SELF-SHRINKERS TO THE MEAN CURVATURE FLOW ASYMPTOTIC TO ISOPARAMETRIC CONES

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Abstract. In this paper we construct an end of a self-similar shrinking solution of the mean curvature flow asymptotic to an isoparametric cone C and lying outside of C. We call a cone C in $\mathbb{R}^{n+1}$ an *isoparametric cone* if C is the cone over a compact embedded isoparametric hypersurface $\Gamma \subset \mathbb{S}^n$. The theory of isoparametic hypersurfaces is extremely rich and there are infinitely many distinct classes of examples, each with infinitely many members.

1. Introduction

A hypersurface Σ in $\mathbb{R}^{n+1}$ is said to be a self-shrinker (centered at (0,0) of space-time) for the mean curvature flow if $\Sigma_t = \sqrt{-t}\,\Sigma$ flows by homothety starting at $t=-1$ until it disappears at time $t=0$. A simple computation shows that Σ_t is a self-shrinker if and only if Σ satisfies the equation

$$H = -\frac{1}{2} X \cdot \nu \ . \tag{1.1}$$

where H is the mean curvature, X is the position vector and ν is the unit normal of Σ. The study of self-similar shrinking solutions to the mean curvature flow is now well understood to be an important and essential feature of the classification of possible singularities that may develop. In fact the monotonicity formula of Huisken [8] and a rescaling argument of Ilmanen and White imply that suitable blowups of singularities of the mean curvature flow are self-shrinkers [4]. By a theorem of Wang [21], if C is a smooth regular cone with vertex at the origin, there is at most one self-shrinker with an end asymptotic to C.

There are relatively few constructions in the literature of self-shrinkers asymptotic to a cone C; see for example [11], [10], [14, 15, 16]. In this paper we will construct,

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for infinitely many families of special mean convex cones C with interesting topology, a corresponding end of a self-shrinker which is asymptotic to C and lies outside of C. A closed connected compact hypersurface $\Gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is called an isoparametric hypersurface if its principal curvatures are constant. Equivalently, Γ is part of a family of parallel hypersurfaces in $\mathbb{S}^n$ which have constant mean curvature. By a theorem of Cecil and Ryan [3], Γ is taut and so is automatically embedded. We will say that a cone C in $\mathbb{R}^{n+1}$ is an *isoparametric cone* if C is the cone over a compact embedded isoparametric hypersurface Γ. The theory of isoparametric hypersurfaces in $\mathbb{S}^n$ is extremely rich and beautiful. Cartan classified all isoparametric hypersurfaces in $\mathbb{S}^n$ with $g \leq 3$ distinct principal curvatures. For $g = 1$ they are the totally umbilic hyperspheres while for $g = 2$ they are a standard product of spheres $\mathbb{S}^p(a) \times \mathbb{S}^q(b)$ where $a^2 + b^2 = 1$ and $n = p + q + 1$. For $g = 3$ Cartan showed that all the principal curvatures have the same multiplicity $m = 1, 2, 4$ or 8 and Γ must be a tube of constant radius over a standard embedding of a projective plane $\mathbb{FP}^2$ into $\mathbb{S}^{3m+1}$ where $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), $\mathbb{O}$ (Cayley numbers). In the process of proving this result, he showed that any such Γ with g distinct principal curvatures of the same multiplicity can be defined by the restriction to $\mathbb{S}^n$ of a homogeneous harmonic polynomial F of degree g on $\mathbb{R}^{n+1}$ satisfying $|\nabla F|^2 = g^2|x|^{2g-2}$. Münzner [12], [13] found a remarkable structural generalization of this last result of Cartan. Let $\Gamma \subset \mathbb{S}^n$ be an isoparametric hypersurface with g distinct principal curvatures. Then there is a homogeneous polynomial F of degree g defined in all of $\mathbb{R}^{n+1}$ (the Cartan-Münzner polynomial) satisfying $|\nabla F|^2 = g^2 r^{2g-2}$, $\Delta F = \frac{m_- - m_+}{2} g^2 r^{g-2}$, where $r = |x|$. The restriction f of F to $\mathbb{S}^n$ has range $[-1, 1]$ and satisfies

$$|\nabla_{S^n} f|^2 = g^2(1 - f^2),\ \Delta_{S^n} f + g(n + g - 1)f = \frac{m_- - m_+}{2} g^2\ .$$

Each member Γ_t of the isoparametric family determined by Γ has the same focal sets $M_\pm := f^{-1}(\pm 1)$ which are smooth minimal submanifolds of codimension $m_- + 1$, $m_+ + 1$ respectively. In proving this result, Münzner shows that if the principal curvatures of Γ are written as $\cot \theta_k$, $0 < \theta_1 < \ldots < \theta_g < \pi$, then $\theta_k = \theta_1 + \frac{k-1}{g}\pi$ with multiplicities $m_k = m_{k+2}$ subscripts mod g. Thus for g odd, all multiplicities are the same while for g even, there are at most two distinct multiplicities m_-, m_+. Moreover each Γ_t separates $\mathbb{S}^n$ into two connected components $D_\pm$ such that $D_\pm$ is a disk bundle with fibers of dimension $m_\pm + 1$ over $M_\pm$. From this he is able to deduce using algebraic topology that $g \in \{1, 2, 3, 4, 6\}$. Earlier, Takagi and Takahashi [20] had classified all the homogeneous examples based on the work of Hsiang and Lawson [7] and had found that $g \in \{1, 2, 3, 4, 6\}$.

For $g = 4$ using representations of Clifford algebras, Ozeki and Takeuchi [18, 19] found two classes of examples, each with infinitely many members of inhomogeneous solutions. Later these methods were greatly generalized by Ferus, Karcher and Münzner [5], who showed there are infinitely many distinct classes of solutions (in odd dimensional spheres $\mathbb{S}^{2l-1} \subset \mathbb{R}^{2l}$) each with infinitely many members. Their examples contain almost all known homogeneous and inhomogeneous examples. For $g = 6$ Abresch [1] showed that $m_- = m_+ = 1$ or 2 so examples occur only in dimensions $n = 7$ or 13.

From the isoparametric function f we easily compute the mean curvature $H(f)$ of the level set $f = t$:

$$H(f) = \frac{1}{|\nabla_{S^n} f|}\Delta_{S^n} f = \frac{1}{\sqrt{1-f^2}}\{g(\frac{m_- - m_+}{2}) - (n+g-1)f\} \ , \tag{1.2}$$

and so the level set $f = \frac{g(m_- - m_+)}{2(n+g-1)}$ is the unique minimal isoparametric hypersurface of the family. Our main theorem may be stated as follows.

Theorem 1.1. *Let C be an isoparametric cone over $\Gamma \in \mathbb{S}^n$. Then there is a radial graph $S = \{e^{\varphi(d(z))} : z \in A\}$, where $d(z)$ is the distance function to Γ, $A = \{z : 0 < d(z) \le d_0 + \varepsilon\}$, which is an end of a self-shrinker to the mean curvature flow (that is satisfies* (1.1)*) and is smoothly asymptotic to C. Here the parallel hypersurface $d(z) = d_0$ is the unique minimal hypersurface of the family.*

An outline of the paper is as follows. In section 2 we derive the equations for a radial graph over a annular neighborhood A of the exterior of a mean convex hypersurface $\Gamma \subset \mathbb{S}^n$, which satisfies equation 1.1 and is asymptotic to the cone over Γ. If we specialize to Γ an isoparametric hypersurface, our problem then reduces to a singular ode problem for $\varphi(d(z))$. We then reformulate the problem as a singular initial value problem for a nonlinear second order ordinary differential equation to make it more tractable. Even so, the problem is quite nonstandard and requires a novel approximation to obtain a globally smooth solution. This is carried out in sections 3 and 4 where we prove existence and uniqueness of a smooth solution. The proof of Theorem 1.1 follows from these results as sketched at the end of section 4.

