HYPERSURFACES OF CONSTANT CURVATURE IN HYPERBOLIC SPACE II.

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1. Introduction

In this paper we continue our study of complete hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature with a prescribed asymptotic boundary at infinity. Given $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$ and a smooth symmetric function f of n variables, we seek a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1.1}$$

$$\partial\Sigma = \Gamma \tag{1.2}$$

where $\kappa[\Sigma] = (\kappa_1, \ldots, \kappa_n)$ denotes the hyperbolic principal curvatures of Σ and $\sigma \in (0, 1)$ is a constant.

We will use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{1}{x_{n+1}^2} \sum_{i=1}^{n+1} dx_i^2\,. \tag{1.3}$$

Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (1.2) may be understood in the Euclidean sense.

As in in our earlier work [14, 12, 6, 9], we will take $\Gamma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek Σ as the graph of a function $u(x)$ over Ω, i.e.

$$\Sigma = \{(x, x_{n+1}) : x \in \Omega,\ x_{n+1} = u(x)\}.$$

Then the coordinate vector fields and upper unit normal are given by

$$X_i = e_i + u_i e_{n+1},\ \mathbf{n} = u\nu = u\frac{(-u_i e_i + e_{n+1})}{w},$$

Research of both authors were supported in part by NSF grants.

where $w = \sqrt{1 + |\nabla u|^2}$. The first and second fundamental forms g_{ij} and h_{ij} are related to their Euclidean counterparts g^e_{ij} and h^e_{ij} by

$$g_{ij} = \langle X_i, X_j \rangle = \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{g^e_{ij}}{u^2}, \tag{1.4}$$

$$\begin{aligned} h_{ij} &= \langle \nabla_{X_i} X_j, \nu \rangle \\ &= \frac{1}{u^2 w}(\delta_{ij} + u_i u_j + u u_{ij}) \\ &= \frac{h^e_{ij}}{u} + \frac{g^e_{ij}}{u^2 w}. \end{aligned} \tag{1.5}$$

The hyperbolic principal curvatures κ_i of Σ are the roots of the characteristic equation

$$\det(h_{ij} - \kappa g_{ij}) = u^{-n} \det(h^e_{ij} - \frac{1}{u}(\kappa - \frac{1}{w}) g^e_{ij}) = 0.$$

Therefore,

$$\kappa_i = u \kappa^e_i + \frac{1}{w}. \tag{1.6}$$

We will present other more explicit and useful expressions for the κ_i in Section 2.

The function f is assumed to satisfy the fundamental structure conditions:

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \;\text{ in } K, \;\; 1 \leq i \leq n, \tag{1.7}$$

$$f \text{ is a concave function in } K, \tag{1.8}$$

and

$$f > 0 \;\text{ in } K, \;\; f = 0 \;\text{ on } \partial K \tag{1.9}$$

where $K \subset \mathbb{R}^n$ is an open symmetric convex cone such that

$$K^+_n := \big\{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \big\} \subset K. \tag{1.10}$$

In addition, we shall assume that f is normalized

$$f(1, \ldots, 1) = 1 \tag{1.11}$$

and

$$f \text{ is homogeneous of degree one.} \tag{1.12}$$

Since f is symmetric, by (1.8), (1.11) and (1.12) we have

$$f(\lambda) \leq f(\mathbf{1}) + \sum f_i(\mathbf{1})(\lambda_i - 1) = \sum f_i(\mathbf{1}) \lambda_i = \frac{1}{n} \sum \lambda_i \;\text{ in } K \tag{1.13}$$

and

$$\sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1-\lambda_i) \geq f(\mathbf{1}) = 1 \text{ in } K. \tag{1.14}$$

Lemma 1.1. *Suppose f satisfies (1.7)-(1.12). Then*

$$\sum_{i\neq r} f_i\lambda_i^2 \geq \frac{1}{n-1}(2f|\lambda_r| + f_r\lambda_r^2) \quad \textit{if } \lambda_r < 0 \tag{1.15}$$

and so

$$\sum_{i\neq r} f_i\lambda_i^2 \geq \frac{1}{n}\sum f_i\lambda_i^2 \quad \textit{if } \lambda_r < 0. \tag{1.16}$$

Proof. Suppose $\lambda_r < 0$ and order the eigenvalues with $\lambda_1 > 0$ the largest and $\lambda_n < 0$ the smallest. Then as a consequence of the concavity condition (1.8) we have

$$f_n \geq f_i \text{ for all } i \text{ and so } f_n\lambda_n^2 \geq f_r\lambda_r^2. \tag{1.17}$$

By (1.12),

$$\sum_{i\neq n} f_i\lambda_i = f + f_n|\lambda_n|.$$

By Schwarz inequality and (1.17),

$$f^2 + 2ff_n|\lambda_n| + f_n^2\lambda_n^2 \leq \sum_{i\neq n} f_i \sum_{i\neq n} f_i\lambda_i^2 \leq (n-1)f_n \sum_{i\neq n} f_i\lambda_i^2.$$

Therefore,

$$\sum_{i\neq n} f_i\lambda_i^2 \geq \frac{1}{n-1}(2f|\lambda_n| + f_n\lambda_n^2).$$

Using (1.17) this implies

$$\sum_{i\neq r} f_i\lambda_i^2 \geq \sum_{i\neq n} f_i\lambda_i^2 \geq \frac{1}{n-1}(2f|\lambda_n| + f_n\lambda_n^2) \geq \frac{1}{n-1}(2f|\lambda_r| + f_r\lambda_r^2) \tag{1.18}$$

completing the proof. □

All of the above assumptions (1.7)-(1.12) are fairly standard. In the present work, the following more technical assumption is important.

$$\liminf_{R\to+\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.19}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

The assumption (1.19) is fairly mild. For $f = H_k^{\frac{1}{k}}$ corresponding to the "higher order mean curvatures", where H_k is the k-th normalized elementary function,

$$\liminf_{R\to\infty} f(\mathbf{1} + O(\varepsilon) + Re_n) = \infty$$

while for $f = (H_{k,l})^{\frac{1}{k-l}} = (\frac{H_k}{H_l})^{\frac{1}{k-l}}$, $k > l$, the class of curvature quotients,

$$\liminf_{R\to\infty} f(\mathbf{1} + O(\varepsilon) + Re_n) = (1 + O(\varepsilon))\Big(\frac{k}{l}\Big)^{\frac{1}{k-l}}.$$

Problem (1.1)-(1.2) reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1.20}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.21}$$

The exact formula of G will be given in Section 2.

