

# On complete mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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## 1 Introduction

In this paper we study complete constant mean curvature  $\frac{1}{2}$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ . Recall that the famous half-space theorem of Hoffman-Meeks says that a properly immersed minimal surface in  $\mathbb{R}^3$  that lies in a half-space must be a plane. Our first result is an analogous result for a complete properly embedded constant mean curvature  $\frac{1}{2}$  surface  $\Sigma$  in  $\mathbb{H}^2 \times \mathbb{R}$  (possibly with compact boundary).

**Theorem 1.1.** *Let  $\Sigma$  be a properly embedded constant mean curvature  $\frac{1}{2}$  surface in  $\mathbb{H}^2 \times \mathbb{R}$ . Suppose  $\Sigma$  is asymptotic to a horocylinder  $C$ , and on one side of  $C$ . If the mean curvature vector of  $\Sigma$  has the same direction as that of  $C$  at points of  $\Sigma$  converging to  $C$ , then  $\Sigma$  is equal to  $C$  (or a subset of  $C$  if  $\partial\Sigma \neq \emptyset$ ).*

Our second main result concerns complete  $H = 1/2$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  transverse to the vertical Killing field  $Z = \frac{\partial}{\partial t}$ . We prove such surfaces are entire graphs.

**Theorem 1.2.** *Let  $\Sigma$  be a complete immersed surface in  $\mathbb{H}^2 \times \mathbb{R}$  of constant mean curvature  $H = 1/2$ . If  $\Sigma$  is transverse to  $Z$  then  $\Sigma$  is an entire vertical graph over  $\mathbb{H}^2$ .*

Finally, we apply Theorem 1.2, together with work of Fernandez-Mira [4], and Wan-Au [3] and Wan [10], to understand such entire graphs.

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**Theorem 1.3.** *To each quadratic holomorphic differential on  $\mathbb{C}$  or the unit disk, one associates an entire  $H = 1/2$  graph.*

Our proof of Theorem 1.1 while in the same spirit as that of [5] is technically more complicated as we must prove the existence of continuous families of catenoid like  $H = \frac{1}{2}$  surfaces that converge nicely to a horocylinder. For this purpose it is convenient to use the half-space model  $\mathbb{H}^2 = \{(x, y) : y > 0\}$  with metric  $ds^2 = \frac{dx^2 + dy^2}{y^2}$  so that the product space  $\mathbb{H}^2 \times \mathbb{R}$  with coordinates  $(x, y, t)$  is endowed with the metric  $d\sigma^2 = ds^2 + dt^2$ . Following Sa Earp [8], we will consider “horizontal” graphs  $y = g(x, t)$ ,  $g > 0$  with  $H = \frac{1}{2}$ . Some standard computations show that on  $S = \text{graph } g$ ,

**Lemma 1.4.** *The coefficients of the metric are given by*

$$g_{11} = \frac{1}{g^2}(1 + g_x^2), \quad g_{12} = \frac{g_x g_t}{g^2}, \quad g_{22} = \frac{g^2 + g_t^2}{g^2}$$

$$g^{11} = \frac{g^2}{W^2}(g^2 + g_t^2), \quad g^{12} = -\frac{g^2}{W^2}(g_x g_t), \quad g^{22} = \frac{g^2}{W^2}(1 + g_x^2)$$

*The coefficients of the second fundamental form are*

$$b_{11} = \frac{1}{W}(g_{xx} + \frac{1 + g_x^2}{g}), \quad b_{12} = \frac{g_{xt}}{W}, \quad b_{22} = \frac{1}{W}(g_{tt} - \frac{g_t^2}{g})$$

*In addition*

$$|\nabla_S g|^2 = g^2(1 - \frac{g^2}{W^2})$$

*The mean curvature equation is given by*

$$1 = g^{ij}b_{ij} = \frac{g^2}{W^3}\{(g^2 + g_t^2)g_{xx} - 2g_x g_t g_{xt} + (1 + g_x^2)g_{tt} + g(1 + g_x^2)\} \quad (1)$$

*and finally we have*

$$\Delta_S g = \frac{g^2}{W}(1 - \frac{g}{W} + \frac{gg_x^2}{W}) > 0 \quad (2)$$

$$\Delta_S \frac{1}{g} = \frac{W - g}{gW} + \frac{g_t^2}{gW^2} > 0 \quad (3)$$

*where  $W^2 = g^2(1 + g_x^2) + g_t^2$ .*

In particular, we see from (1) that  $g$  satisfies the strange looking equation

$$(g^2 + g_t^2)g_{xx} - 2g_x g_t g_{xt} + (1 + g_x^2)g_{tt} = -g(1 + g_x^2) + \frac{W^3}{g^2} \quad (4)$$

Also note the somewhat surprising result that both  $g$  and  $\frac{1}{g}$  are subharmonic on  $S$  which will play an important role in the proofs.

We remark that the mean curvature vector of  $S$  is given by

$$\vec{H} = \frac{1}{2W}(-g^2 g_x, g^2, -g_t)$$

and that the constant solutions  $g = \tau > 0$  correspond to the horocylinders  $C(\tau) = \{(x, y, t) : y = \tau\}$ . The induced metric on each  $C(\tau)$  is complete and isometric to the flat  $\mathbb{R}^2$ , and the mean curvature of each  $C(\tau)$  is  $H = 1/2$ . Our catenoid like horizontal graphs are given by the following theorem.