In Appendix A we prove that $g(d) = e^{-2\varphi(d)}$ or equivalently its inverse function $\gamma(s)$ is in the Gevrey class $\mathcal{G}^2$. This result is relevant to the study of degenerate Cauchy problems with quadratic degeneracy of which we were unable to find any results in the literature. It is also related to the question of what is the minimum smoothness of a mean convex cone C so that there exists an end of a self-shrinker asymptotic to C. We expect that the answer is Gevrey class $\mathcal{G}^2$ smoothness and we hope to show in future work that our method extends to this case. This section is not needed for the proof of the main Theorem 1.1, so may be skipped by the general reader. Appendix B contains several Faà di Bruno variants and lemmas used to obtain the inductive estimates of section 3 and the Gevrey regularity result of Appendix A.

2. The self-shrinker equation for a radial graph asymptotic to a mean convex cone

Let Γ be a compact embedded hypersurface in the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with strictly positive mean curvature H with respect to the inner orientation and let C be the cone over Γ. Let A be an annular neighborhood of the outside of Γ and let $S = \{e^{v(z)}z : z \in A\}$ be a radial graph over A that is asymptotic to C. Notice that to satisfy this last condition we need both $v(z) \to \infty$ and $d(z)e^{v(z)} \to 0$ as $d(z) \to 0$ where $d(z) > 0$ is the distance function to Γ in A.

We let σ_{ij} denote the metric of $\mathbb{S}^n$ with respect to local coordinates with $(\sigma^{ij}) = (\sigma_{ij})^{-1}$. We use the shorthand $v^i = \sigma^{ik}v_k$, $v_{ij} = \nabla_i\nabla_j v$ for covariant derivative with respect to the metric σ_{ij}. Then [6] the outward unit normal to S is $\nu = \frac{z-\nabla v}{w}$, $w = \sqrt{1+|\nabla v|^2}$ and the mean curvature with respect to the orientation ν is given by

$$H^S = \frac{a^{ij}v_{ij} - n}{e^v w} \tag{2.1}$$

where $a^{ij} := \sigma^{ij} - \frac{v^i v^j}{w^2}$. The self-shrinker equation $H^S = -\frac{1}{2}X \cdot \nu$ is then

$$\frac{a^{ij}v_{ij} - n}{e^v w} = -\frac{e^v}{2w}\ ,$$

or

$$a^{ij}v_{ij} = n - \frac{1}{2}e^{2v} \tag{2.2}$$

with singular boundary condition $v(z) \to \infty$ as $z \to \Gamma$. It is not too difficult to see that the singularity of $v(z)$ must be asymptotically $\log \frac{H}{d}$ in order that S be smoothly asymptotic to C.

This is a seemingly ill-posed problem since by the uniqueness theorem of Lu Wang [21], we cannot hope to impose boundary conditions on $\partial A \setminus \Gamma$ so we are stuck with a Cauchy problem for a singular equation that resembles a backward heat equation. In this paper we simplify the general problem by reducing it to a singular ordinary differential equation. In order to accomplish this we assume that Γ is part of an *isoparametric* family of parallel hypersurfaces Γ_d with constant principle curvatures or equivalently constant mean curvature. More precisely if $f : \Gamma \to \mathbb{S}^n$, then $\Gamma_d = f_d(\Gamma)$ where $f_d(x) := \cos d\, f(z) - \sin d\, N(z)$ and $N(z)$ is chosen with the outer orientation so we are considering only the outer part of the family relative to Γ. Except for the unique minimal hypersurface of the family, each parallel hypersurface has (if properly oriented) positive constant mean curvature. If we denote by $H(d)$ the mean curvature of the parallel hypersurface Γ_d at distance d with orientation consistent with Γ), then

$$H(d) > 0,\ H'(d) < 0,\ 0 < d < d_1,\ H(d_1) = 0$$

and we may look for a solution of the form $v = \varphi(d(z))$. Substitution of this ansatz into (2.2) gives

$$(\sigma^{ij} - \frac{\varphi'^2 d^i d^j}{1+\varphi'^2})(\varphi' d_{ij} + \varphi'' d_i d_j) = n - \frac{1}{2}e^{2\varphi}.$$

Since $|\nabla d|^2 = \sigma^{ij} d_i d_j = 1$, $\sum_i d_i d_{ij} = 0$ and $\sigma^{ij} d_{ij} = \Delta d = H(d)$, we arrive after simplification to the equation

$$\frac{\varphi''}{1+\varphi'^2} + \varphi' H(d) = n - \frac{1}{2}e^{2\varphi}. \tag{2.3}$$

Because of the expected form of the singularity of φ, we change variables by setting $\varphi := -\frac{1}{2}\log g(d)$. Then (2.3) transforms to

$$2\,\frac{g'^2 - gg''}{g'^2 + 4g^2} + \frac{1}{2g}(1 - H(d)g') - n = 0 \tag{2.4}$$

with boundary condition $g(0) = 0$ and the implied compatability condition (assuming we can find a smooth enough solution) $g'(0) = H(0) > 0$. To simplify our analysis in the next section, we set $s = g(d)$ with inverse $d = \gamma(s)$. Then after simplification, we arrive at the singular initial value problem

$$4\frac{s^2\gamma''(s) + s\gamma'(s)}{1 + 4s^2\gamma'^2(s)} + (1 - 2ns)\gamma'(s) - H(\gamma) = 0,\ \gamma(0) = 0\ ,\ s > 0 \tag{2.5}$$

and we seek a smooth solution on a uniform interval $[0, s_0]$ with $0 < \gamma(s) < d_0$ and $\gamma'(s) > 0$. Formally it is not hard to see that all the derivatives $g^{(k)}(0)$ are determined.

That is, there exists a formal power series solution of (2.5)

$$\sum_{k=1}^{\infty} \frac{A_k}{k!} s^k \tag{2.6}$$

since it is straightforward (since all the compositions in (2.6) are well defined) to see that A_{k+1} is recursively determined from $A_1 \dots, A_k$. In fact,

$$A_{k+1} = (-4k^2 + (2n + h'(0)k)A_k + P(A_1, \dots, A_{k-1}) \tag{2.7}$$

where P is a polynomial. This will be used in section 3.

3. Approximation and apriori estimates

In this section we study a special ε-regularized initial value problem

$$\begin{aligned} &\varepsilon\gamma''(s) + 4\frac{s^2\gamma''(s) + s\gamma'(s)}{1 + 4s^2\gamma'^2(s)} + (1 - 2ns)\gamma'(s) - H(\gamma) = 0,\ s > 0 \\ &\gamma(0) = 0,\ \gamma'(0) = B(\varepsilon) := \sum_{i=0}^{N} B_i\varepsilon^i,\ B_0 = H(0) \end{aligned} \tag{3.1}$$

for $\gamma = \gamma(s, \varepsilon)$, where $\varepsilon > 0$ is small, N is a fixed large integer. The B_i will be determined recursively as polynomial functions of the A_k so that $\gamma^{(k+2)}(0)$ is bounded in terms of N but independent of ε and $\lim_{\varepsilon\to 0} \gamma^{(k)}(0) = A_k$, $k = 1, \dots, N + 1$, where the A_k are the coefficients in the formal power series solution (2.6). For any choice of $B_1, \dots, B_N$, there is an analytic solution $\gamma(s) = \gamma(s; \varepsilon)$ of the initial value problem (3.1) on some small interval $0 < s < s_0(\varepsilon)$.

Proposition 3.1. *There exists $B_1, \dots, B_N$ such that*

$$\gamma^{(k)}(0) = A_k + ((-1)^{k-1}B_k + \phi_k(B_1, \dots, B_{k-1}))\varepsilon + O(\varepsilon^2),\ k = 1, \dots, N + 1 \tag{3.2}$$

Moreover $|B_k|$, $|\gamma^{(k)}(0)| \le C = C(H, N)$, $k = 1, \dots, N + 1$ where C is independent of ε.