We seek solutions of equation (1.20) satisfying $\kappa[u] \equiv \kappa[\text{graph}(u)] \in K$. Following the literature we define the class of *admissible* functions

$$\mathcal{A}(\Omega) = \{u \in C^2(\Omega) : \kappa[u] \in K\}.$$

By [2] condition (1.7) implies that equation (1.20) is elliptic for admissible solutions. Our goal is to show that the Dirichlet problem (1.20)-(1.21) admits smooth admissible solutions for all $0 < \sigma < 1$. Due to the special nature of the problem, there are substantial technical difficulties to overcome and we have not yet succeeded in finding solutions for all $\sigma \in (0, 1)$. However we shall prove

Theorem 1.2. *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that the Euclidean mean curvature $\mathcal{H}_{\partial\Omega} \geq 0$ and $\sigma \in (0, 1)$ satisfies $\sigma > \sigma_0$, where σ_0 is the unique zero in $(0, 1)$ of*

$$\phi(a) := \frac{8}{3}a + \frac{22}{27}a^3 - \frac{5}{27}(a^2 + 3)^{\frac{3}{2}}. \tag{1.22}$$

(Numerical calculations show $0.3703 < \sigma_0 < 0.3704$.)

Under conditions (1.7)-(1.12) and (1.19), there exists a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)-(1.2) with uniformly bounded principal curvatures

$$|\kappa[\Sigma]| \leq C \quad \textit{on } \Sigma. \tag{1.23}$$

Moreover, Σ *is the graph of a unique admissible solution* $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ *of the Dirichlet problem (1.20)-(1.21). Furthermore,* $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ *and*

$$\begin{aligned} &\sqrt{1+|Du|^2} \leq \frac{1}{\sigma}, \;\; u|D^2u| \leq C \quad \text{in } \Omega, \\ &\sqrt{1+|Du|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.24}$$

Theorem 1.2 holds for a large family of $f = \frac{1}{N}\sum_{l=1}^N f_l$ where each f_l consisting of sums and "concave products" (that is of the form $(f_1 \cdots f_{N_l})^{\frac{1}{N_l}}$) where each f_l satisfies (1.7)-(1.12) and one of the f_l satisfies (1.19).

As we shall see in Section 2, equation (1.20) is degenerate where $u = 0$. It is therefore natural to approximate the boundary condition (1.21) by

$$u = \varepsilon > 0 \quad \text{on } \partial\Omega. \tag{1.25}$$

When ε is sufficiently small, the Dirichlet problem (1.20),(1.25) is solvable for all $\sigma \in (0,1)$.

Theorem 1.3. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$ *with* $\mathcal{H}_{\partial\Omega} \geq 0$ *and suppose* f *satisfies (1.7)-(1.12) and (1.19). Then for any* $\sigma \in (0,1)$ *and* $\varepsilon > 0$ *sufficiently small, there exists an admissible solution* $u^\varepsilon \in C^\infty(\bar{\Omega})$ *of the Dirichlet problem (1.20),(1.25). Moreover,* u^ε *satisfies the* a priori *estimates*

$$\sqrt{1+|Du^\varepsilon|^2} \leq \frac{1}{\sigma} \quad \text{in } \Omega \tag{1.26}$$

$$u^\varepsilon|D^2u^\varepsilon| \leq \frac{C}{\varepsilon^2} \quad \text{in } \Omega \tag{1.27}$$

where C *is independent of* ε*.*

The organization of the paper is as follows. Sections 2 and 3 summarize the basic information about vertical graphs and the linearized operator that we will need in the sequel. In section 4 we use the assumption on the mean curvature of the boundary to prove that the linearized operator is always invertible and that u satisfies the sharp global gradient bound $\frac{1}{w} \geq \sigma$. These are essential for the boundary second derivative estimates which we derive in Section 5. Here we make use of Lemma 1.1 and a careful analysis of the linearized operator to derive the mixed normal-tangential estimates. We then use assumption (1.19) to establish pure normal second derivative estimate. In Section 6 we prove a maximum principle for the maximal hyperbolic principal curvature using radial graphs as in our earlier paper [9]. It is here that we have had to restrict the allowable range of $\sigma \in (0,1)$. Otherwise our approach is completely general and we expect Theorem 1.2 is valid for all $\sigma \in (0,1)$.

2. Vertical graphs and the Linearized operator

Suppose Σ is locally represented as the graph of a function $u \in C^2(\Omega)$, $u > 0$, in a domain $\Omega \subset \mathbb{R}^n$:

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : \ x \in \Omega\}.$$

oriented by the upward (Euclidean) unit normal vector field ν to Σ:

$$\nu = \Big(\frac{-Du}{w}, \frac{1}{w}\Big), \ \ w = \sqrt{1 + |Du|^2}.$$

The Euclidean metric and second fundamental form of Σ are given respectively by

$$g^e_{ij} = \delta_{ij} + u_i u_j, \ \ h^e_{ij} = \frac{u_{ij}}{w}.$$

According to [3], the Euclidean principal curvatures $\kappa^e[\Sigma]$ are the eigenvalues of the symmetric matrix $A^e[u] = \{a^e_{ij}\}$:

$$a^e_{ij} := \frac{1}{w}\gamma^{ik} u_{kl} \gamma^{lj}, \tag{2.1}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{u_i u_j}{w(1+w)}. \tag{2.2}$$

Note that the matrix $\{\gamma^{ij}\}$ is invertible with inverse

$$\gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1+w} \tag{2.3}$$

which is the square root of $\{g^e_{ij}\}$, i.e., $\gamma_{ik}\gamma_{kj} = g^e_{ij}$. By (1.6) the hyperbolic principal curvatures $\kappa[u]$ of Σ are the eigenvalues of the matrix $A[u] = \{a_{ij}[u]\}$:

$$a_{ij}[u] := \frac{1}{w}\Big(\delta_{ij} + u\gamma^{ik} u_{kl} \gamma^{lj}\Big). \tag{2.4}$$