**Theorem 1.5.** *Let  $U$  be the annulus  $U = B_{R_2} \setminus B_{R_1}$  with  $R_2 \geq 2R_1$ . Then for  $\varepsilon > 0$  sufficiently small (depending only on  $R_1$ ), there exists constant mean curvature  $H = 1/2$  horizontal graphs  $g^+$  and  $g^-$  satisfying (4) in  $U$  with Dirichlet boundary data  $g^\pm = 1 \pm \varepsilon$  on  $\partial B_{R_1}$ ,  $g^\pm = 1$  on  $\partial B_{R_2}$ . Moreover  $g^\pm$  is unique and varies continuously with the parameters  $\varepsilon, R_1, R_2$  and  $g^\pm$  tends to  $1 \pm \varepsilon$  uniformly on compact subsets as  $R_2$  tends to  $\infty$ .*

Assuming Theorem 1.5 the proof of Theorem 1.1 goes as follows. After an isometry, we can assume that there is a sequence of point  $p_i = (x_i, y_i, t_i) \in \Sigma$  with  $y_i \rightarrow 1$  and  $\langle \vec{H}, \frac{\partial}{\partial y} \rangle > 0$ . In the first case either  $\Sigma$  is contained in  $C(1)$  or is contained in  $y > 1$ . For  $\varepsilon > 0$  small we consider the slab  $S^+$  bounded by  $C(1)$  and  $C(1 + \varepsilon)$ . Then by the maximum principle  $\Sigma^+ = \Sigma \cap S^+$  has a non compact component with boundary  $\partial \Sigma^+ \subset C(1 + \varepsilon)$ . Let  $D(\tau, R)$  denote the disk in  $C(\tau)$  defined by  $D(\tau, R) = \{(x, \tau, t) : x^2 + t^2 \leq R^2\}$ . We can find a disk  $D(1, 3R_1)$  such that  $D(1, 3R_1) \times [1, 1 + \varepsilon] \cap S^+ = \{\emptyset\}$ . Let  $\Gamma(1, R) = \partial D(1, R)$ . Then for each  $R \geq 2R_1$ , there is a horizontal graph  $g_R^+$  bounded by  $\Gamma(1 + \varepsilon, R_1) \cup \Gamma(1, R)$  in the slab  $S^+$ . By the maximum principle, this family of graphs foliates the unbounded component of  $S^+ \setminus \text{graph}(g_{2R_1}^+)$  but converges to  $C(1 + \varepsilon)$ . Thus there is a first point of contact at an interior point. Since the mean curvature vectors are pointing up, this violates the maximum principle and  $\Sigma^+$  cannot exist. In the second case we redo exactly

the same argument exchanging the roles of  $C(1 - \varepsilon)$  and  $C(1 + \varepsilon)$ .

We will prove the existence part of Theorem 1.5 in section 2 using the Schauder fixed point theorem. Because of the complicated dependence of equation (4) on  $g$ , the uniqueness of the solutions is not obvious. This will be proved in section 3 from which the continuous dependence follows by standard elliptic theory. The proof of Theorem 1.2 is given in section 4 using compactness and analytic continuation arguments. The final step of the proof uses the special case of Theorem 1.1 when  $\Sigma$  is a horizontal graph. We give an independent simple proof for this case. Finally in section 5 we describe the construction of Fernandez and Mira [4] of entire  $H = \frac{1}{2}$  vertical graphs starting from holomorphic quadratic differentials on  $\mathbb{C}$  or the unit disk  $U$ .

## 2 The existence part of Theorem 1.5.

Let  $U = B_{R_2} \setminus B_{R_1}$  be an annulus with  $R_2 \geq 4R_1$  and fix  $h = 1 \pm \frac{\varepsilon}{\log \frac{R_2}{R_1}} \log \frac{R_2}{r}$  where  $r^2 = x^2 + t^2$ . We expect the solution  $g$  to be close to  $h$  so we define the weighted  $C^{2+\alpha}$  norm

$$|v|_{2,\alpha;U}^* = \sup_X \{ |v(X)| + r(X)|Dv(X)| + r^2(X)|D^2v(X)| + r_X^{2+\alpha}[D^2v]_{\alpha;X} \}$$

where  $X = (x, t)$  and  $[D^2v]_{\alpha;X}$  is the Hölder coefficient of  $D^2v$  at  $X$ .

**Definition 2.1.** *We say  $g$  is an admissible solution of (4) if  $g \in \mathcal{A}_\varepsilon$  where*

$$\mathcal{A}_\varepsilon = \{g \in C^{2,\alpha}(U) , g = h \text{ on } \partial U : |g - h|_{2,\alpha;U}^* \leq \sqrt{\varepsilon}\}$$

We note that  $\mathcal{A}_\varepsilon$  is a convex and compact subset of the Banach space  $\mathcal{B} = C^{2,\beta}(U)$  ,  $\beta < \alpha$ . We will reformulate our existence problem as a fixed point of a continuous operator  $T : \mathcal{A}_\varepsilon \rightarrow \mathcal{A}_\varepsilon$  by rewriting equation (4) in the form

$$(g^2 + g_t^2)g_{xx} - 2g_x g_t g_{xt} + (1 + g_x^2)g_{tt} + g g_x^2 - \left(\frac{W}{g^2} + \frac{1}{W + g}\right)(g^2 g_x^2 + g_t^2) = 0 \quad (5)$$

**Remark 2.2.** *Note that (5) implies that any solution  $g^\pm$  solving the Dirichlet problem of Theorem 1.5 satisfies  $1 - \varepsilon \leq g^- \leq 1$  and  $1 \leq g^+ \leq 1 + \varepsilon$  in  $U$ .*

We now define the operator  $w = Tg$  as the solution of the linear Dirichlet problem

$$\begin{aligned} L_g w &:= aw_{xx} + 2bw_{xt} + cw_{tt} + dw_x + ew_t = 0 \text{ in } U \\ w &= h \text{ on } \partial U \end{aligned} \quad (6)$$

where  $a = g^2 + g_t^2$ ,  $b = -g_x g_t$ ,  $c = 1 + g_x^2$ ,  $d = gg_x - g^2(\frac{W}{g^2} + \frac{1}{W+g})g_x$  and  $e = -(\frac{W}{g^2} + \frac{1}{W+g})g_t$ . Note that for  $g \in \mathcal{A}_\varepsilon$ , if  $L_g u = f$  in  $D \subset \bar{U}$  where  $D$  is “of scale  $R$ ”, ( i.e. if  $X \in D$ , then  $c_1 R \leq |X| \leq c_2 R$  for uniform constants  $c_1, c_2$ ) then  $\tilde{u} = u(RX)$  satisfies

$$\tilde{L}\tilde{u} = \tilde{a}\tilde{u}_{xx} + 2\tilde{b}\tilde{u}_{xt} + \tilde{c}\tilde{u}_{tt} + R\tilde{d}\tilde{u}_x + R\tilde{e}\tilde{u}_t = R^2\tilde{f} \text{ in } \tilde{D} \quad (7)$$

where  $\tilde{D}$  is of scale 1 and  $\tilde{a}(X) = a(RX)$ ,  $\tilde{b}(X) = b(RX)$ , etc. Hence for  $\varepsilon$  sufficiently small,  $\tilde{L}$  is uniformly close to  $\Delta$  with Hölder continuous coefficients.

