B'(v) = (n-1)H_0(v)$ at all differentiable points $v \in (0, \mathrm{vol}(B))$ [96, Lem. 5]. Then it follows from (108), e.g., see [47, p. 189], that $\mathcal{I}_B' \geq \mathcal{I}_{\mathbf{R}^n}'$ almost everywhere on $[0, \mathrm{vol}(B))$. Hence

$$
\mathcal{I}_B(v) \geq \mathcal{I}_{\mathbf{R}^n}(v),
\tag{109}
$$

for all $v \in [0, \mathrm{vol}(B))$ as desired. So we have established the isoperimetric inequality (2) for Cartan-Hadamard manifolds. It remains then to show that equality holds in (2) only for Euclidean balls. To this end we first record that:

Lemma 11.2. *Suppose that equality in* (1) *holds for a bounded set* $\Omega \subset M$. *Then* Γ *is strictly convex,* $\mathcal{C}^\infty$, *and has constant mean curvature* H_0. *Furthermore, the principal curvatures of* Γ *are all equal to* H_0 .

Proof. If equality holds in (1), then we have equality in (109) for some ball $B \subset M$ large enough to contain Ω, and $v = \mathrm{vol}(\Omega)$. This in turn forces equality to hold successively in (108), and (107). Now equality between the third and fourth lines in (107) yields that

$$
\mathcal{H}^{n-1}(\Gamma \cap \partial B) = 0,
\tag{110}
$$

$$
\Gamma = \Gamma_0.
\tag{111}
$$

Then equality between the second and third lines in (107) yields that

$$
H^{n-1} = GK \equiv H_0^{n-1},
\tag{112}
$$

on $(\Gamma\cap B)\setminus \mathrm{sing}(\Gamma)$. By (111), Γ is convex. Thus as in the proof of part (iii) of Lemma 11.1, for every point $p\in\Gamma\cap B$, $T_p\Gamma$ is a hyperplane. So $\Gamma\cap B$ is $\mathcal{C}^\infty$. On the other hand by part (ii) of Lemma 11.1, near ∂B, Γ is locally a $\mathcal{C}^{1,1}$ graph and thus every point of Γ has a Hölder continuous unit normal. Furthermore, Γ has H^{n-1} almost everywhere constant mean curvature H_0, by (110). It follows that Γ is $\mathcal{C}^\infty$ in a neighborhood of ∂B; see [109, Thm. 27.4] for details of this well-known argument. Finally (112) implies that all principal curvatures are equal to H_0 at all points of Γ. □

We also need the following basic fact:

Lemma 11.3. *Let Γ_i be a sequence of $\mathcal{C}^2$ convex hypersurfaces in M which converge to a convex hypersurface Γ with respect to the Hausdorff distance. Suppose that the principal curvatures of Γ_i are bounded above by a uniform constant. Then Γ is $\mathcal{C}^{1,1}$.*

Proof. Let p a point of M, and set $\widehat{\Gamma} := \exp_p^{-1}(\Gamma)$, $\widehat{\Gamma}_i := \exp_p^{-1}(\Gamma_i)$. Then $\widehat{\Gamma}_i$ will still be $\mathcal{C}^2$, and their principal curvatures are uniformly bounded above. It follows then from Blaschke's rolling theorem [94] that a ball of radius ε rolls freely inside $\widehat{\Gamma}_i$. Thus a ball of radius ε rolls freely inside $\widehat{\Gamma}$, or $\mathrm{reach}(\widehat{\Gamma})>0$. Hence $\widehat{\Gamma}$ is $\mathcal{C}^{1,1}$ by Lemma 2.6, which in turn yields that so is Γ. □