Let $\mathcal{S}$ be the vector space of $n \times n$ symmetric matrices and

$$\mathcal{S}_K = \{A \in \mathcal{S} : \lambda(A) \in K\},$$

where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of A. Define a function F by

$$F(A) = f(\lambda(A)), \ \ A \in \mathcal{S}_K. \tag{2.5}$$

Throughout the paper we denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \tag{2.6}$$

The matrix $\{F^{ij}(A)\}$, which is symmetric, has eigenvalues $f_1, \ldots, f_n$, and therefore is positive definite for $A \in \mathcal{S}_K$ if f satisfies (1.7), while (1.8) implies that F is concave for $A \in \mathcal{S}_K$ (see [2]), that is

$$F^{ij,kl}(A)\xi_{ij}\xi_{kl} \le 0, \quad \forall \ \{\xi_{ij}\} \in \mathcal{S}, \ A \in \mathcal{S}_K. \tag{2.7}$$

We have

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i, \tag{2.8}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{2.9}$$

The function G in equation (1.20) is determined by

$$G(D^2u, Du, u) = F(A[u]) \tag{2.10}$$

where $A[u] = \{a_{ij}[u]\}$ is given by (2.4). Let

$$\mathcal{L} = G^{st}\partial_s\partial_t + G^s\partial_s + G_u \tag{2.11}$$

be the linearized operator of G at u, where

$$G^{st} = \frac{\partial G}{\partial u_{st}}, \ G^s = \frac{\partial G}{\partial u_s}, \ G_u = \frac{\partial G}{\partial u}.$$

We shall give the exact formula for G^s later but note that

$$\begin{aligned} G^{st} &= \frac{u}{w} F^{ij}\gamma^{is}\gamma^{jt} \\ G^{st}u_{st} &= uG_u = F^{ij}a_{ij} - \frac{1}{w}\sum F^{ii} \end{aligned} \tag{2.12}$$

and

$$G^{pq,st} := \frac{\partial^2 G}{\partial u_{pq}\partial u_{st}} = \frac{u^2}{w^2} F^{ij,kl}\gamma^{is}\gamma^{tj}\gamma^{kp}\gamma^{ql} \tag{2.13}$$

where $F^{ij} = F^{ij}(A[u])$, etc. It follows that, under condition (1.7), equation (1.20) is elliptic for u if $A[u] \in \mathcal{S}_K$, while (1.8) implies that $G(D^2u, Du, u)$ is concave with respect to D^2u.

For later use, the eigenvalues of $\{G^{ij}\}$ and $\{F^{ij}\}$ (which are the f_i) are related by

Lemma 2.1. *Let $0 < \mu_1 \leq \ldots \leq \mu_n$ and $0 < f_1 \leq \ldots \leq f_n$ denote the eigenvalues of $\{G^{ij}\}$ and $\{F^{ij}\}$ respectively. Then*

$$w\mu_k \leq u f_k \leq w^3 \mu_k, \ \ 1 \leq k \leq n. \tag{2.14}$$

Proof. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ we have from (2.12)

$$uF^{ij}\xi_i\xi_j = wG^{kl}\gamma_{ik}\gamma_{lj}\xi_i\xi_j = wG^{kl}\xi'_k\xi'_l$$

where

$$\xi'_i = \gamma_{ik}\xi_k = \xi_i + \frac{(\xi \cdot Du)u_i}{1+w} \ .$$

Note that

$$|\xi|^2 \leq |\xi'|^2 = |\xi|^2 + |\xi \cdot Du|^2 \leq w^2|\xi|^2$$

where $\xi' = (\xi'_1, \ldots, \xi'_n)$. Since both $\{G^{ij}\}$ and $\{F^{ij}\}$ are positive, (2.14) follows from the minimax characterization of eigenvalues. □

3. Height estimates and the asymptotic angle condition

Let $B_R(a)$ be a ball of radius R centered at $a = (a', -\sigma R) \in \mathbb{R}^{n+1}$ where $\sigma \in (0,1)$ and $S = \partial B_R(a) \cap \mathbb{H}^{n+1}$. Then $\kappa_i[S] = \sigma$ for all $1 \leq i \leq n$ with respect to its outward normal. These so called equidistant spheres serve as useful barriers. We have the following estimates which were derived in [6], [9] using these barriers..

Lemma 3.1. *Suppose f satisfies (1.7), (1.9) and (1.12). Let Σ be a hypersurface in $\mathbb{H}^{n+1}$ with $\kappa[\Sigma] \in K$ and*

$$\sigma_1 \leq f(\kappa[\Sigma]) \leq \sigma_2$$

where $0 \leq \sigma_1 \leq \sigma_2 \leq 1$ are constants, and $\partial\Sigma \subset P(\varepsilon) \equiv \{x_{n+1} = \varepsilon\}$, $\varepsilon \geq 0$. Let Ω be the region in $\mathbb{R}^n$ bounded by the projection of $\partial\Sigma$ to $\mathbb{R}^n = \{x_{n+1} = 0\}$ (such that $\mathbb{R}^n \setminus \Omega$ is unbounded), and u denote the height function of Σ.

(i) For any point $(x, u) \in \Sigma$,

$$\frac{\varepsilon\sigma_2}{1+\sigma_2} + d(x)\sqrt{\frac{1-\sigma_2}{1+\sigma_2}} \leq u \leq \frac{L}{2}\sqrt{\frac{1-\sigma_1}{1+\sigma_1}} + \varepsilon, \tag{3.1}$$

where $d(x)$ and L denote the distance from $x \in \mathbb{R}^n$ to $\partial\Omega$ and the (Euclidean) diameter of Ω, respectively.