Proof. We differentiate (3.1) k times making use of the identities

$$(\frac{d}{ds})^l (s\frac{d}{ds})^p u(s) = (s\frac{d}{ds} + l)^p u^{(l)}(s),$$

$$4\frac{s^2\gamma''(s) + s\gamma'(s)}{1 + 4s^2\gamma'^2(s)} = 2s\frac{d}{ds} \arctan\left(2s\gamma'(s)\right) .$$

Thus

$$(3.3)\qquad \begin{aligned} &(\frac{d}{ds})^k[(1-2ns)\gamma'(s)](0) = \gamma^{(k+1)}(0) - 2nk\gamma^{(k)}(0),\\ &(\frac{d}{ds})^k(2s\frac{d}{ds}\arctan(2s\gamma'(s)))(0) = 2k(\frac{d}{ds})^k\arctan(2s\gamma'(s)))(0), \end{aligned}$$

so that

$$(3.4)\qquad \begin{aligned} \varepsilon\gamma^{(k+2)}(0) &= -\gamma^{(k+1)}(0) + 2nk\gamma^{(k)}(0) + (\frac{d}{ds})^k\{H(\gamma) - 2k\arctan(2s\gamma'(s))\}(0)\\ &:= -\gamma^{(k+1)}(0) + A^{\varepsilon}_{k+1} \end{aligned}$$

where the last term on the right hand side of (3.4) contains derivatives of H and γ evaluated at 0 of order at most k. A more explicit expression for the right hand side can be obtained using the formulas (B.8), (B.9) of Appendix B. Note that the coefficients A_k of the formal solution of (2.6) are obtained by setting $\varepsilon = 0$ and $A_1 = \gamma'(0) = H(0)$. This leads to the simple but important observation:

$$(3.5)\qquad A_{k+1} = F(A_1, \dots, A_k) := A^{\varepsilon}_{k+1}|_{\varepsilon=0}\ .$$

It is straightforward to verify that

$$(3.6)\qquad |A_k| \le C^k(H)(k!)^2\ ;$$

in other words the formal solution is in the Gevrey class $\mathcal{G}^2$ (of order 2) and in fact no better. Similarly for the ε regularized problem with initial condition $\gamma'(0) = B(\varepsilon)$, the derivatives $\gamma^{(k)}(0)$ are recursively well defined by (3.4). Note that A^{ε}_{k+1} is a polynomial in ε with coefficients depending on the $\{B_i\}$. By our choice $B_0 = H(0)$, $\gamma''(0) = -\sum_{i=1}^N B_i\varepsilon^{i-1}$ and we choose $B_1 = -A_2$. Suppose we have chosen $B_0 = H(0)$, $B_1 = -A_2, \dots, B_k$ so that

$$\gamma^{(l)}(0) = A_l + ((-1)^{l-1}B_l + \phi_l(B_1, \dots, B_{l-1}))\varepsilon + O(\varepsilon^2),\ l = 2, \dots, k+1\ .$$

Claim: $\gamma^{(k+2)}(0) = (-1)^{k+1}B_{k+1} + \phi(B_1, \dots, B_k) + O(\varepsilon)$.
Assuming the claim, we complete the induction by defining B_{k+1} by the relation

$$A_{k+2} = (-1)^{k+1}B_{k+1} + \phi(B_1, \dots, B_k)\ .$$

To prove the claim we will need a more explicit expression for A^{ε}_{k+1} so that we can expand $\varepsilon\gamma^{k+2}(0)$ in powers of ε.

By (3.4) and (3.5) and our induction hypothesis, the constant term in the expansion of $\varepsilon\gamma^{(k+2)}$ in powers of ε is automatically 0 and the ε term begins with $(-1)^{k+1}B_k + \phi_{k+1}(B_1, \dots, B_{k-1})$ coming from $\gamma^{(k+1)}(0)$. It remains to verify that the ε terms of

A^ε_{k+1} depends only on $B_1, \dots, B_{k-1}$. But this is now obvious from the formulas (B.8), (B.9) and our induction hypothesis. Thus

$$\begin{aligned} \varepsilon\gamma^{k+2}(0) &= (-A_{k+1} + A^\varepsilon_{k+1}|_{\varepsilon=0}) + ((-1)^{k+1}B_k + \phi(B_1, \dots, B_{k-1}))\varepsilon + O(\varepsilon^2) \\ \gamma^{k+2}(0) &= (-1)^{k+1}B_k + \phi(B_1, \dots, B_{k-1}) + O(\varepsilon)\ , \end{aligned} \tag{3.7}$$

completing the proof. □

Now that we have control of the first $N+1$ derivatives of γ at the origin, we will inductively derive energy estimates that imply $0 < \gamma(s) < d_0, \gamma'(s) > 0$, $|\gamma^{(k)}(s)| \le C_k$, $k = 1, \dots, N$ independent of ε (as $\varepsilon \to 0$) on a uniform interval $(0, s_0)$. We assume, based on Proposition 3.1, that ε is chosen so small that

$$\frac{1}{2}H(0) \le \gamma'(0) \le 2H(0),\ , |\gamma^{(k)}(0) - A_k| \le 1,\ \varepsilon|\gamma^{(k)}(0)|^2 \le 1,\ k = 1, \dots, N+1\ . \tag{3.8}$$

In the following discussion, C will denote a constant depending on H and dimension n which may change from line to line. We multiply (3.1) by $\gamma'(s)$ and integrate to obtain

$$\varepsilon\frac{\gamma'^2(s) - \gamma'^2(0)}{2} + \frac{1}{2}\log\left(1 + 4s^2\gamma'^2(s)\right) + \int_0^s (1 - 2nt)\gamma'^2(t)dt = \mathcal{H}(\gamma(s)) \tag{3.9}$$

where $\mathcal{H}'(t) = H(t)$, $\mathcal{H}(0) = 0$. We restrict $0 \le s \le \frac{1}{4n}$. Since $\mathcal{H}(\gamma(s)) \le C\gamma(s)$ we conclude that

$$\gamma^2(s) = (\int_0^s \gamma'(t)dt)^2 \le s\int_0^s \gamma'^2(t)dt \le s(\varepsilon\gamma'^2(0) + 2C\gamma(s))\ .$$

Hence $\gamma(s) \le 2Cs + \gamma'(0)\sqrt{\varepsilon s} \le C(s + \sqrt{\varepsilon s})$. Thus by (3.9) and (3.1), we obtain the preliminary estimates

Lemma 3.2.

$$\begin{aligned} &\gamma(s) \le C(s + \sqrt{\varepsilon s}) \\ &0 \le s\gamma'(s) \le \max\left(\gamma'(0)s, C(s + \sqrt{\varepsilon s})^{\frac{1}{2}}\right) \le C(s + \sqrt{\varepsilon s})^{\frac{1}{2}}, \\ &s^2|\gamma''(s)| \le \gamma'(s) + C,\ \int_0^s (\gamma'(t))^2 dt \le C \end{aligned} \tag{3.10}$$

We differentiate (3.1) $k \ge 1$ times making use of the identity

$$(\frac{d}{ds})^l(s\frac{d}{ds})^p u(s) = (s\frac{d}{ds} + l)^p u^{(l)}(s)$$

to obtain

$$\begin{aligned}\varepsilon\gamma^{(k+2)}+4\sum_{l=0}^{k}\binom{k}{l}\left(s^2\gamma^{(l+2)}+(2l+1)s\gamma^{(l+1)}+l^2\gamma^{(l)}\right)(\frac{d}{ds})^{k-l}\eta(s\gamma')\\ +(1-2ns)\gamma^{(k+1)}-2nk\gamma^{(k)}-(\frac{d}{ds})^k H(\gamma)=0\ ,\end{aligned}\tag{3.11}$$

where $\eta(x)=\frac{1}{1+4x^2}$. An explicit formula for $(\frac{d}{ds})^{k-l}\eta(s\gamma')$ is given in formula (B.6) of the Appendix. We multiply (3.11) by $\gamma^{(k+1)}$ and integrate isolating the crucial terms involving $(\gamma^{(k+1)})^2$:

$$\begin{aligned}&\frac{\varepsilon}{2}[(\gamma^{(k+1)})^2(s)-(\gamma^{(k+1)})^2(0)]+2s^2\eta(s\gamma')(\gamma^{(k+1)})^2(s)\\ &+4\sum_{l=1}^{k-2}\binom{k}{l}\int_0^s\gamma^{(k+1)}(t)[t^2\gamma^{(l+2)}+(2l+1)t\gamma^{(l+1)}+l^2\gamma^{(l)}](\frac{d}{dt})^{k-l}\eta(t\gamma')\ dt\\ &+\int_0^s[-2\frac{d}{dt}(t^2\eta(t\gamma'))+4(2k+1)t\eta(t\gamma')+4kt^2\frac{d}{dt}\eta(t\gamma')](\gamma^{(k+1)})^2\ dt\\ &+4\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma')+8t\gamma'\eta^2(t\gamma')(t\gamma^{(k+1)}+k\gamma^{(k)})\}\ dt\\ &-32\int_0^s[t^2\gamma'(t^2\gamma''+t\gamma')]\eta^2(t\gamma')(\gamma^{(k+1)})^2\ dt-32k\int_0^s[t\gamma'(t^2\gamma''+t\gamma')]\eta^2\gamma^{(k)}\gamma^{(k+1)}\ dt\\ &+\int_0^s(1-2nt)(\gamma^{(k+1)})^2(t)\ dt+\int_0^s[4k^2\eta(t\gamma')+4k(2k-1)t\frac{d}{dt}\eta(t\gamma')-2nk]\gamma^{(k+1)}\gamma^{(k)}\ dt\\ &+4k(k-1)^2\int_0^s\gamma^{(k+1)}\gamma^{(k-1)}\frac{d}{dt}\eta(t\gamma')\ dt-\int_0^s(\frac{d}{dt})^kH(\gamma)\gamma^{(k+1)}\ dt=0.\end{aligned}\tag{3.12}$$

where we have used (B.6) with $m=k$ to rewrite the term $4\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')(\frac{d}{dt})^k\eta(t\gamma')\ dt$.

For k=1 using the preliminary estimates of Lemma 3.2, we obtain

$$I\geq\int_0^s(1-C(t+\sqrt{\varepsilon t}))(\gamma'')^2\ dt\ .$$

Therefore we obtain for $k=1$

$$\frac{\varepsilon}{2}(\gamma''(s)^2-\gamma''(0)^2)+2s^2\eta(s\gamma')\gamma''^2(s)+I-\int_0^s[8\eta(t\gamma')+H'(\gamma)]\gamma'\gamma''\ dt=0\tag{3.13}$$

Using the estimate (3.9) in (3.13) yields for $0\leq s\leq s_0$ (s_0 a uniform constant)

$$\frac{\varepsilon}{2}\gamma''(s)^2+\frac{2s^2\gamma''^2(s)}{1+4(s\gamma')^2}+\int_0^s(\gamma'')^2\ dt\leq C\tag{3.14}$$

In particular we have the improved estimates

$$|\gamma'(s)-\gamma'(0)|\le C\sqrt{s},\ \ s|\gamma''(s)|\le C \tag{3.15}$$

and using this in Lemma 3.2 yields

$$s^2|\gamma''(s)|\le s^2|\gamma''(0)|+C\sqrt{s}. \tag{3.16}$$

Collecting all terms I in (3.12) involving $\int_0^s(\cdot)(\gamma^{(k+1)})^2\,dt$, we obtain

$$I=\int_0^s\{(1-2nt)+8kt\eta(t\gamma')-[(32(k+1)t^2\gamma'(t)(t^2\gamma''+\gamma']\eta^2(t\gamma')\}(\gamma^{(k+1)})^2\,dt. \tag{3.17}$$

Inserting (3.15), (3.16) in (3.17) yields

$$I=\int_0^s(1+o(1))(\gamma^{(k+1)})^2\,dt,\ \ k\ge 2 \tag{3.18}$$

uniformly in k.

Theorem 3.3. *Let $\gamma=\gamma(s;\varepsilon,N)$ be a solution of the ε-regularized initial value problem* (3.1) *on the interval* $[0,s]$*. There exists s_0 sufficiently small independent of ε and N so that*

$$\|\gamma^{(j)}\|_{L^2[0,s]}+|(\frac{d}{ds})^{(j-1)}(s\gamma'(s))|\,dt\le C(N),\ 1\le j\le N+1,\ s\in[0,s_0]. \tag{3.19}$$

Proof. We will prove (3.19) by induction. We have already proved the estimates (3.19) for $j=1,2$, so $s\gamma''+\gamma'(s)$ and $\frac{d}{ds}\eta(s\gamma')$ are bounded. Moreover the crucial relation (3.18) holds. In the following discussion $C(N)$ will be a generic constant independent of ε and s_0 which may change from line to line. Suppose that (3.19) holds for $j=1,\dots,k$. Note that

$$s^2\gamma^{(l+2)}+(2l+1)s\gamma^{(l+1)}+l^2\gamma^{(l)}=s(\frac{d}{ds})^{(l+1)}(s\gamma'(s))+l(\frac{d}{ds})^{(l)}(s\gamma'(s))$$

is uniformly bounded for $1\le l\le k-2$ and so is $(\frac{d}{ds})^{k-l}\eta(s\gamma')$ by the Faà di Bruno formula (B.6) of Appendix B. Note also that (B.6) implies

$$(\frac{d}{ds})^k\eta(s\gamma'(s)=-8s\gamma'(s)\eta^2(s\gamma^{(k+1)}(s)+k\gamma^{(k)}(s))+\text{bounded terms}\ .$$

Hence the term

$$\begin{aligned}&4\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma')+8(t\gamma')\eta^2(t\gamma')(t\gamma^{(k+1)}+k\gamma^{(k)}\}\,dt\\ &=O(C(N)\int_0^s|\gamma^{(k+1)}|\,dt)=O(\frac{1}{10}\int_0^s(\gamma^{(k+1)})^2\,dt+C(N))\ .\end{aligned} \tag{3.20}$$

Moreover the terms involving $\int_0^s (\cdot)\gamma^{(k+1)}\gamma^{(j)}dt$, $j = k$ or $j = k-1$ satisfy the same bounds. Similarly, $(\frac{d}{ds})^k = H'(\gamma))\gamma^{(k)} + O(C(N)$ so

$$|\int_0^s (\frac{d}{dt})^k H(\gamma)\gamma^{(k+1)}\ dt| \leq \frac{1}{10}\int_0^s (\gamma^{(k+1)})^2\ dt + C(N)\ .$$

Thus for s_0 sufficiently small we derive from (3.12)

$$\|\gamma^{(k+1)}||_{L^2[0,s]} + |(\frac{d}{ds})^{(k)}(s\gamma'(s))|\ dt \leq C(N),\ s \in [0, s_0]\ ,$$

and the induction is complete.

□

4. Existence and uniqueness

In this section we apply the results of Section 3 to prove the existence of a $C^\infty[0, s_0]$ solution of the singular initial value problem (2.5). We start by providing an elementary proof of the uniqueness of $C^2[0, s_0]$ solutions to the original singular initial value problem (2.5).

Let $\gamma_1(s)$, $\gamma_2(s)$ be two solutions of (3.1) and set $u(s) = \gamma_1(s) - \gamma_2(s)$, $\gamma_\theta(s) := \gamma_2(s) + \theta u(s)$. Then using the mean value theorem, we see that

$$(1 - 2ns)u'(s) - b(s)u(s) + 4s\frac{d}{ds}(2sa(s)u'(s)) = 0 \tag{4.1}$$

where

$$b(s) := \int_0^1 H'(\gamma_\theta(s))d\theta,\ a(s) := \int_0^1 \frac{d\theta}{1 + 4s^2(\gamma_\theta'(s))^2}\ . \tag{4.2}$$

Multiplying (4.1) by u' and integrating by parts gives

$$4a(s)s^2u'^2(s) + \int_0^s [(1 - 2nt) + 4t^2a'(t)]u'^2(t)\ dt = \int_0^s b(t)u(t)u'(t)\ dt \tag{4.3}$$

Since $a'(s) = O(s)$, $(1 - 2nt + 4t^2a'(t)) > 0$ and $a(s) \geq \frac{1}{2}$ for $0 < t < s$ small enough. Moreover,

$$|\int_0^s b(t)u(t)u'(t)\ dt| \leq C\left(\int_0^s u^2(t)\ dt\int_0^s u'^2(t)\ dt\right)^{\frac{1}{2}} \leq Cs\int_0^s u'(t)^2\ dt\ . \tag{4.4}$$

where C is independent of s. Combining (4.3) and (4.4) gives

$$\int_0^s u'^2(t)\ dt \leq Cs\int_0^s u'^2(t)\ dt\ ,$$

so $u(s) \equiv 0$. We have proved

Theorem 4.1. *Assume $H \in C^1[0, d_0)$. Then there is at most one $C^2[0, s_0]$ solution to the singular initial value problem* (2.5)*.*

We can now apply Theorem 3.3 of the previous section to find a smooth solution to the singular initial value problem (2.5).