**Proposition 2.3.** *Let  $w = Tg$  for  $g \in \mathcal{A}_\varepsilon$ . Then for  $\varepsilon$  sufficiently small,  $w \in \mathcal{A}_\varepsilon$ .*

**Proof.** Set  $u = w - h$ ; then

$$L_g u = [(1 - g^2 - g_t^2)h_{xx} + 2g_x g_t h_{xt} - g_x^2 h_{tt} - dh_x - eh_t] := f \quad (8)$$

By the maximum principle,  $1 \leq w \leq 1 + \varepsilon$  (or  $1 - \varepsilon \leq w \leq 1$ ) so  $|u| \leq \varepsilon$ .

We now write  $U = U_1 \cup U_2 \cup U_3$  where

$$U_1 = \{X : R_1 \leq |X| \leq \frac{3}{2}R_1\},$$

$$U_2 = \{X : \frac{3}{2}R_1 < |X| < \frac{3}{4}R_2\},$$

$$U_3 = \{X : \frac{3}{4}R_2 \leq |X| \leq R_2\}.$$

Fix  $Y \in U$ . If  $Y \in U_1$ , then  $B_{\frac{R_1}{4}}(R_1\frac{Y}{|Y|}) \cap U \subset B_{\frac{R_1}{2}}(R_1\frac{Y}{|Y|}) \cap U \subset U_1$ . If  $Y \in U_3$ , then  $B_{\frac{R_2}{4}}(R_2\frac{Y}{|Y|}) \cap U \subset B_{\frac{R_2}{2}}(R_2\frac{Y}{|Y|}) \cap U \subset U_3$ . Finally if  $Y \in U_2$ , then  $B_{\frac{R}{16}}(Y) \subset B_{\frac{R}{8}}(Y) \subset U$  for  $R = |Y|$ . So each of the three domains is of scale  $R$  and we can apply Schauder interior or boundary estimates to  $\tilde{L}\tilde{u} = R^2\tilde{f}$  in  $\tilde{D}$  to obtain

$$\|\tilde{u}\|_{2,\alpha;\tilde{D}} \leq C(\|\tilde{u}\|_{0;\tilde{D}} + \|R^2\tilde{f}\|_{0,\alpha;\tilde{D}}) \leq C\varepsilon. \quad (9)$$

since from (8) follows  $\|\tilde{f}\|_{0,\alpha;\bar{D}} \leq C\varepsilon^{\frac{3}{2}}$ . Undoing the scaling gives

$$\|u\|_{2,\alpha;D}^* \leq C\varepsilon .$$

Since  $u = w - h$ , it follows that for  $\varepsilon$  small enough,  $w \in \mathcal{A}_\varepsilon$  and the proposition is proved.

We are now in a position to apply the Schauder fixed point theorem to our operator  $w = Tg$  to find a solution  $g^\pm \in \mathcal{A}$  to (5) which is equivalent to our original equation (4).

### 3 Completion of the proof of Theorems 1.5.

In this section we refine our estimates in order to prove uniqueness, continuous dependence and convergence to a constant as  $R_2 \rightarrow \infty$ .

Let  $g^\pm$  be an admissible solution of (4) and let  $\phi$  satisfy

$$\begin{aligned} \Delta_S \phi &= 0 \text{ in } U \\ \phi &= 1 \text{ on } \partial B_{R_1} \\ \phi &= 0 \text{ on } \partial B_{R_2} \end{aligned}$$

Then since  $g^\pm = 1 \pm \varepsilon\phi$  on  $\partial U$  and both  $g^\pm$  and  $\frac{1}{g^\pm}$  are subharmonic on  $S$  (see Lemma 1.4) we have

$$\frac{1}{1 \mp \frac{\varepsilon\phi}{1 \pm \varepsilon}} \leq g^\pm \leq 1 \pm \varepsilon\phi \quad (10)$$

**Proposition 3.1.**  $0 \leq 1 - \phi \leq C \frac{\log \frac{r}{R_1}}{\log \frac{R_2}{R_1}}$  where  $C$  is independent of  $R_2$ .

**Proof.** On  $U$ ,  $\phi$  satisfies the uniformly elliptic divergence form equation

$$L\phi := \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} (a^{ij} \phi_{x_j}) = 0 \quad (11)$$

where  $a^{ij} = g^{-2} W g^{ij}$  is close to  $\delta_{ij}$ . We extend  $g \equiv 1$  for  $|X| > R_2$  so  $L$  is also extended as a uniformly elliptic divergence form operator with Lipschitz continuous coefficients. We now recall the generalized Kelvin transform (see [9] p.262). Let  $Y = \frac{X}{|X|^2}$  be inversion in the unit circle mapping  $U$  to the

annulus  $\tilde{U} = \{Y : \frac{1}{R_2} \leq |Y| \leq \frac{1}{R_1}\}$  and define  $\tilde{\phi}(Y) = \phi(X)$ . Then  $\tilde{\phi}$  satisfies the uniformly elliptic divergence form equation

$$\tilde{L}\tilde{\phi} := \sum_{i,j=1}^2 \frac{\partial}{\partial y_i} (\tilde{a}^{ij} \tilde{\phi}_{y_j}) = 0 \quad (12)$$

in  $\tilde{U}$  where

$$\tilde{a}^{kl}(Y) = a^{ij}(X) \left( \delta_{ki} - 2 \frac{x_k x_i}{|X|^2} \right) \left( \delta_{lj} - 2 \frac{x_l x_j}{|X|^2} \right) \quad (13)$$

Note that because the matrix  $(\delta_{ki} - 2 \frac{x_k x_i}{|X|^2})$  is unitary, the eigenvalues of  $\tilde{a}^{kl}(Y)$  are the same as those of  $a^{ij}(X)$ .

Now let  $\tilde{G}(Y)$  be the positive Green's function for  $\tilde{L}$  in  $B_{\frac{1}{R_1}}$  with pole at the origin. Then by the maximum principle,

$$1 - \tilde{\phi} \leq \frac{\tilde{G}(Y)}{\min_{|Y|=\frac{1}{R_2}} \tilde{G}} \leq C \frac{\log \frac{1}{R_1 |Y|}}{\log \frac{R_2}{R_1}} \quad (14)$$

where  $C$  is universal. This last inequality is the classical comparison theorem of Littman, Stampacchia and Weinberger for fundamental solutions [6]. Returning to the original variables gives the desired inequality.