(ii) Assume that $\partial\Sigma \in C^2$. For $\varepsilon > 0$ sufficiently small,

$$\sigma_1 - \frac{\varepsilon\sqrt{1-\sigma_1^2}}{r_1} - \frac{\varepsilon^2(1+\sigma_1)}{r_1^2} < \nu^{n+1} < \sigma_2 + \frac{\varepsilon\sqrt{1-\sigma_2^2}}{r_2} + \frac{\varepsilon^2(1-\sigma_2)}{r_2^2} \quad \text{on } \partial\Sigma \tag{3.2}$$

where r_1 and r_2 are the maximal radii of exterior and interior spheres to $\partial\Omega$, respectively. In particular, if $\sigma_1 = \sigma_2 = \sigma$ then $\nu^{n+1} \to \sigma$ on $\partial\Sigma$ as $\varepsilon \to 0$.

4. The approximating problems and the continuity method

We study the approximating Dirichlet problem

$$\begin{aligned} G(D^2u, Du, u) &= \sigma \quad \text{in } \Omega \\ u &= \varepsilon \quad \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

using the continuity method.

Consider for $0 \le t \le 1$ the family of Dirichlet problems

$$\begin{aligned} G(D^2u^t, Du^t, u^t) = \sigma^t &:= t\sigma + (1-t) \quad \text{in } \Omega, \\ u^t &= \varepsilon \quad \text{on } \partial\Omega, \\ u^0 &\equiv \varepsilon. \end{aligned} \tag{4.2}$$

For Ω a $C^{2+\alpha}$ domain, we find (starting from $u^0 \equiv 0$) a smooth family of solutions u^t, $0 \le t \le 2t_0$ by the implicit function theorem since $G_u|_{u^0} \equiv 0$. We shall show in a moment that these solutions are unique. By elliptic regularity it is now well understood that if we can find uniform estimates in C^2 for $0 < t_0 \le t \le 1$ then we can solve (4.1).

By Lemma 3.1 we obtain the C^0 estimate and boundary gradient estimate

$$\varepsilon \le u^t \le C \text{ in } \Omega, \;\; |Du^t| \le C \text{ on } \partial\Omega. \tag{4.3}$$

4.1. **The C^1 estimate.**

The following proposition shows that we have uniform C^1 estimates in the continuity method and that the linearized operator $\mathcal{L}$ satisfies the maximum principle.

Proposition 4.1. *Let $u^t \in C^{2+\alpha}(\overline{\Omega})$ be a family of admissible solutions of (4.2) for $0 \le t \le t^*$. Suppose $\mathcal{H}_{\partial\Omega} \ge 0$. Then $G_u|_{u^t} \le 0$ so we have uniqueness. Hence w^t assumes its maximum on $\partial\Omega$ and $w^t \le \frac{1}{\sigma^t}$ on $\overline{\Omega}$ for all $0 \le t \le t^*$.*

Proof. We suppress the t dependence for convenience. By (2.12) and (1.14),

$$uG_u = \sigma - \frac{1}{w}\sum f_i \le \sigma - \frac{1}{w}.$$

For $t = 0$, $\sigma^0 = 1$, $u^0 \equiv \varepsilon$, $\kappa_i = 1$, $f_i = \frac{1}{n}$ and so $\mathcal{L} = \frac{\varepsilon}{n}\Delta$. For t near 0, $\mathcal{L}u_k = 0$ with $\mathcal{L}$ close to Δ in the C^2 topology and so (as is well known) each derivative u_k achieves its maximum on $\partial\Omega$. In particular, w assumes its maximum on $\partial\Omega$. Let $0 \in \partial\Omega$ be the point where w assumes its maximum. Choose coordinates $(x_1, \dots, x_n)$ at 0 with x_n the inner normal direction for $\partial\Omega$. Then at 0,

$$u_\alpha = 0,\ 1 \le \alpha < n,\ u_n > 0,\ u_{nn} \le 0,$$

and

$$\sum u_{\alpha\alpha} = -u_n(n-1)\mathcal{H}_{\partial\Omega} \le 0.$$

Note that by (1.13), the hyperbolic mean curvature of graph $(u) \ge \sigma$. Therefore,

$$\frac{n}{\varepsilon}\Big(\sigma - \frac{1}{w}\Big) \le \frac{1}{w}\Big(\sum_{\alpha<n} u_{\alpha\alpha} + \frac{u_{nn}}{w^2}\Big) \le -(n-1)\frac{u_n}{w}\mathcal{H}_{\partial\Omega} \le 0.$$

Hence $\sigma - \frac{1}{w} \le 0$ or $w \le \frac{1}{\sigma}$. Thus $G_u \le 0$ so $\mathcal{L}$ satisfies the maximum principle. Consequently, the same estimates must continue to hold as we increase t up to t^*. □

In Section 5, we will make use of Proposition 4.1 to complete the proof of the C^2 estimates (see Theorem 5.1 and Corollary 5.8). Since the linearized operator is invertible, we have unique smooth solvability all the way to $t = 1$ completing the proof of Theorem 1.3. Using the global maximum principle, Theorem 6.1 of Section 6 and Theorem 5.1, we obtain uniform estimates for the hyperbolic principal curvatures so we can let ε tend to zero to obtain Theorem 1.2 as a consequence of Theorem 1.3.

5. Boundary estimates for second derivatives

In this section we establish boundary estimates for second derivatives of admissible solutions to the Dirichlet problem (4.2) for all $t_0 \le t \le 1$. Clearly it suffices to consider the case $t = 1$. Throughout this section let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \ge 0$, and $u \in C^3(\bar{\Omega})$ an admissible solution of the Dirichlet problem

$$\begin{cases} G(D^2u, Du, u) = \sigma, & \text{on } \overline{\Omega}, \\ u = \varepsilon, & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where G is defined in (2.10).

Theorem 5.1. *Suppose that f satisfies (1.7)-(1.12) and (1.19). If ε is sufficiently small,*

$$u|D^2u| \le C \quad \text{on } \partial\Omega \tag{5.2}$$

where C is independent of ε.

The notation of this section follows that of Section 2. Let $\mathcal{L}'$ denote the partial linearized operator of G at u:

$$\mathcal{L}' = \mathcal{L} - G_u = G^{st}\partial_s\partial_t + G^s\partial_s$$

where G^{st}, G_u are defined in (2.12) and

$$G^s := \frac{\partial G}{\partial u_s} = -\frac{u_s}{w^2}F^{ij}a_{ij} - \frac{2}{w}F^{ij}a_{ik}\Big(\frac{wu_k\gamma^{sj} + u_j\gamma^{ks}}{1+w}\Big) + \frac{2}{w^2}F^{ij}u_i\gamma^{sj} \tag{5.3}$$

by the formula (2.21) in [8], where $F^{ij} = F^{ij}(A[u])$ and $a_{ij} = a_{ij}[u]$.