Now suppose that equality holds in (1) for some region Ω in a Cartan-Hadamard manifold M. Then equality holds successively in (108), (107), and (106). So we have $\mathcal{G}(\Gamma_0)=n\omega_n$. But we know from Lemma 11.2 that Γ is convex, or $\Gamma_0=\Gamma$. So

$$\mathcal{G}(\Gamma)=n\omega_n. \tag{113}$$

Let $\lambda_1:=\mathrm{reach}(\Gamma)$, as defined in Section 2. Furthermore note that, by Lemma 11.2, Γ is $\mathcal{C}^\infty$. Thus $\lambda_1>0$ by Lemma 2.6. Set $u:=\widehat{d_\Gamma}$. Then $\Gamma_\lambda := u^{-1}(-\lambda)$ will be a $\mathcal{C}^\infty$ hypersurface for $\lambda\in[0,\lambda_1)$ by Lemma 2.5. For any point p of Γ, let p_λ be the point obtained by moving p the distance of λ along the inward geodesic orthogonal to Γ at p, and set $R_{\ell n\ell n}(\lambda):=R_{\ell n\ell n}(p_\lambda)$. We claim that

$$R_{\ell n\ell n}(\lambda)\equiv 0. \tag{114}$$

To see this note that for λ sufficiently small Γ_λ is positively curved by continuity. Let $\overline{\lambda}$ be the supremum of $x<\lambda_1$ such that Γ_λ is positively curved on $[0,x)$. By Theorem 1.2, $\mathcal{G}(\Gamma_\lambda)\geq n\omega_n$. Thus, by (113) and Corollary 4.11,

$$0\geq n\omega_n-\lim_{\lambda\to\overline{\lambda}}\mathcal{G}(\Gamma_\lambda)=\mathcal{G}(\Gamma)-\lim_{\lambda\to\overline{\lambda}}\mathcal{G}(\Gamma_\lambda)=-\int_{\Omega\setminus D_{\overline{\lambda}}} R_{rnrn}\frac{GK}{\kappa_r}\,d\mu\geq 0,$$

where $D_{\overline{\lambda}}$ is the limit of the regions bounded by Γ_λ as $\lambda\to\overline{\lambda}$. So $R_{rnrn}(\lambda)\equiv 0$ for $\lambda<\overline{\lambda}$. Now, if we set $\kappa_\ell(\lambda):=\kappa_\ell(p_\lambda)$, then Riccati's equation for principal curvatures of parallel hypersurfaces [78, Cor. 3.5], gives

$$\kappa_\ell'(\lambda) = \kappa_\ell^2(\lambda) + R_{\ell n \ell n}(\lambda) = \kappa_\ell^2(\lambda), \tag{115}$$

for $\lambda < \overline{\lambda}$. Solving this equation yields $\kappa_\ell(\lambda) = \kappa_\ell(0)/(1 - \lambda\kappa_\ell(0))$. Since, by Lemma 11.2, $\kappa_\ell(0) = H_0$, it follows that Γ_λ has constant principal curvatures equal to

$$H_\lambda := \frac{H_0}{1 - \lambda H_0}, \tag{116}$$

for $\lambda < \overline{\lambda}$. Now suppose that $\overline{\lambda} < \lambda_1$. Then $\Gamma_{\overline{\lambda}}$ will be a $\mathcal{C}^2$ hypersurface, and therefore, by continuity, it will have constant principal curvature $H_{\overline{\lambda}} := \lim_{\lambda \to \overline{\lambda}} H_\lambda$. Since $\Gamma_{\overline{\lambda}}$ is a closed hypersurface, $H_{\overline{\lambda}} > 0$. So $\Gamma_{\overline{\lambda}}$ has positive curvature, which is not possible if $\overline{\lambda} < \lambda_1$. Thus we conclude that $\overline{\lambda} = \lambda_1$, which establishes (114) as claimed. So (115) and consequently (116) now hold for all $\lambda \in [0, \lambda_1)$.