Combining inequality (10) and Proposition 3.1 gives

**Corollary 3.2.** *Let  $g^\pm$  be an admissible solution of (4). Then*

$$(1 + \varepsilon) \left( 1 - C\varepsilon \frac{\log \frac{r}{R_1}}{\log \frac{R_2}{R_1}} \right) \leq g^+ \leq 1 + \varepsilon$$

$$1 - \varepsilon \leq g^- \leq 1 - \varepsilon + C\varepsilon \frac{\log \frac{r}{R_1}}{\log \frac{R_2}{R_1}}$$

We use Corollary 3.2 to prove the uniqueness of admissible solutions completing the proof of Theorem 1.5. (Note that Corollary 3.2 shows that  $g^\pm$  tends to  $1 \pm \varepsilon$  uniformly on compact subsets as  $R_2$  tends to  $\infty$ .)

**Proposition 3.3.** *For  $\varepsilon = \varepsilon(R_1)$  sufficiently small, the solutions  $g^\pm$  are unique.*

**Proof.** Note that if  $g$  is a solution of (4) in  $U$  then  $g^\lambda = \frac{1}{\lambda}g(\lambda x, t)$  is also a solution in  $U^\lambda = \{(x, t) : R_1^2 \leq \lambda^2 x^2 + t^2 \leq R_2^2\}$ . To fix the ideas we give the proof of uniqueness for  $g^+$  which is straightforward. Suppose that  $g_1$  and  $g_2$  are two admissible solutions (we drop the  $+$  for convenience). Then for  $\lambda = \frac{1}{1+\varepsilon}$ ,  $g_2^\lambda < g_1$  in  $U \cap U^\lambda$ . As we increase  $\lambda$  back toward  $\lambda = 1$ ,  $g_2^\lambda < g_1$  on  $\partial(U \cap U^\lambda)$ . Thus a first contact may only occur in the interior, which is impossible by the maximum principle. Thus  $g_2 < g_1$  in  $U$ . Reversing the roles of  $g_1$  and  $g_2$  proves uniqueness.

Observe that this argument seems to have a problem when we consider  $g^-$  because it might happen that there is a first point of contact on the inner boundary of  $U \cap U^\lambda$ , i.e

$$g_2^\lambda = \frac{1-\varepsilon}{\lambda} = g_1 \text{ for some point on } \lambda^2 x^2 + t^2 = R_1^2, \quad (15)$$

We show this cannot happen. For at such a contact point,  $\lambda^2(x^2 + t^2) = R_1^2 + (\lambda^2 - 1)t^2 \leq R_1^2$  so  $r \leq \frac{R_1}{\lambda}$ . Hence by (15) and Corollary 3.2 we have

$$\frac{1-\varepsilon}{\lambda} = g_1 \leq 1 - \varepsilon + C\varepsilon \frac{\log \frac{1}{\lambda}}{\log \frac{R_2}{R_1}} \leq 1 - \varepsilon + \frac{C\varepsilon}{\log \frac{R_2}{R_1}} \left( \frac{1}{\lambda} - 1 \right)$$

Therefore,

$$1 - \varepsilon \leq \frac{C\varepsilon}{\log \frac{R_2}{R_1}} \quad (16)$$

which is impossible if  $\varepsilon$  is chosen so that  $\frac{\varepsilon}{1-\varepsilon} < \frac{\log 2}{C}$ . Thus there is no first contact on the boundary and  $g_2 > g_1$  in  $U$ . This proves uniqueness and also the continuous dependence in the parameters. In particular we can now let  $g^\pm$  be an arbitrary solution which is not necessarily admissible. We do know however (see Remark 2.2) that  $1 - \varepsilon < g^- < 1$  and  $1 < g^+ < 1 + \varepsilon$  in  $U$ . Now let  $g_2^\pm$  be our continuous family of admissible solutions. If we make  $R_2$  very large  $g_2^+ > g^+$  and  $g_2^- < g^-$  when restricted to the original annulus by Corollary 3.2. Thus decreasing  $R_2$  back to its original value shows these inequalities persist. Similarly we can decrease the parameter  $\varepsilon$  close to zero and obtain the reverse inequalities proving the general uniqueness of solutions.



## 4 The proof of Theorem 1.2

**Proof.** The mean curvature vector of  $\Sigma$  never vanishes so  $\Sigma$  is orientable. Let  $\nu$  be a unit vector field along  $\Sigma$  in  $\mathbb{H} \times \mathbb{R}$ . The function  $u = \langle \nu, Z \rangle$  is a non-zero jacobian function on  $\Sigma$ , so  $\Sigma$  is strongly stable and thus has bounded curvature. We can assume  $u > 0$  and  $\langle \nu, \vec{H} \rangle > 0$ .

Hence there is  $\delta > 0$  such that for each  $p \in \Sigma$ ,  $\Sigma$  is a graph (in exponential coordinates) over the disk  $D_\delta \subset T_p \Sigma$  of radius  $\delta$ , centered at the origin of  $T_p \Sigma$ . This graph, denoted by  $G(p)$ , has bounded geometry. The  $\delta$  is independent of  $p$  and the bound on the geometry of  $G(p)$  is uniform as well.

We denote by  $F(p)$  the surface  $G(p)$  translated to the origin  $O \in \mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$  (The translation that takes  $p$  to  $O$ ).

For  $q \in \mathbb{H}^2 \times \mathbb{R}$ , we denote by  $\Gamma_\delta(q)$  a horizontal horocycle arc of length  $2\delta$ , centered at  $q$ .

Claim 1: Let  $p_n \in \Sigma$ , satisfy  $u(p_n) \rightarrow 0$  as  $n \rightarrow \infty$  ( $T_{p_n}(\Sigma)$  are becoming vertical). There is a subsequence of  $p_n$  (which we also denote by  $\{p_n\}$ ) such that  $F(p_n)$  converges to  $\Gamma_\delta(O) \times [-\delta, \delta]$ , for some horocycle  $\Gamma_\delta(O)$ . The convergence is in the  $C^2$ -topology.