Since $F = \{F^{ij}\}$ and $A = \{a_{ij}\}$ are simultaneously diagonalizable we have

Lemma 5.2. *Suppose that f satisfies (1.7), (1.8), (1.11) and (1.12). Then*

$$|G^s| \leq \frac{\sigma}{w} + \frac{2}{w}\Big(\sum F^{ii} + \sum f_i|\kappa_i|\Big). \tag{5.4}$$

Since $\gamma^{sj}u_s = u_j/w$,

$$G^s u_s = \Big(\frac{1}{w^2} - 1\Big)F^{ij}a_{ij} - \frac{2}{w^2}F^{ij}a_{ik}u_k u_j + \frac{2}{w^3}F^{ij}u_i u_j. \tag{5.5}$$

It follows from (2.6), (2.8) and (2.12) that

$$\mathcal{L}'u = \frac{1}{w^2}F^{ij}a_{ij} - \frac{1}{w}\sum F^{ii} - \frac{2}{w^2}F^{ij}a_{ik}u_k u_j + \frac{2}{w^3}F^{ij}u_i u_j. \tag{5.6}$$

Lemma 5.3. *Suppose that f satisfies (1.7), (1.8), (1.11) and (1.12). Then*

$$\mathcal{L}\Big(1 - \frac{\varepsilon}{u}\Big) \leq -\frac{(1-\sigma)\varepsilon}{u^2 w}\sum F^{ii} - \frac{2\varepsilon}{u^2w^2}F^{ij}a_{ik}u_k u_j \quad \text{in } \Omega. \tag{5.7}$$

Proof. By (5.6), (2.12) and (1.12),

$$\begin{aligned} \mathcal{L}\Big(1 - \frac{\varepsilon}{u}\Big) &= \frac{\varepsilon}{u^2}\mathcal{L}'u - \frac{2\varepsilon}{u^3}G^{st}u_s u_t + G_u(1 - \frac{\varepsilon}{u}) \\ &= \frac{\varepsilon}{u^2}\Big(\frac{\sigma}{w^2} - \frac{1}{w}\sum F^{ii}\Big) + G_u(1 - \frac{\varepsilon}{u}) - \frac{2\varepsilon}{u^2w^2}F^{ij}a_{ik}u_k u_j. \end{aligned} \tag{5.8}$$

Since $G_u \leq 0$ by Proposition 4.1, (5.7) now follows from (1.14). □

We now refine Lemma 5.3. For the symmetric matrix $A = A[u]$ we can uniquely define the symmetric matrices (see [13])

$$|A| = \{AA^T\}^{\frac{1}{2}},\ A^+ = \frac{1}{2}(|A| + A),\ A^- = \frac{1}{2}(|A| - A) \tag{5.9}$$

which all commute and satisfy $A^+A^- = 0$. Moreover, $F = \{F^{ij}\}$ commutes with $|A|$, $A^\pm$ and so all are simultaneously diagonalizable. Write $A^\pm = \{a^\pm_{ij}\}$ and define

$$L = \mathcal{L} - \frac{2}{w^2} F^{ij} a^-_{ik} u_k \partial_j. \tag{5.10}$$

Lemma 5.4. *Suppose that f satisfies (1.7), (1.8), (1.11) and (1.12). Then*

$$L\Big(1 - \frac{\varepsilon}{u}\Big) \leq -\frac{(1-\sigma)\varepsilon}{u^2 w} \sum F^{ii} \quad \text{in } \Omega. \tag{5.11}$$

Proof. Since $\{F^{ij}\}$ is positive definite and simultaneously diagonalizable with $A^\pm$,

$$F^{ij} a^\pm_{ik} \xi_j \xi_k \geq 0, \ \ \forall\, \xi \in \mathbb{R}^n.$$

Therefore,

$$F^{ij} a_{ik} u_k u_j = F^{ij} (a^+_{ik} - a^-_{ik}) u_k u_j \geq -F^{ij} a^-_{ik} u_k u_j \tag{5.12}$$

Combining (5.12) and Lemma 5.3 we obtain (5.11). □

The following lemma is well-known [1], [9].

Lemma 5.5. *Suppose that f satisfies (1.7), (1.8), (1.11) and (1.12). Then*

$$\mathcal{L}(x_i u_j - x_j u_i) = 0, \ \mathcal{L} u_i = 0, \qquad 1 \leq i, j \leq n. \tag{5.13}$$

Proof of Theorem 5.1. Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of $\mathbb{R}^n$ and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph

$$x_n = \rho(x') = \frac{1}{2} \sum_{\alpha,\beta<n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \qquad x' = (x_1, \ldots, x_{n-1}). \tag{5.14}$$

Since $u = \varepsilon$ on $\partial\Omega$, we see that $u(x', \rho(x')) = \varepsilon$ and

$$u_{\alpha\beta}(0) = -u_n \rho_{\alpha\beta} \qquad \alpha, \beta < n. \tag{5.15}$$

Consequently,

$$|u_{\alpha\beta}(0)| \leq C|Du(0)|, \qquad \alpha, \beta < n \tag{5.16}$$

where C depends only on the (Euclidean maximal principal) curvature of $\partial\Omega$.

As in [1] we consider for fixed $\alpha < n$ the operator

$$T_\alpha = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta). \tag{5.17}$$

Using Lemma 5.5 and the boundary condition $u = \varepsilon$ on $\partial\Omega$ we have

$$\begin{aligned} &\mathcal{L}T_\alpha u = 0, \\ &|T_\alpha u| + \frac{1}{2}\sum_{l<n} u_l^2 \le C \ \text{ in } \Omega \cap B_\varepsilon(0) \\ &|T_\alpha u| + \frac{1}{2}\sum_{l<n} u_l^2 \le C|x|^2 \ \text{ on } \partial\Omega \cap B_\varepsilon(0). \end{aligned} \tag{5.18}$$

Now define

$$\phi = \pm T_\alpha u + \frac{1}{2}\sum_{l<n} u_l^2 - \frac{C}{\varepsilon^2}|x|^2$$

where C is chosen large enough (independent of ε) so that $\phi \le 0$ on $\partial(\Omega \cap B_\varepsilon(0))$. This is possible by (5.18).