Next note that, since by Lemma 2.5 u is $\mathcal{C}^2$ on $\Omega \setminus \mathrm{cut}(\Gamma)$, then for all $p \in \Gamma$ and $\lambda < \mathrm{dist}(p, \mathrm{cut}(\Gamma))$ we have,

$$H_\lambda = u_{\ell\ell}(p_\lambda),$$

by (13). Now we claim that, for all points p of Γ, $p_{\lambda_1} \in \mathrm{cut}(\Gamma)$. Suppose not. Then $\lambda_1 < \mathrm{dist}(p, \mathrm{cut}(\Gamma))$ for some $p \in \Gamma$. So it follows, by Lemma 2.5, that u is $\mathcal{C}^2$ near p_{λ_1}. Consequently,

$$H_{\lambda_1} = u_{\ell\ell}(p_{\lambda_1}) < \infty.$$

Furthermore note that by (116), $H_\lambda \leq H_{\lambda_1}$, for all $\lambda \in (0, \lambda_1)$. So principal curvatures of Γ_λ are uniformly bounded above for $\lambda < \lambda_1$. Consequently Lemma 11.3 yields that Γ_{λ_1} is $\mathcal{C}^{1,1}$, and therefore is disjoint from $\mathrm{cut}(\Gamma)$, which is not possible by definition of λ_1. So we have shown that $\Gamma_{\lambda_1} \subset \mathrm{cut}(\Gamma)$, as claimed. Moreover, we have $\Gamma_{\lambda_1} \subset \partial\mathrm{cut}(\Gamma)$, since Γ_λ is disjoint from $\mathrm{cut}(\Gamma)$ for $\lambda < \lambda_1$. On the other hand, $\mathrm{cut}(\Gamma) \subset \Omega_\lambda$, for $\lambda \in [0, \lambda_1)$, where $\Omega_{\lambda_1} \subset \Omega$ is the region bounded by Γ_λ. Thus it follows that $\mathrm{cut}(\Gamma) \subset \Omega_{\lambda_1}$. So $\partial\mathrm{cut}(\Gamma) \subset \partial\Omega_{\lambda_1} = \Gamma_{\lambda_1}$. We conclude then that $\partial\mathrm{cut}(\Gamma) = \Gamma_{\lambda_1}$. But $\mathrm{cut}(\Gamma)$ is nowhere dense [126, Thm. 1]. So

$$\mathrm{cut}(\Gamma) = \partial\,\mathrm{cut}(\Gamma) = \Gamma_{\lambda_1}.$$

Thus every point of $\mathrm{cut}(\Gamma)$ is at the constant distance λ_1 from Γ, or $u = -\lambda_1$ on $\mathrm{cut}(\Gamma)$, and so $\lambda_1 = -m_0$, the minimum value of u on Ω. But since Γ is strictly convex, u has a unique minimum point x_0. Thus $\mathrm{cut}(\Gamma) = \{x_0\}$.

So $\Gamma = \partial B_{\lambda_1}$ or $\Omega = B_{\lambda_1}$, a geodesic ball of radius λ_1 centered at x_0. Furthermore, the condition $R_{\ell n \ell n} = 0$ now means that all "radial curvatures" of M with respect to x_0 are zero on B_{λ_1}, i.e., along each geodesic segment which connects x_0 to ∂B_{λ_1}, the sectional curvatures of M with respect to the planes tangent to that geodesic vanish. Finally, it is well-known (e.g. see [90, Lem. 5.6] or [82, Thm. C]) that vanishing of radial curvatures yields that B_{λ_1} is flat. Indeed if the radial curvatures vanish, then $J'' = 0$ for any Jacobi field J along radial geodesics of B_{λ_1}, since the Jacobi equation depends only on sectional curvatures with respect to planes tangent to the geodesic. Consequently,

$$J(t) = \sum t(J_i)'(0)E_i(t)$$

for a parallel frame E_i along the geodesic. So if we set $\widehat{B}_{\lambda_1} := \exp_{x_0}^{-1}(B_{\lambda_1})$, it follows that $\exp_{x_0}\colon \widehat{B}_{\lambda_1} \to B_{\lambda_1}$ is an isometry.

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