Proof of Claim 1. Choose a subsequence  $p_n$  so that the oriented tangent planes  $T_O(F(p_n))$  converge to a vertical plane  $P$ . Let  $\Gamma_\delta(O)$  be the horocycle arc through  $O$  whose curvature vector has the same direction as the curvature vector of the (limit) curvature vectors of  $F(p_n)$ .

Since the  $F(p_n)$  have bounded geometry and they are graphs over  $D_\delta(p_n) \subset T_{p_n}(F(p_n))$ , the surfaces  $F(p_n)$  are bounded horizontal graphs over  $\Gamma_\delta(O) \times [-\delta, \delta]$  for  $n$  large. Thus a subsequence of these graphs converges to an  $H = 1/2$  surface  $F$ ;  $F$  is tangent to  $\Gamma_\delta(O) \times [-\delta, \delta]$  at  $O$  and a horizontal graph over this. It suffices to show  $F = \Gamma_\delta(O) \times [-\delta, \delta]$ .

Were this not the case, then the intersection near  $O$ , of  $F$  and  $\Gamma_\delta(O) \times [-\delta, \delta]$  would consist of  $m$  smooth curves passing through  $O$ ,  $m \geq 2$ , meeting transversally at  $O$ . In a neighborhood of  $O$ , these curves separate  $F$  into  $2m$  components. Adjacent components lie on opposite sides of  $\Gamma_\delta(O) \times [-\delta, \delta]$ .

Hence in a neighborhood of  $O$  in  $F$ , the mean curvature vector of  $F$  alternates from pointing up in  $\mathbb{H}^2 \times \mathbb{R}$  to pointing down (or vice-versa), as one goes from one component to the other. But  $F(p_n)$  converges to  $F$  in the  $C^2$ -topology, so  $F(p_n)$ ,  $n$  large, would also have points where the mean curvature vector points up and down in  $\mathbb{H}^2 \times \mathbb{R}$ . This contradicts that

$F(p_n)$  is transverse to  $Z$ , and claim 1 is proved. Notice that we have proved that whenever  $F(p_n)$  converges to a local surface  $F$ ,  $F$  is necessarily some  $\Gamma_\delta(O) \times [-\delta, \delta]$ . This prove Claim 1.

Now let  $p \in \Sigma$  and assume  $\Sigma$  in a neighborhood of  $p$  is a vertical graph of a function  $f$  defined on  $B_R$ ,  $B_R$  the open ball of radius  $R$  of  $\mathbb{H}^2$ , centered at  $O \in \mathbb{H}^2$ . Denote by  $S(R)$  the graph of  $f$  over  $B_R$ . If  $\Sigma$  is not an entire graph then we let  $R$  be the largest such  $R$  so that  $f$  exists. Since  $\Sigma$  has constant mean curvature,  $f$  has bounded gradient on relatively compact subsets of  $B_R$ .

Let  $q \in \partial B_R$  be such that  $f$  does not extend to any neighborhood of  $q$  (to an  $H = 1/2$  graph).

Claim 2: For any sequence  $q_n \in B_R$ , converging to  $q$ , the tangent planes  $T_{p_n}(S(R))$ ,  $p_n = (q_n, f(q_n))$ , converge to a vertical plane  $P$ .  $P$  is tangent to  $\partial B_R$  at  $q$  (after vertical translation to height zero in  $\mathbb{H}^2 \times \mathbb{R}$ ).

Proof of Claim 2. Let  $F(n)$  denote the image of  $G(p_n)$  under the vertical translation taking  $p_n$  to  $q_n$ . Observe first, that  $T_{q_n}(F(n))$  converges to the vertical, for any subsequence of the  $q_n$ . Otherwise the graph of bounded geometry  $G(p_n)$ , would extend to a vertical graph beyond  $q$ , for  $q_n$  close enough to  $q$ . hence  $f$  would extend; a contradiction.

Now we can prove  $T_{q_n}(F_n)$  converges to the vertical plane  $P$  passing through  $q$  and tangent to  $\partial B_R$  at  $q$ . Suppose some subsequence  $q_n$  satisfies  $T_{q_n}(F_n)$  converges to a vertical plane  $Q$ ,  $Q \neq P$ ,  $q \in Q$ . By Claim 1, the  $F_n$  converge in the  $C^2$ -topology, to  $\Gamma_\delta(q) \times [-\delta, \delta]$ , where  $\Gamma_\delta(q)$  is a horocycle arc centered at  $q$ . Since  $Q \neq P$ , and  $\Gamma_\delta(q)$  is tangent to  $Q$  at  $q$ , there are points of  $\Gamma_\delta(q)$  in  $B_R$ . Such a point is the limit of points on  $F_n$ . Then the gradient of  $f$  at these points of  $F_n$  diverges, which contradicts interior gradient estimates of  $f$ . This proves Claim 2.

Now applying Claim 1 and Claim 2, we know that for any sequence  $q_n \in B_R$  converging to  $q$ , the  $F(q_n)$  converge to  $\Gamma_\delta(q) \times [-\delta, \delta]$ .

Claim 3: For any  $q_n \rightarrow q$ ,  $q_n \in B_R$ , we have  $f(q_n) \rightarrow +\infty$  or  $f(q_n) \rightarrow -\infty$ .

Proof of Claim 3. Let  $\gamma$  be a compact horizontal geodesic of length  $\varepsilon$  starting at  $q$ , entering  $B_R$  at  $q$ , and orthogonal to  $\partial B_R$  at  $q$ . Let  $C$  be the graph of  $f$  over  $\gamma$ . Notice that  $C$  has no horizontal tangents at points near  $q$  since the tangent planes of  $S(R)$  are converging to  $P$ . So assume  $f$  is increasing along  $\gamma$  as one converges to  $q$ . If  $f$  were bounded above, then  $C$

would have a finite limit point  $(q, c)$  and  $C$  would have finite length up till  $(q, c)$ . Since  $\Sigma$  is complete,  $(q, c) \in \Sigma$ . But then  $\Sigma$  has a vertical tangent plane at  $(q, c)$ ; a contradiction. This proves Claim 3.