By (5.4), (5.18), (2.12) and Lemma 2.1

$$\mathcal{L}\phi \ge \sum_{l<n} G^{ij}u_{li}u_{lj} - \frac{C}{\varepsilon}\Big(\sum f_i + \sum f_i|\kappa_i|\Big) \ \text{ in } \Omega \cap B_\varepsilon(0). \tag{5.19}$$

Following Ivochkina, Lin and Trudinger [10] we have

Proposition 5.6. *At each point in $\Omega \cap B_\varepsilon(0)$ there is an index r such that*

$$\sum_{l<n} G^{ij}u_{li}u_{lj} \ge c_0 u \sum_{i\ne r} f_i(\kappa_i^e)^2 \ge \frac{c_0}{2u}\Big(\sum_{i\ne r} f_i\kappa_i^2 - \frac{2}{w^2}\sum f_i\Big) \tag{5.20}$$

Proof. Let P be an orthogonal matrix that simultaneously diagonalizes $\{F^{ij}\}$ and A^e. By (2.12) and (2.1),

$$\begin{aligned} \sum_{l<n} G^{ij}u_{li}u_{lj} &= \frac{u}{w}\sum_{l<n} F^{st}\gamma^{is}\gamma^{jt}u_{li}u_{lj} \\ &= uw\sum_{l<n} F^{st}a^e_{sq}a^e_{pt}\gamma_{pl}\gamma_{ql} \\ &= uw\sum_{l<n} f_i(\kappa_i^e)^2 P_{pi}\gamma_{pl}P_{qi}\gamma_{ql} \\ &= uw\sum_{l<n} f_i(\kappa_i^e)^2 b_{li}^2, \end{aligned} \tag{5.21}$$

where $B = \{b_{rs}\} = \{P_{ir}\gamma_{is}\}$ and $\det B = \det(B^T) = w$.

Suppose for some i, say $i = 1$ that

$$\sum_{l<n} b_{l1}^2 < \theta^2.$$

Expanding $\det B$ by cofactors along the first column gives

$$1 \leq w = \det B = b_{11}C^{11} + \dots b_{n-1\ 1}C^{1n-1} + b_{n1}\det M \leq c_1\theta + c_2 \det M,$$

where the C^{1j} are cofactors and M is the $n-1$ by $n-1$ matrix

$$M = \begin{bmatrix} b_{12} & \dots & b_{n-1\ 2} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{n-1\ n} \end{bmatrix}. \tag{5.22}$$

So $\det M \geq \frac{1-c_1\theta}{c_2}$. Now expanding $\det M$ by cofactors along row $r \geq 2$ gives

$$\det M \leq c_3 \left(\sum_{l<n} b_{lr}^2\right)^{\frac{1}{2}}$$

by Schwarz inequality. Hence

$$\sum_{l<n} b_{lr}^2 \geq \Big(\frac{1-c_1\theta}{c_2c_3}\Big)^2. \tag{5.23}$$

Choosing $\theta < \frac{1}{2c_1}$, (5.23) and (5.21) imply

$$\sum_{l<n} G^{ij}u_{li}u_{lj} \geq c_0 u \sum_{i\neq r} f_i(\kappa_i^e)^2 \ \text{ for some } r.$$

Finally using $\kappa_i^e = \frac{1}{u}(\kappa_i - \frac{1}{w})$, (5.20) follows. □

Proposition 5.7. *Let L be defined by* (5.10)*. Then*

$$L\phi \geq -C_1\Big(G^{ij}\phi_i\phi_j + \frac{1}{\varepsilon}\sum f_i\Big)$$

for a controlled constant C_1 independent of ε.

Proof. By the generalized Schwarz inequality ,

$$\begin{aligned} \frac{2}{w^2}|F^{ij}a^-_{jk}u_i\phi_k| &\leq 2\Big(uF^{ij}\phi_i\phi_j\Big)^{\frac{1}{2}}\Big(\frac{1}{u}F^{ij}a^-_{il}a^-_{kj}\frac{u_ku_l}{w^2}\Big)^{\frac{1}{2}} \\ &\leq \frac{c_0}{8nu}\sum_{\kappa_i<0} f_i\kappa_i^2 + CG^{ij}\phi_i\phi_j \end{aligned} \tag{5.24}$$

where we have used Lemma 2.1 to compare $uF^{ij}\phi_i\phi_j$ to $G^{ij}\phi_i\phi_j$.

Since

$$\sum f_i|\kappa_i| < \sigma + 2\sum_{\kappa_i<0} f_i|\kappa_i|,$$

using (5.24), (5.19), Proposition 5.6 and Lemma 1.1 we have

$$\begin{aligned} L\phi &\geq \frac{c_0}{2u}\sum_{i\neq r} f_i\kappa_i^2 - \frac{c_0}{4nu}\sum_{\kappa_i<0} f_i\kappa_i^2 - C\Big(G^{ij}\phi_i\phi_j + \frac{1}{\varepsilon}\sum f_i\Big) \\ &\geq -C_1\Big(G^{ij}\phi_i\phi_j + \frac{1}{\varepsilon}\sum f_i\Big) \end{aligned} \tag{5.25}$$

for a controlled constant C_1 independent of ε. □

Let $h = (e^{C_1\phi} - 1) - A(1 - \frac{\varepsilon}{u})$ with C_1 the constant in Proposition 5.7 and A large compared to C_1. By Proposition 5.7 and Lemma 5.4,

$$h \leq 0 \ \text{ on } \partial(\Omega \cap B_\varepsilon(0))$$

and

$$Lh \geq 0 \ \text{ in } \Omega \cap B_\varepsilon(0).$$

By the maximum principle $h \leq 0$ in $\Omega \cap B_\varepsilon(0)$. Since $h(0) = 0$, we have $h_n(0) \leq 0$ which gives

$$|u_{\alpha n}(0)| \leq \frac{A}{C_1\varepsilon}u_n(0). \tag{5.26}$$

Finally, $|u_{nn}(0)|$ can be estimated as in [9] using hypothesis (1.19). For completeness we include the argument here. We may assume $[u_{\alpha\beta}(0)]$, $1 \leq \alpha, \beta < n$, to be diagonal. Note that $u_\alpha(0) = 0$ for $\alpha < n$. We have at $x = 0$

$$A[u] = \frac{1}{w}\begin{bmatrix} 1 + uu_{11} & 0 & \dots & \frac{uu_{1n}}{w} \\ 0 & 1 + uu_{22} & \dots & \frac{uu_{2n}}{w} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{uu_{n1}}{w} & \frac{uu_{n2}}{w} & \dots & 1 + \frac{uu_{nn}}{w^2} \end{bmatrix}.$$