Theorem 4.2. *There exists s_0 depending only on H such that the initial value problem* (2.5) *has a unique solution $\gamma \in C^\infty[0, s_0]$. Moreover, $\gamma^{(k)}(0) = A_k$, $k = 1, 2, \ldots$.*

Proof. According to Theorem 3.3 , for any integer N, there is an analytic solution $\gamma_N(s; \varepsilon)$ of the ε regularized initial value problem (3.1) on a uniform interval $[0, s_0)$ independent of ε and N. Moreover,

$$\|\gamma_N\|_{C^{N+1}[0,s_0]} \leq C(N), \ \lim_{\varepsilon \to 0} \gamma_N^{(k)}(0, \varepsilon) = A_k, \ k = 1, \ldots, N \ .$$

Hence taking limits and using Theorem 4.1, there is a unique solution $\gamma(s) \in C^N[0, s_0)$ of (2.5) with $\gamma^{(k)}(0) = A_k$, $k = 1, 2, \ldots, N$. Since N is arbitrary, the theorem follows. □

Proof of Theorem 1.1: According to Theorem 4.1 and the calculations of section 3, there exists a unique smooth solution $\varphi(d) = -\frac{1}{2} \log g(d)$ to (2.3) on a small interval $0 < d < d_1$. That is to say, the radial graph $\Sigma = \{e^{\varphi(d(z))} : z \in A\}$, where $d(z)$ is the distance function to Γ in $A = \{z : 0 < d(z) \leq d_1 + \varepsilon\}$ is an end of a self-shrinker to the mean curvature flow. Moreover according to Theorem A.1, $g(d) = e^{-2\varphi(d)}$ is in the Gevrey class $\mathcal{G}^2$. It remains to show that $\varphi(d)$ exists on the interval $(0, d_0 + \varepsilon)$ for some small $\varepsilon > 0$, where the parallel hypersurface to Γ, $d(z) = d_0$ is the unique minimal hypersurface of the family. To see this we multiply (2.3) by $2\varphi'(d)$ and integrate from $\frac{d_1}{2}$ to d to obtain

$$\log\left(1 + \varphi'^2(d)\right) + 2 \int_{\frac{d_1}{2}}^{d} H(t) \varphi'^2(t) dt = 2n\varphi(d) - \frac{1}{2} e^{2\varphi(d)} + C(d_1) \ . \tag{4.5}$$

Note that the right hand side of (4.5) tends to negative infinity as $\varphi \to \pm\infty$ while the left hand side remains strictly positive while $H \geq 0$. It follows easily that the solution continues past $d = d_0$, completing the proof.

Appendix A. Further Gevrey regularity of the solution

In this section we show that the solution of the singular initial value problem (2.5) is in the Gevrey class $\mathcal{G}^2[0,s_0]$ for a uniform constant s_0. For the following calculations we will need to assume that $\|\gamma''\|_{L^2[0,s]} + |\frac{d}{ds}(s\gamma'(s))|$ is small on $[0,s_0]$. We can achieve this by the rescaling $\tilde{\gamma}(s) = \gamma(\lambda s)$, $0 \le s \le \frac{s_0}{\lambda}$ with λ small. For $\frac{d}{ds}[s\tilde{\gamma}'](s) = \lambda \frac{d}{ds}[s\gamma'](\lambda s)$, $\|\tilde{\gamma}''\|_{L^2[0,\frac{s_0}{\lambda}]} = \lambda^{\frac{3}{2}}\|\gamma''\|_{L^2[0,s_0]}$ and $\tilde{\gamma}(s)$ satisfies

$$\begin{aligned} & 4\frac{s^2\tilde{\gamma}''(s) + s\tilde{\gamma}'(s)}{1+4s^2\tilde{\gamma}'^2(s)} + (\frac{1}{\lambda} - 2ns)\tilde{\gamma}'(s) - H(\tilde{\gamma}) = 0,\ 0 \le s \le \frac{s_0}{\lambda} \\ & \tilde{\gamma}(0) = 0\ . \end{aligned} \tag{A.1}$$

The only difference between (A.1) and the original equation (2.5) is in the coefficient $(\frac{1}{\lambda} - 2ns)$ instead of $1-2ns$ which only improves the estimates. Thus there is no loss of generality in working with (2.5) and assuming the necessary smallness conditions.

Theorem A.1. *There exists $s_0 > 0$ small and M large (depending on H) so that if $u \in C^\infty[0,s_0]$ is the unique solution of* (2.5)*, then*

$$\|\gamma^{(j)}\|_{L^2[0,s]} + |(\frac{d}{ds})^{j-1}(s\gamma'(s))| \le M^{j-3}\frac{(j-2)!^2}{(j-1)},\ \text{on } [0,s_0]. \tag{A.2}$$

for all integers $j \ge 2$.

We will prove (2.5) by induction assuming that it holds for $j = 2,\dots,k$. Note that for M to be chosen later depending only on H, the starting case $j=2$ can be achieved by our smallness assumptions.

Proposition A.2. *Let $k \ge 2$ and assume the induction hypothesis* (A.2) *for $j = 2,\dots,k$. Then*

$$\begin{aligned} & i.\ \text{for } 1 \le n \le k-1,\ |(\frac{d}{ds})^n(\eta(s\gamma'(s))| \le 16M^{n-2}\frac{(n-1)!^2}{n}\ , \\ & ii.\ |(\frac{d}{ds})^k(\eta(s\gamma'(s)) + 8s\gamma'(s)\eta^2[\gamma^{(k+1)}(s) + k\gamma^{(k)}(s)]| \le CM^{k-4}\frac{k!(k-2)!}{(k-1)^2}\ , \end{aligned} \tag{A.3}$$

Proof. To prove part i. we use the alternate version of the Faà di Bruno formula from Lemma B.2:

$$(\frac{d}{ds})^n(\eta(s\gamma'(s)) = \sum_{l=1}^{n}\binom{n}{l}(-2)^l\Lambda_{l+1}(s\gamma'(s))\left\{(\frac{d}{dh})^{n-l}[\int_0^1 \gamma''(s+h\theta)\ d\theta + \gamma'(s+h)]^l\right\}_{h=0}.$$

Under the induction hypothesis (2.5), Lemma B.4 implies (we assume $\frac{C}{M} \leq \frac{1}{2}$ below)

$$
\begin{aligned}
|(\frac{d}{ds})^n(\eta(s\gamma'(s))| &\leq \sum_{l=1}^{n} \binom{n}{l} 2^l l! (\frac{C}{M})^{l-1} M^{n-l-1} \frac{(n-l)!^2}{(n-l+1)^2} \\
&= \frac{M^n}{C} n! \sum_{l=1}^{n} (\frac{2C}{M^2})^l \frac{(n-l)!}{(n-l+1)^2} \leq 16 M^{n-2} \frac{(n-1)!^2}{n}
\end{aligned} \tag{A.4}
$$

The proof of part ii. is essentially the same. □

Lemma A.3. *Let $k \geq 2$ and assume the induction hypothesis* (A.2) *for $j = 2, \ldots, k$. Then*

$$
\begin{aligned}
&i.\ \|(\frac{d}{ds})^k[H(\gamma(s))]\|_{L^2[0,s]} \leq C_1 M^{k-3} \frac{(k-2)!^2}{k-1} + \sqrt{s_0} C_1^2 C M^{k-4} (k-2)!^2 \ , \\
&ii.\ \quad |\int_0^s (\frac{d}{dt})^k H(\gamma)\gamma^{(k+1)}\ dt| \leq C_2 M^{k-3} (k-2)!^2 \|\gamma^{(k+1)}\|_{L^2[0,s]} \ ,
\end{aligned} \tag{A.5}
$$

where C is a universal constant and $C_1 = C_1(H)$, $C_2 = C_2(H)$.