Now choose  $q_n \in \gamma$ ,  $q_n \rightarrow q$ , and  $F(q_n)$  converges to  $\Gamma_\delta(q) \times [-\delta, \delta]$ . Let  $\Gamma$  be the horocycle containing  $\Gamma_\delta(q)$ , and parametrize  $\Gamma$  by arc length; denote  $q(s) \in \Gamma$  the point at distance  $s$  on  $\Gamma$  from  $q = q(0)$ ,  $-\infty < s < +\infty$ . Denote by  $\gamma(s)$  a horizontal geodesic arc orthogonal to  $\Gamma$  at  $q(s)$ ,  $q(s)$  the mid-point of  $\gamma(s)$ . Assume the length of each  $\gamma(s)$  is  $2\varepsilon$  and  $\cup_{s \in \mathbb{R}} \gamma(s) = N_\varepsilon(\Gamma)$  is the  $\varepsilon$ -tubular neighborhood of  $\Gamma$ .

Let  $\gamma^+(s)$  be the part of  $\gamma(s)$  on the mean convex side of  $\Gamma$ ; so  $\gamma = \gamma^+(0)$ . More precisely, the mean curvature vector of  $\Sigma$  points up in  $\mathbb{H}^2 \times \mathbb{R}$ , and  $f \rightarrow +\infty$  as one approaches  $q$  along  $\gamma$ , so  $\Gamma$  is convex towards  $B_R$ .

Claim 4: For  $n$  large, each  $F(q_n)$  is disjoint from  $\Gamma \times \mathbb{R}$ . Also, for  $|s| \leq \delta$ ,  $F(q_n) \cap \gamma^+(s)$  is a vertical graph over an interval of  $\gamma^+(s)$ .

Proof of Claim 4. Choose  $n_0$  so that for  $n \geq n_0$ ,  $C_n(s) = F(q_n) \cap (\gamma(s) \times \mathbb{R})$  is one connected curve of transverse intersection, for each  $s \in [-\delta, \delta]$ . Since the  $F(q_n)$  are  $C^2$ -close to  $\Gamma_\delta(q) \times [-\delta, \delta]$ ,  $C_n(s)$  has no horizontal or vertical tangents and is a graph over an interval in  $\gamma(s)$ .

We now show this interval is in  $\gamma^+(s) - q(s)$ . Suppose not, so  $C_n(s)$  goes beyond  $\Gamma \times \mathbb{R}$  on the concave side. Recall that  $C = \gamma \cap P^\perp$  is the graph of  $f$  and  $f \rightarrow +\infty$  as one goes up on  $C$ . We have  $p_n = (q_n, f(q_n))$ . Fix  $n \geq n_0$  and choose new points  $q_k$ ,  $k \geq n$ , so that  $f(q_{k+1}) - f(q_k) = \delta$ ; clearly  $q_k \rightarrow q$  as  $k \rightarrow \infty$ . Lift each  $C_k(s)$  to  $G(p_k)$  by the vertical translation of  $F(q_k)$  by  $f(q_k)$ . By construction,  $C_{k+1}(s)$  is the analytic continuation of  $C_k(s)$  in  $\Sigma \cap (\gamma(s) \times \mathbb{R})$ , for each  $s \in [-\delta, \delta]$ , and for all  $k \geq n+1$ . The curve  $C(s) = \cup_{k \geq n} C_k(s)$  is a vertical graph over an interval in  $\gamma(s)$ . It has points on the concave side of  $\Gamma \times \mathbb{R}$  for some  $s_0 \in [-\delta, \delta]$ . For  $s = 0$ ,  $C(0) = C$  stays on the convex side of  $\Gamma \times \mathbb{R}$ . So for some  $s_1$ ,  $0 < s_1 \leq s_0$ ,  $C(s_1)$  has a point on  $\Gamma \times \mathbb{R}$  and also inside the concave side of  $\Gamma \times \mathbb{R}$ .

But the  $F(q_k)$  converge uniformly to  $\Gamma_\delta(q) \times [-\delta, \delta]$  as  $k \rightarrow \infty$ , so the curve  $C(s_1)$  converges to  $q(s_1) \times \mathbb{R}$  as the height goes to  $\infty$ . This obliges  $C(s_1)$  to have a vertical tangent on the concave side of  $\Gamma \times \mathbb{R}$ , a contradiction. This proves Claim 4.

Now we choose an  $\varepsilon_1 < \varepsilon$  (which we call  $\varepsilon$  as well) so that  $\cup_{s \in [-\delta, \delta]} C(s)$  is a vertical graph of a function  $g$  on  $\cup_{s \in [-\delta, \delta]} (\gamma^+(s) - q(s))$ , (the  $\gamma^+(s)$  now have length  $\varepsilon_1$ );  $g$  is an extension of  $f$ .

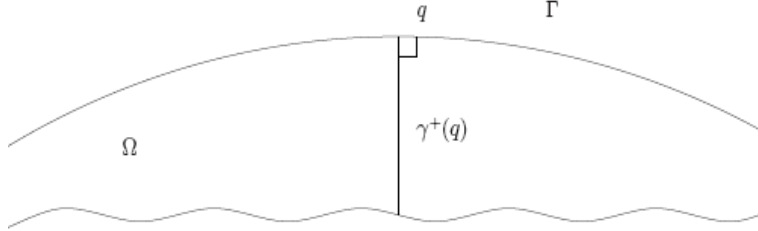


Figure 1

The graph of  $g$  on each  $\gamma^+ \times \mathbb{R}$  is the curve  $C(s)$ , and the graph of  $g$  converges to  $\Gamma_\delta(q) \times \mathbb{R}$  as the height goes to infinity.

Now we begin this process again replacing  $C$  by the curves  $C(\delta)$  and then  $C(-\delta)$ . Analytic continuation yields an extension  $h$  of  $g$  to a domain  $\Omega$  contained in the open  $\varepsilon$ -tubular neighborhood of  $\Gamma \times \mathbb{R}$ , on the convex side of  $\Gamma$ .  $\Omega$  is an open neighborhood of  $\Gamma$  in this mean convex side. The graph  $h \rightarrow \infty$  as one approaches  $\Gamma$  in  $\Omega$ ; it converges to  $\Gamma \times \mathbb{R}$  as the height goes to infinity.

Claim 5: There is an  $\varepsilon > 0$ , such that  $\Omega$  contains the  $\varepsilon$  tubular neighborhood of  $\Gamma$  on the convex side.