By Lemma 1.2 in [2], if $\varepsilon u_{nn}(0)$ is very large, the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A[u]$ are asymptotically given by

$$\begin{aligned} \lambda_\alpha &= \frac{1}{w}(1 + \varepsilon u_{\alpha\alpha}(0)) + o(1), \ \alpha < n \\ \lambda_n &= \frac{\varepsilon u_{nn}(0)}{w^3}\Big(1 + O\Big(\frac{1}{\varepsilon u_{nn}(0)}\Big)\Big). \end{aligned} \tag{5.27}$$

By (5.16) and assumptions (1.12)-(1.19), for all $\varepsilon > 0$ sufficiently small,

$$\sigma = \frac{1}{w}F(wA[u](0)) \geq \frac{1}{w}\Big(1 + \frac{\varepsilon_0}{2}\Big)$$

if $\varepsilon u_{nn}(0) \geq R$ where R is a uniform constant. By the hypothesis (1.19) and Proposition 4.1 however,

$$\sigma \geq \frac{1}{w}\Big(1+\frac{\varepsilon_0}{2}\Big) \geq \sigma\Big(1+\frac{\varepsilon_0}{2}\Big) > \sigma$$

which is a contradiction. Therefore

$$\varepsilon|u_{nn}(0)| \leq R$$

and the proof is complete. □

Applying the maximum principle for the largest principal curvature $\kappa_{\max}$ obtained in Theorem 5.2 of [9] we obtain

Corollary 5.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \geq 0$, and $u \in C^3(\bar{\Omega}) \cap C^4(\Omega)$ an admissible solution of problem (5.1). Suppose that f satisfies (1.7)-(1.12) and (1.19). Then, if ε is sufficiently small,*

$$u|D^2u| \leq \frac{C}{\varepsilon^2} \quad \text{in } \overline{\Omega} \tag{5.28}$$

where C is independent of ε.

Note that Corollary 5.8 suffices to complete the proof of Theorem 1.3 but that we cannot use it to pass to the limit as ε tends to zero. In the following section we address this problem.

6. Global estimates for second derivatives

In this section we prove a maximum principle for the largest hyperbolic principal curvature $\kappa_{\max}(x)$ of solutions of $f(\kappa[u]) = \sigma$. We make extensive use of our previous calculations in Section 5 of [9].

Let Σ be the graph of u. For $x \in \Omega$ let $\kappa_{\max}(x)$ be the largest principal curvature of Σ at the point $X = (x, u(x)) \in \Sigma$. We define

$$M_0 = \max_{x \in \overline{\Omega}} \frac{\kappa_{\max}(x)}{\eta - a},$$

where $\eta = \nu^{n+1} = \mathbf{e} \cdot \nu$ is the upward (Euclidean) unit normal to Σ and $\inf \eta \geq \sigma > a$.

Theorem 6.1. *Suppose that f satisfies (1.7)-(1.12) and $\sigma \in (0,1)$ satisfies $\sigma > \sigma_0$, where σ_0 is the unique zero in $(0,1)$ of*

$$\phi(a) := \frac{8}{3}a + \frac{22}{27}a^3 - \frac{5}{27}(a^2+3)^{\frac{3}{2}}. \tag{6.1}$$

Let $u \in C^4(\Omega)$ be an admissible solution of (5.1) *such that $\nu^{n+1} = \frac{1}{w} \geq \sigma$. Then at an interior maximum of M_0,*

$$\kappa_{\max} \leq \frac{C}{\sigma - \sigma_0}$$

where C is independent of ε. Numerical calculations show $0.3703 < \sigma_0 < 0.3704$.

Proof. Suppose M_0 is attained at an interior point $x_0 \in \Omega$ and let $X_0 = (x_0, u(x_0))$. After a horizontal translation of the origin in $\mathbb{R}^{n+1}$, we may write Σ locally near X_0 as a radial graph

$$X = e^v \mathbf{z}, \quad \mathbf{z} \in \mathbb{S}^n_+ \subset \mathbb{R}^{n+1} \tag{6.2}$$

with $X_0 = e^{v(\mathbf{z}_0)}\mathbf{z}_0$, $\mathbf{z}_0 \in \mathbb{S}^n_+$, such that $\nu(X_0) = \mathbf{z}_0$. Note that the height function $u = ye^v$, where $y = \mathbf{e} \cdot \mathbf{z}$. (Here $\mathbf{e}$ is the vertical unit vector pointing upwards.)

We choose an orthonormal local frame $\tau_1, \ldots, \tau_n$ around $\mathbf{z}_0$ on $\mathbb{S}^n_+$ such that $v_{ij} = \nabla_{\tau_j}\nabla_{\tau_i} v$ is diagonal at $\mathbf{z}_0$, where ∇ denotes the Levi-Civita connection on $\mathbb{S}^n$. As shown in Section 2.1 of [9], the hyperbolic principal curvatures of the radial graph X are the eigenvalues of the matrix $A^{\mathbf{s}}[v] = \{a^{\mathbf{s}}_{ij}[v]\}$:

$$a^{\mathbf{s}}_{ij}[v] := \frac{1}{w}(y\gamma^{ik} v_{kl} \gamma^{lj} - \mathbf{e} \cdot \nabla v \delta_{ij}) \tag{6.3}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{v_i v_j}{w(1+w)}, \quad w = (1 + |\nabla v|^2)^{\frac{1}{2}}.$$

We can then rewrite equation (5.1) in the form

$$F(A^{\mathbf{s}}[v]) = \sigma. \tag{6.4}$$

Henceforth we write $A[v] = A^{\mathbf{s}}[v]$ and $a_{ij} = a^{\mathbf{s}}_{ij}[v]$.