Proof. According to Corollary B.1

$$
(\frac{d}{ds})^m H(\gamma(s)) = H'(\gamma(s))\gamma^{(m)}(s) + \sum_{l=2}^{m} \binom{m}{l} H^{(l)}(g(s)) \left\{ (\frac{d}{dh})^{m-l} [\int_0^1 \gamma'(s+h\theta)\ d\theta]^l \right\}_{h=0} .
$$

Moreover by the induction hypothesis $u(h;s) := \int_0^1 \gamma'(s+h\theta)\ d\theta$ satisfies

$$
|(\frac{d}{dh})^j[u(h;s)]| \leq \frac{1}{j+1}\sqrt{s-h}\|\gamma^{(j+2)}\|_{L^2[0,s-h]} \leq \sqrt{s_0 - h} M^{j-1} \frac{j!^2}{(j+1)^2}
$$

for $0 \leq j \leq k-2$. Hence we can use Lemma B.3 to conclude as in Proposition A.2 ii. that part i. of (A.5) holds. Part ii. follows immediately from part i. □

We now are in a position to complete the induction. From (3.12) with $\varepsilon = 0$ we have

$$
\begin{aligned}
&2s^2\eta(s\gamma')(\gamma^{(k+1)})^2(s) + \int_0^s (1+o(1))(\gamma^{(k+1)})^2 \, dt \\
&+ 4\sum_{l=1}^{k-2} \binom{k}{l} \int_0^s \gamma^{(k+1)}(t)[t^2\gamma^{(l+2)} + (2l+1)t\gamma^{(l+1)} + l^2\gamma^{(l)}](\frac{d}{dt})^{k-l}\eta(t\gamma') \, dt \\
&+ 4\int_0^s \gamma^{(k+1)}(t^2\gamma'' + t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma') + 8t\gamma'\eta^2(t\gamma')(t\gamma^{(k+1)} + k\gamma^{(k)})\} \, dt \\
&- 32k\int_0^s [t\gamma'(t^2\gamma'' + t\gamma')]\eta^2\gamma^{(k)}\gamma^{(k+1)} \, dt \\
&+ \int_0^s [4k^2\eta(t\gamma') + 4k(2k-1)t\frac{d}{dt}\eta(t\gamma') - 2nk]\gamma^{(k+1)}\gamma^{(k)} \, dt \\
&+ 4k(k-1)^2\int_0^s \gamma^{(k+1)}\gamma^{(k-1)}\frac{d}{dt}\eta(t\gamma') \, dt - \int_0^s (\frac{d}{dt})^k H(\gamma)\gamma^{(k+1)} \, dt = 0.
\end{aligned}
\tag{A.6}
$$

We now estimate the terms of (A.6) in order of difficulty.

Lemma A.4. *Let $k \geq 2$ and assume the induction hypothesis* (A.2) *for $j = 2, \ldots, k$. Then*

$$
\begin{aligned}
&\sum_{l=1}^{k-2} \binom{k}{l} \int_0^s \gamma^{(k+1)}(t)[t^2\gamma^{(l+2)} + (2l+1)t\gamma^{(l+1)} + l^2\gamma^{(l)}](\frac{d}{dt})^{k-l}\eta(t\gamma') \, dt \\
&\leq C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-2}\frac{k!(k-1)!}{(k+1)^2} \ ,
\end{aligned}
\tag{A.7}
$$

where C is a universal constant.

Proof. In the following we use the induction hypothesis and Proposition A.2 part i.

(A.8)
$$\begin{aligned}
&\sum_{l=1}^{k-2}\binom{k}{l}\int_0^s \gamma^{(k+1)}(t)[t^2\gamma^{(l+2)}+(2l+1)t\gamma^{(l+1)}+l^2\gamma^{(l)}](\frac{d}{dt})^{k-l}\eta(t\gamma')\ dt\\
&=\sum_{l=1}^{k-2}\binom{k}{l}\int_0^s \gamma^{(k+1)}(t)[t(\frac{d}{dt})^{(l+1)}(t\gamma'(t))+l(\frac{d}{dt})^{(l)}(t\gamma'(t))](\frac{d}{dt})^{k-l}\eta(t\gamma')\ dt\\
&\le \sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}\sum_{l=1}^{k-2}\binom{k}{l}\{s_0M^{l-1}\frac{l!^2}{l+1}+M^{l-2}(l-1)!^2\}\cdot 16M^{k-l-1}\frac{(k-l-1)!^2}{k-l}\\
&=\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}\cdot 16(s_0+\frac{1}{M})M^{k-2}k!\sum_{l=1}^{k-2}\frac{(l+1)!(k-l-1)!}{(l+1)^2(k-l)^2}\\
&\le 8\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-2}\frac{k!(k-1)!}{(k+1)^2}\sum_{l=1}^{k-2}(\frac{1}{l+1}+\frac{1}{k-l-1})^2\\
&\le 32\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-2}\frac{k!(k-1)!}{(k+1)^2}\sum_{l=1}^{\infty}\frac{1}{l^2}\\
&= C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-2}\frac{k!(k-1)!}{(k+1)^2}
\end{aligned}$$

□

Lemma A.5. *Let $k\ge 2$ and assume the induction hypothesis* (A.2) *for $j=2,\dots,k$. Then*

$$\begin{aligned}
&\int_0^s \gamma^{(k+1)}(t^2\gamma''+t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma')+8t\gamma'\eta^2(t\gamma')(t\gamma^{(k+1)}+k\gamma^{(k)})\}\ dt\\
&\le C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}\ M^{k-4}\frac{k!(k-2)!}{(k-1)^2}\ ,
\end{aligned}\tag{A.9}$$

where $C=C(H)$.

Proof. In the following we use the induction hypothesis and Proposition A.2 part ii.

$$\begin{aligned}
&\int_0^s \gamma^{(k+1)}(t^2\gamma''+t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma')+8t\gamma'\eta^2(t\gamma')(t\gamma^{(k+1)}+k\gamma^{(k)})\}\ dt\\
&\le \|\gamma^{(k+1)}\|_{L^2[0,s]}\sqrt{s_0}\cdot CM^{k-4}\frac{k!(k-2)!}{(k-1)^2}\\
&= C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}\ M^{k-4}\frac{k!(k-2)!}{(k-1)^2}
\end{aligned}\tag{A.10}$$

□

Finally we estimate the remaining terms in (A.6).

Lemma A.6. *Let $k \geq 2$ and assume the induction hypothesis* (A.2) *for $j = 2, \ldots, k$. Then for a constant $C = C(H)$,*

$$\begin{aligned}
&|-32k\int_0^s [t\gamma'(t^2\gamma''+t\gamma')]\eta^2\gamma^{(k)}\gamma^{(k+1)}\ dt \\
&+\int_0^s [4k^2\eta(t\gamma')+4k(2k-1)t\frac{d}{dt}\eta(t\gamma')-2nk]\gamma^{(k+1)}\gamma^{(k)}\ dt \\
&+4k(k-1)^2\int_0^s \gamma^{(k+1)}\gamma^{(k-1)}\frac{d}{dt}\eta(t\gamma')\ dt\,| \\
&\leq C\|\gamma^{(k+1)}\|_{L^2[0,s]}(k^2\|\gamma^{(k)}\|_{L^2[0,s]}+k^3\|\gamma^{(k-1)}\|_{L^2[0,s]}) \\
&\leq C\|\gamma^{(k+1)}\|_{L^2[0,s]}(M^{k-3}\frac{k^2(k-2)!^2}{k-1}+M^{k-4}\frac{k^3(k-3)!^2}{k-2}) \\
&=\frac{C}{M}\{M^{k-2}\frac{(k-1)!^2}{k}\|\gamma^{(k+1)}\|_{L^2[0,s]}\}\ .
\end{aligned} \tag{A.11}$$