Proof of Claim 5. We know there is a domain  $\Omega$  on the convex side of  $\Gamma$ ;  $\Omega$  is a neighborhood of  $\Gamma$  on the convex side. Also the surface  $\Sigma$  contains a graph over  $\Omega$ , composed of curves  $C(q)$ ,  $q \in \Gamma$ , where each curve  $C(q)$  is a graph over an interval  $\gamma^+(q)$ ,  $\gamma^+(q)$  orthogonal to  $\Gamma$  at  $q$ . Also  $C(q)$  is a strictly monotone increasing graph with no horizontal tangents and  $C(q)$  converges to  $\{q\} \times \mathbb{R}^+$ , as one goes up to  $+\infty$ ; cf. figures 1 and 2.

The graph over  $\Omega$  is converging uniformly to  $\Gamma \times \mathbb{R}^+$  as one goes up.

Now suppose that for some  $q \in \Gamma$ ,  $\gamma^+(q)$  is of length less than  $\varepsilon$ . Then  $C(q)$  diverges to  $-\infty$  as one approaches the end-point  $\tilde{q}$  of  $\gamma^+(q)$ ,  $\tilde{q} \neq q$ ; cf. figure 2.

The previous discussion where we showed the graph over  $\Omega$  exists and converges to  $\Gamma \times \mathbb{R}^+$ , now applies to show that there is a horocycle  $\tilde{\Gamma}$  passing through  $\tilde{q}$ ,  $C(q)$  converges to  $\{\tilde{q}\} \times \mathbb{R}^-$  as one tends to  $\tilde{q}$  on  $\gamma^+(q)$ . Also a

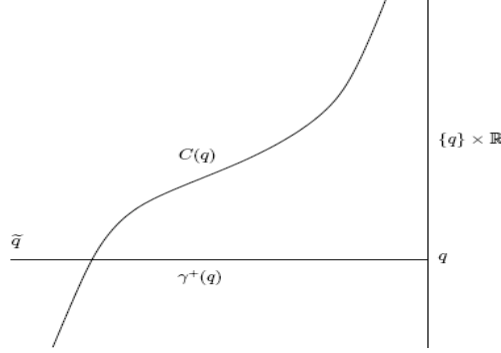


Figure 2

$\delta$ -neighborhood of  $C(q)$  in  $\Sigma$ , converges uniformly to  $\tilde{\Gamma}_\delta(\tilde{q}) \times \mathbb{R}^-$ , as one goes down to  $-\infty$ . We know this  $\delta$ -neighborhood of  $C(q)$  IN  $\Sigma$ , converges uniformly to  $\Gamma_\delta(q) \times \mathbb{R}^+$ , as one goes up to  $+\infty$ .

For each  $q(s) \in \Gamma$ , a distance  $s$  from  $q$  on  $\Gamma$ ,  $|s| \leq \delta$ , the curve  $C(q(s))$  converges uniformly to some  $\{\tilde{q}(\tilde{s})\} \times \mathbb{R}^-$ , as one goes down to  $-\infty$ . By analytic continuation of the  $\delta$ -neighborhoods, one continues this process along  $\gamma$ .

If  $\Gamma \cap \tilde{\Gamma} = \emptyset$ , then the process continues along all  $\Gamma$  and  $\Omega$  is the region bounded by  $\Gamma \cup \text{Gamma}$ , has the same length.

So we can assume  $\Gamma \cap \tilde{\Gamma} = \{p\}$ . Consider the curves  $C(q(s))$ , as  $q(s)$  goes from  $q$  to  $p$  along  $\Gamma$ . They are graphs that become vertical both at  $+\infty$  and  $-\infty$ . Hence the graphs  $C(q(s))$  become vertical at every point as  $q(s) \rightarrow p$ ; cf. figure ??.

Consider the point of  $C(q(s))$  at height 0 in  $\mathbb{H}^2 \times \mathbb{R}$ . As  $q(s) \rightarrow p$ , these points converge to a point of  $\Sigma$  and the tangent plane of  $\Sigma$  is vertical at this point; a contradiction.

We remark that in the case  $f(q_n) \rightarrow -\infty$  (see Claim 3), one works on the concave side of the horocycle  $\Gamma(q)$  and Claims 4 and 5 show there is an  $\varepsilon > 0$  and a graph  $G \subset \Sigma$  over the domain  $\Omega(\varepsilon)$  between  $\Gamma(\varepsilon)$  (the equidistant horocycle to  $\Gamma$  on the concave side of  $\Gamma$ ) and  $\Gamma$ . The graph  $G$  converges uniformly to  $\Gamma \times \mathbb{R}$  as one approaches  $\Gamma$  in  $\Omega(\varepsilon)$ .

To complete the proof of the theorem we apply the half-space theorem to show no such  $H = 1/2$  graph exists. Strictly speaking we can not apply the half-space theorem directly since the graph  $G$  does not have a compact

On complete mean curvature  $\frac{1}{2}$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$

Laurent Hauswirth, Harold Rosenberg and Joel Spruck \*

**1 Introduction**

In this paper we study complete constant mean curvature  $\frac{1}{2}$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ . Recall that the famous half-space theorem of Hoffman-Meeks says that a properly immersed minimal surface in  $\mathbb{R}^3$  that lies in a half-space must lie a plane. Our first result is an analogous result for a complete properly embedded constant mean curvature  $\frac{1}{2}$  surface  $\Sigma$  in  $\mathbb{H}^2 \times \mathbb{R}$  (endlessly with compact boundary).

**THEOREM 1.1.** *Let  $\Sigma$  be a properly embedded constant mean curvature  $\frac{1}{2}$  surface in  $\mathbb{H}^2 \times \mathbb{R}$ . Suppose  $\Sigma$  is asymptotic to a horocylinder  $C$ , and on one side of  $C$ . If the mean curvature vector of  $\Sigma$  has the same direction as that of  $C$  at points of  $\Sigma$  converging to  $C$ , then  $\Sigma$  is equal to  $C$  (or a subset of  $C$  of  $2\pi$  length).*

Our second main result concerns complete  $H = 1/2$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  transverse to the vertical Killing field  $\mathbb{Z} = \frac{\partial}{\partial t}$ . We prove such surfaces are entire graphs.