To complete the proof of Theorem A.1 we need to show

$$\|\gamma^{(k+1)}\|_{L^2[0,s]}+|(\frac{d}{ds})^k(s\gamma'(s))|\leq M^{k-2}\frac{(k-1)!^2}{(k)},\text{ on }[0,s_0]. \tag{A.12}$$

Using Proposition A.2 and Lemmas A.3, A.4, A.5 and A.6 to estimate the error terms in (A.6), we find

$$\begin{aligned}
&\|\gamma^{(k+1)}\|_{L^2[0,s]}+|(\frac{d}{ds})^k(s\gamma'(s))| \\
&\leq C\sqrt{s_0}\{M^{k-2}\frac{k!(k-1)!}{(k+1)^2}+M^{k-4}\frac{k!(k-2)!}{(k-1)^2}\} \\
&+\frac{C}{M}M^{k-2}\frac{(k-1)!^2}{k}+C_2M^{k-3}(k-2)!^2 \\
&\leq M^{k-2}\frac{(k-1)!^2}{k}\{C\sqrt{s_0}+C\sqrt{s_0}\frac{1}{M^2}\} \\
&\leq C\sqrt{s_0}M^{k-2}\frac{(k-1)!^2}{k}\leq M^{k-2}\frac{(k-1)!^2}{k}
\end{aligned} \tag{A.13}$$

for s_0 small and M large. This completes the proof of Theorem A.1.

Appendix B. Faà di Bruno formulas

In this section we recall the Faà di Bruno formulas for the higher derivatives of composite functions of one variable. An excellent reference is the interesting and informative survey article of W. P. Johnson [9]. The usual version of the formula is

(B.1)
$$(\frac{d}{ds})^m f(g(s)) = \sum_{l=1}^{m} \sum_{b\in A_{m,l}} \frac{m!}{b_1!b_2!\cdots b_m!} f^{(l)}(g(s)(\frac{g'(s)}{1!})^{b_1}(\frac{g''(s)}{2!})^{b_2}\cdots(\frac{g^{(m)}(s)}{m!})^{b_m}$$

where the second summation is extended over all admissible partitions

$$A_{m,l} = \{b = (b_1, \ldots, b_m) \in \mathbb{N}^m : b_1 + b_2 + \ldots + b_m = l,\ b_1 + 2b_2 + \ldots + mb_m = m\}\ .$$

An alternate version (attributed by Johnson to J. F. C. Tiburce Abadie) is

$$(\frac{d}{ds})^m f(g(s)) = \sum_{l=1}^{m} \binom{m}{l} f^{(l)}(g(s)) \left\{(\frac{d}{dh})^{m-l}(\frac{g(s+h)-g(s)}{h})^l\right\}_{h=0} \tag{B.2}$$

The following corollary was rediscovered by Yamanaka [22].

Corollary B.1.

$$\begin{aligned}(\frac{d}{ds})^m f(g(s)) &= \sum_{l=1}^{m} \binom{m}{l} f^{(l)}(g(s)) \left\{(\frac{d}{dh})^{m-l}[\int_0^1 g'(s+h\theta)\ d\theta]^l\right\}_{h=0} \\ &= f'(g(s))g^{(m)}(s) + \sum_{l=2}^{m} \binom{m}{l} f^{(l)}(g(s)) \left\{(\frac{d}{dh})^{m-l}[\int_0^1 g'(s+h\theta)\ d\theta]^l\right\}_{h=0}\end{aligned} \tag{B.3}$$

We apply these formulas to our situation. Let $\eta(x) = \frac{1}{1+4x^2}$, then

$$(\frac{d}{dx})^p \eta(x) = (-2)^p p! \Lambda_{p+1}(x),\ \Lambda_{p+1}(x) := \frac{\sin[(p+1)\arcsin\frac{1}{\sqrt{1+4x^2}}]}{(1+4x^2)^{\frac{p+1}{2}}} \tag{B.4}$$

Hence also,

$$(\frac{d}{dx})^{p+1} \arctan 2x = 2(\frac{d}{dx})^p \eta(x) = (-1)^p 2^{p+1} p! \Lambda_{p+1}(x) \tag{B.5}$$

Therefore,

(B.6)

$$(\frac{d}{ds})^m\eta(s\gamma') = \sum_{l=1}^{m}\sum_{b\in A_{m,l}}\frac{m!}{b_1!b_2!\cdots b_m!}(-2)^l l!\Lambda_{l+1}(s\gamma')\cdot(\frac{s\gamma''+\gamma'}{1!})^{b_1}\cdots(\frac{s\gamma^{(m+1)}+m\gamma^{(m)}}{m!})^{b_m}$$

$$= -8s\gamma'\eta^2(s\gamma^{(m+1)}(s)+m\gamma^{(m)}(s)) + \sum_{l=2}^{m}\sum_{b\in A_{m,l}}\frac{m!}{b_1!b_2!\cdots b_{m-1}!}(-2)^l l!\Lambda_{l+1}(s\gamma')$$

$$\cdot(\frac{s\gamma''+\gamma'}{1!})^{b_1}\cdots(\frac{s\gamma^{(m)}+(m-1)\gamma^{(m-1)}}{(m-1)!})^{b_{m-1}} .$$

$$\begin{aligned}(\frac{d}{ds})^m \arctan(2s\gamma'(s)) = \sum_{l=1}^{m}\sum_{b\in A_{m,l}}\frac{m!}{b_1!b_2!\cdots b_m!}(-1)^{l-1}2^l(l-1)!\Lambda_l(s\gamma')\\ \cdot(\frac{s\gamma''+\gamma'}{1!})^{b_1}\cdots(\frac{s\gamma^{(m+1)}+m\gamma^{(m)}}{m!})^{b_m}\end{aligned} \tag{B.7}$$

$$\begin{aligned}&(\frac{d}{ds})^k(s\frac{d}{ds})\arctan(2s\gamma'(s)(0) = (s\frac{d}{ds}+k)(\frac{d}{ds})^k\arctan(2s\gamma'(s)(0)\\ &= k(\frac{d}{ds})^k\arctan(2s\gamma'(s)(0)\\ &= k\sum_{l=1}^{k}(-1)^{l-1}2^l(l-1)!\sum_{b\in A_{k,l}}\frac{k!}{b_1!b_2!\cdots b_k!}\cdot(\frac{\gamma'(0)}{1!})^{b_1}\cdots(\frac{\gamma^{(k)}(0)}{(k-1)!})^{b_k}\end{aligned} \tag{B.8}$$

Similarly,

$$(\frac{d}{ds})^k H(\gamma)(0) = \sum_{l=1}^{k}H^{(l)}(0)\sum_{b\in A_{k,l}}\frac{k!}{b_1!\cdots b_k!}\cdot(\frac{\gamma'(0)}{1!})^{b_1}(\frac{\gamma''(0)}{2!})^{b_2}\cdots(\frac{\gamma^{(k)}(0)}{k!})^{b_k} . \tag{B.9}$$

Lemma B.2.

(B.10)

$$(\frac{d}{ds})^n\eta(s\gamma') = \sum_{l=1}^{n}\binom{n}{l}(-2)^l l!\Lambda_{l+1}(s\gamma'(s))\left\{(\frac{d}{dh})^{n-l}[\int_0^1\gamma''(s+h\theta)\ d\theta+\gamma'(s+h)]^l\right\}_{h=0}$$

$$= -8s\gamma'(s)\eta^2(s\gamma'(s))(\frac{d}{ds})^n[s\gamma']$$

$$+\sum_{l=2}^{n}\binom{n}{l}(-2)^l l!\Lambda_{l+1}(s\gamma'(s))\left\{(\frac{d}{dh})^{n-l}[\int_0^1\gamma''(s+h\theta)\ d\theta+\gamma'(s+h)]^l\right\}_{h=0} .$$

Proof. By Corollary B.1 and (B.4) it suffices to calculate $\int_0^1 g'(s+h\theta)\,d\theta$ for $g(s) = s\gamma'(s)$, $g'(s) = \gamma'(s) + s\gamma''(s)$, Hence

$$\int_0^1 g'(s+h\theta)\,\theta = \int_0^1 \{(s+h\theta)\gamma''(s+h\theta)+\gamma'(s+h\theta)\}\,d\theta = \int_0^1 s\gamma''(s+h\theta)\,d\theta+\gamma'(s+h)$$

after an integration by parts. □

Lemma B.3. *Assume* $|(\frac{d}{ds})^j u(s)| \le M^{j-1}\frac{j!^2}{(j+1)^2}$ *on* $[0, s_0]$ *for* $0 \le j \le m$. *Then*

$$|(\frac{d}{ds})^j (u(s))^l| \le (\frac{C}{M})^{l-1} M^{j-1}\frac{j!^2}{(j+1)^2} \text{ on } [0, s_0] \tag{B.11}$$

for $0 \le j \le m$ *where* l *is a positive integer and* C *is a universal constant.*

Proof. We use induction on l. The case $l = 1$ is trivial. Set $u_l(s) = u^l(s)$ and suppose that for (B.11) holds for some positive integer l. Then

$$\begin{aligned}
|(\frac{d}{ds})^j u_{l+1}(s)| &= |\sum_{i=0}^{j}\binom{j}{i}(\frac{d}{ds})^i u_l(s)(\frac{d}{ds})^{j-i}u(s)|\\
&\le \sum_{i=0}^{j}\binom{j}{i}(\frac{C}{M})^{l-1}M^{i-1}\frac{i!^2}{(i+1)^2}M^{j-i-1}\frac{(j-i)!^2}{(j-i+1)^2}\\
&\le (\frac{C}{M})^{l-1}M^{j-2}m!\sum_{i=0}^{j} i!(j-i)!\frac{1}{(j+2)^2}(\frac{1}{i+1}+\frac{1}{j-i+1})^2\\
&\le (\frac{C}{M})^{l}M^{j-1}\frac{j!^2}{(j+1)^2}\frac{4}{C}\sum_{i=0}^{\infty}\frac{1}{i^2} \le (\frac{C}{M})^{l}M^{j-1}\frac{j!^2}{(j+1)^2}\ ,\\
&\text{for } C = 4\sum_{i=0}^{\infty}\frac{1}{i^2}\ .
\end{aligned} \tag{B.12}$$

□

Lemma B.4. *Assume that*

$$|(\frac{d}{ds})^{j+1}(s\gamma'(s)| \le M^{j-1}\frac{j!^2}{j+1} \tag{B.13}$$

on $[0, s_0]$ *for* $j \ge 1$ *and define*

$$u(h;s) = \int_0^1 s\gamma''(s+h\theta)\ d\theta + \gamma'(s+h),\ \ u_l(h;s) = (u(h;s))^l\ .$$

Then

$$|(\frac{d}{dh})^j u_l(h;s)|_{h=0}| \le (\frac{C}{M})^{l-1} M^{j-1} \frac{j!^2}{(j+1)^2} \ . \tag{B.14}$$

Proof. Note that

$$\begin{aligned}
(\frac{d}{dh})^j u(h;s) &= \int_0^1 s\theta^j \gamma^{(j+2)}(s+h\theta)\ d\theta + \gamma^{(j+1)}(s+h) \\
&= \int_0^1 \theta^j (s+h\theta)\gamma^{(j+2)}(s+h\theta)\ d\theta - \int_0^1 \theta^{j+1}\frac{d}{d\theta}[\gamma^{(j+1)}(s+h\theta)]\ d\theta + \gamma^{(j+1)}(s+h) \\
&= \int_0^1 \theta^j \{(s+h\theta)\gamma^{(j+2)} + (j+1)\gamma^{(j+1)}(s+h\theta)\}\ d\theta \\
&= \int_0^1 \theta^j (\frac{d}{ds})^{j+1}[s\gamma'(s)](s+h\theta)\ d\theta \ .
\end{aligned} \tag{B.15}$$

Therefore (B.15) and assumption (B.13) imply

$$|(\frac{d}{dh})^j u(h;s)| \le M^{j-1}\frac{j!^2}{j+1}\int_0^1 \theta^j\ d\theta = M^{j-1}\frac{j!^2}{(j+1)^2} \ .$$

Hence we can apply Lemma B.3 to conclude (B.14).

□

References

[1] U. Abresch, Isoparametric hypersurfaces with four or six distinct principal curvatures. Necessary conditions on the multiplicities, *Math. Ann.*, **264** (1983), 283–302.

[2] E. Cartan, Sur des familles remarquables d' hypersurfaces isoparamétriques dans les espaces sphèriques, *Math. Z.* **45** (1939), 335–367.

[3] T. E. Cecil and P. J. Ryan,Tight and Taut Immersions of Manifolds, *Res. Notes Math* vol. 107, Pitman, Boston, MA (1985)

[4] T. H. Colding and W. P. Minicozzi, Smooth compactness of self-shrinkers, *Comment. Math. Helv.* **87** (2012), 463–475.

[5] D. Ferus, H. Karcher and H.F. Münzner, Cliffordalgebren and neue isoparametrische Hyperflaechen, *Math. Zeit.* **177** (1981), pp. 479–502

[6] B. Guan and J. Spruck, Boundary-value problems on $\mathbb{S}^n$ for surfaces of constant Gauss curvature, *Ann. of Math.* **138** (1993), 601–624.

[7] W. Y. Hsiang and H. B. Lawson, Minimal submanifolds of low cohomogenity, *J. Diff. Geom.* **5** (1971), 1–38.

[8] G. Huisken, Asymptotic behavior of the singularities of the mean curvature flow, *J. Diff. Geom.* **20** (1990), 285–299.

[9] W. P. Johnson, The Curious History of Faà di Bruno's Formula *The American Mathematical Monthly* **109 2002**, 217–234.

[10] N. Kapouleas, S. J. Kleene and N. M. Møller, Mean curvature self-shrinkers of high genus: non-compact examples, arXiv:1106.5454v3, to appear in *J. Reine Angew. Math.*

[11] S. Kleene and N. Møller, Self-shrinkers with a rotational symmetry, *Trans. Amer. Math. Soc.* **366** (2014), 3943–3963.

[12] H.F. Münzner, Isoparametrische Hyperflächen in Sphären I, *Math. Ann* **251** (1980), 57–71.

[13] H.F. Münzner, Isoparametrische Hyperflächen in Sphären II, *Math. Ann* **256** (1981), 215–232.

[14] X.H. Nguyen, Construction of complete embedded self-similar surfaces under mean curvature flow. Part III., *Duke Math. J.* **163** (2014), 2023–2056.

[15] X.H. Nguyen, Construction of complete embedded self-similar surfaces under mean curvature flow. Part II., *Adv. Differential Equations* **15** (2010), 503–530.

[16] X.H. Nguyen, Construction of complete embedded self-similar surfaces under mean curvature flow. Part I., *Trans. Amer. Math. Soc.* **361** (2009), 1683–1701.

[17] K. Nomizu, Some results in É. Cartan's theory of isoparametric families of hypersurfaces, *Bull. Amer. Math. Soc* **79** (1973), 1184–1188.

[18] Ozeki, M. Takeuchi, On some types of isoparametric hypersurfaces in spheres I, *Tôhoku Math. J.*, **27** (1975), 515–559.

[19] H. Ozeki, M. Takeuchi, On some types of isoparametric hypersurfaces in spheres II, *Tôhoku Math. J.*, **28** (1976), 755

[20] R. Takagi and T. Takahashi, On the principal curvatures of homogeneous hypersurfaces in a sphere, Diff. Geom. in honor of K. Yano,

[21] L. Wang, Uniqueness of self-similar shrinkers with asymptotically conical ends, *J. Amer. Math. Soc.* **27** (2014), 613–638.

[22] T. Yamanaka, A new higher order chain rule and Gevrey class, *Ann. Global Anal. Geom* **7**(1989), 179–203.

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