**THEOREM 1.2.** *Let  $\Sigma$  be a complete immersed surface in  $\mathbb{H}^2 \times \mathbb{R}$  of constant mean curvature  $H = 1/2$ . If  $\Sigma$  is transverse to  $\mathbb{Z}$  then  $\Sigma$  is an entire vertical graph over  $\mathbb{H}^2$ .*

Finally, we apply Theorem 1.2, together with work of Fernandez-Mira [4], and Xiao-Au [3] and Xiao [10], to understand such entire graphs.

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Figure 3

boundary. But this graph is proper in the tubular neighborhood of the horocylinder, so the proof shows the graph can not exist.

## 5 Quadratic holomorphic differentials, harmonic maps and entire graphs $H=1/2$

We will now describe how to obtain entire  $H = 1/2$  graphs starting with a holomorphic quadratic differential  $Q = \phi(z)dz^2$ . This originates from the work of Fernandez-Mira [4], Wan [10] and Au-Wan [3].

Abresch and Rosenberg [1, 2] constructed a holomorphic quadratic differential  $Q_0$  associated to the surface; this  $Q_0$  generalizes the Hopf differential associated to constant mean curvature surfaces of  $\mathbb{R}^3$ . When  $H = 1/2$  and the surface is a graph Fernandez-Mira [4] proved there exists a harmonic map from the surface to  $\mathbb{H}^2$  whose associated holomorphic quadratic differential is  $Q = -Q_0$ . In addition, given a harmonic map  $G$  from a surface to  $\mathbb{H}^2$  plus some additional data (described below) they construct graphs  $H = 1/2$  on  $\mathbb{H}^2 \times \mathbb{R}$  with this harmonic map as Gauss map.

For a given holomorphic  $Q = \phi(dz)^2$ , Wan [10] on the disk, Wan and Au [3] for  $\mathbb{C}$ , construct a unique harmonic map  $G : \Sigma \rightarrow \mathbb{H}^2$ , such that the Jacobian  $J(G) \geq 0$  and the metric  $\tau|dz| := 4(\sigma \circ G)^2|G_z|^2$  is complete. To do that, they construct a CMC  $H = 1/2$  in  $M^{2,1}$  the Minkowski space with Gauss map  $G$  and metric  $\tau|dz|^2$ .

Let  $G : \Sigma \longrightarrow \mathbb{H}^2$  be a harmonic map where  $\Sigma$  is  $\mathbb{C}$  or the unit disk. Then  $Q(z) = \phi dz^2$  is a quadratic holomorphic differential associated to  $G$  by the relation  $\phi = (\sigma \circ G)^2 G_z G_{\bar{z}}$ . Here we note  $\mathbb{H}^2 = (D^2, \sigma)$  where  $\sigma$  is the conformal factor of the hyperbolic metric on the disk. We define the function  $\omega = \frac{1}{2} \log \frac{|G_z|}{|G_{\bar{z}}|}$  and we express the Jacobian  $J(G) = \sigma^2(|G_z|^2 - |G_{\bar{z}}|^2) = 2\sinh(2\omega)|\phi|$ .

Fernandez-Mira construct multi-graph immersions  $\psi : \Sigma \longrightarrow \mathbb{H}^2 \times \mathbb{R}$  with  $H = 1/2$ , depending on the data  $\{Q, \tau\}$ ;  $\tau$  as above. We note the unit normal vector of  $\psi$  by  $\eta = (\widehat{N}, u)$ , with  $0 < |u| \leq 1$ . They show that the metric  $ds^2 = \lambda|dz|^2$  can be expressed as

$$\lambda = \frac{2\tau}{u^2} = 2\tau + 4|h_z|^2 \text{ and } u = \sqrt{\frac{\tau}{\tau + 2|h_z|^2}}$$

where  $h$  is the solution of a differential equation depending on  $\tau$  and  $\phi$ . By the above relation between  $\lambda$  and  $\tau$ , it is clear that the metric  $ds = \lambda|dz|^2$  is complete.

Thus associated to a holomorphic quadratic differential  $Q = \phi(z)dz^2$ , one obtains a complete multigraph  $H = 1/2$  in  $\mathbb{H}^2 \times \mathbb{R}$ ; hence an entire graph by Theorem 1.2. We give an independent proof below that the curvature  $K_\lambda$  is bounded (using the fact that the Jacobian of  $G$  is non negative). This condition is  $\omega \geq 0$  on  $\Sigma$ .

**Lemma 5.1.** *If  $G$  satisfies  $J(G) > 0$  and  $\tau = 4(\sigma \circ G)^2 |G_z|^2$  is non zero then the curvature of the associate constant mean curvature  $H = 1/2$  immersion  $\psi$  in  $\mathbb{H}^2 \times \mathbb{R}$  is bounded:*

$$|K_\lambda| \leq C.$$

**Proof.** In the Fernandez-Mira paper we have (formula (2.5)), for the metric  $ds^2 = \lambda|dz|^2$  of the immersion  $\psi$ , with mean curvature  $H$ :

$$\lambda(\log \lambda)_{z\bar{z}} = 2(|p|^2 - \lambda^2(H^2 - 1)/4 - \lambda|h_z|^2).$$

Here  $pdz^2 = -\langle \psi_z, \eta_z \rangle dz^2$  is the Hopf differential of  $\psi$  (the (2,0)-part of its complexified second fundamental form). Moreover, we have  $\phi = 2Hp + h_z^2$  (see [4]). Then with  $H = 1/2$  and  $\frac{|h_z|^2}{\lambda} = \frac{1 - u^2}{4} \leq 1/4$ :

$$\begin{aligned}
|K_\lambda| &= \frac{1}{2\lambda} |(\log \lambda)_{z\bar{z}}| \leq \frac{|p|^2}{\lambda^2} + 3/16 + \frac{|h(z)|^2}{\lambda} \\
&\leq \frac{|p + h_z^2|^2}{\lambda^2} + 7/16 + \frac{|h(z)|^4}{\lambda^2} \leq 1/2 + \frac{4u^4|\phi|^2}{\tau^2}
\end{aligned}$$

Notice that  $\frac{4u^4|\phi|^2}{\tau^2} = \frac{u^4}{4e^{4\omega}} \leq C$  since  $\omega \geq 0$ .

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