Since $\nu(X_0) = \mathbf{z}_0$, $\nabla v(\mathbf{z}_0) = 0$ and hence

$$a_{ij} = y v_{ij} = \kappa_i \delta_{ij} \tag{6.5}$$

at $\mathbf{z}_0$ by (6.3), where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at X_0. We may assume

$$\kappa_1 = \kappa_{\max}(X_0). \tag{6.6}$$

The function $\frac{a_{11}}{\eta-a}$, which is defined locally near $\mathbf{z}_0$, then achieves its maximum at $\mathbf{z}_0$. By Proposition 5.3 and Lemma 5.4 of [9] we have at $\mathbf{z}_0$:

$$\begin{aligned}\sigma(y-a)\kappa_1^2 + a\kappa_1 \sum f_i\kappa_i^2 + (a - 2(1-y^2)(y-a))\kappa_1 \sum f_i \\ \leq 2\sigma\kappa_1 + \frac{2a\kappa_1}{\alpha} \sum f_i(\kappa_i - \alpha)\, y_i^2 \\ - \frac{2a^2\kappa_1^2}{\alpha^2(y-a)} \sum_{i=2}^n \frac{f_i - f_1}{\kappa_1 - \kappa_i} (\kappa_i - \alpha)^2\, y_i^2\end{aligned} \tag{6.7}$$

where $\alpha = \frac{a\kappa_1}{\kappa_1 - (y-a)}$. The last term in (6.7) comes from the "concavity term"

$$-F^{ij,kl} a_{ij,1} a_{kl,1} \geq \sum_{i=2}^n \frac{f_i - f_1}{\kappa_1 - \kappa_i} (a_{i1,1})^2 \tag{6.8}$$

where, since $(\frac{a_{11}}{\eta-a})_i = 0$,

$$a_{i1,1} = a_{11,i} + \frac{(\kappa_i - \kappa_1) y_i}{y} = -\frac{a\kappa_1(\kappa_i - \alpha) y_i}{\alpha y (y-a)}.$$

We also recall from [6], [9] the identity

$$\sum y_i^2 = 1 - y^2$$

which has been used in (6.7).

It was shown in Section 6 of [9] that the coefficient $\gamma(y)$ of $\kappa_1 \sum f_i$ in (6.7), namely

$$\gamma(y) = a - 2(1-y^2)(y-a) \tag{6.9}$$

satisfies

$$\gamma(y) > \frac{7}{3}a - \frac{4}{27}a^3 - \frac{4}{27}(a^2+3)^{\frac{3}{2}} > 0 \ \text{ on } (a,1) \tag{6.10}$$

if $a^2 > \frac{1}{8}$. From (6.7) it is clear that we need only concern ourselves with

$$I = \{i : \kappa_i > \alpha > a\}.$$

Fix $\theta \in (0,1)$ to be chosen later and let

$$\begin{aligned}J &= \{i \in I : f_1 \leq \theta f_i\}, \\ K &= \{i \in I : f_1 > \theta f_i\}.\end{aligned}$$

Then

$$a\kappa_1 \sum_{i\in J} f_i\kappa_i^2 > a^3\kappa_1 \sum_{i\in J} f_i \tag{6.11}$$

and

$$\frac{2a\kappa_1}{\alpha}\sum_{i\in K} f_i(\kappa_i-\alpha)\,y_i^2 - a\kappa_1^3 f_1 \le \kappa_1^2(\frac{2}{\theta}-a\kappa_1)f_1 < 0, \tag{6.12}$$

provided $\kappa_1 > \frac{2}{a\theta}$. On the other hand,

$$\begin{aligned}
&\sum_{i\in J} f_i(\kappa_i-\alpha)\,y_i^2 - \frac{a\kappa_1}{\alpha(y-a)}\sum_{i\in J}\frac{f_i-f_1}{\kappa_1-\kappa_i}(\kappa_i-\alpha)^2\,y_i^2 \\
&\quad\le \sum_{i\in J} f_i y_i^2\Big((\kappa_i-\alpha)-\frac{(1-\theta)a}{\alpha(y-a)}(\kappa_i-\alpha)^2\Big) \\
&\quad\le \frac{\alpha(y-a)(1-y^2)}{4(1-\theta)a}\sum_{i\in J} f_i \\
&\quad= \frac{\alpha(a-\gamma(y))}{8(1-\theta)a}\sum_{i\in J} f_i.
\end{aligned} \tag{6.13}$$

Combining (6.7), (6.11), (6.12) and (6.13) we obtain

$$\sigma(y-a)\kappa_1^2 + \phi_\theta(y)\kappa_1\sum_{i\in J} f_i \le 2\sigma\kappa_1 \tag{6.14}$$

where the coefficient of $\kappa_1\sum_{i\in J} f_i$ in (6.14) is

$$\phi_\theta(y) = \gamma(y) - \frac{a-\gamma(y)}{4(1-\theta)} + a^3.$$

Note that by (6.10),

$$\begin{aligned}
\phi_0(y) &= \frac{5}{4}\{\gamma(y)+\frac{4}{5}a^3-\frac{a}{5}\} \\
&> \frac{5}{4}\{\frac{7}{3}a-\frac{4}{27}a^3-\frac{4}{27}(a^2+3)^{\frac{3}{2}}+\frac{4}{5}a^3-\frac{a}{5}\} \\
&= \frac{8}{3}a+\frac{22}{27}a^3-\frac{5}{27}(a^2+3)^{\frac{3}{2}} := \phi(a).
\end{aligned} \tag{6.15}$$

For $a\in(0,1)$ it is easily checked that $\phi'(a)>0$, $\phi(0)<0$, $\phi(1)>0$. Let σ_0 be the unique zero of $\phi(a)$ in $(0,1)$. Numerical calculations show that $0.3703<\sigma_0<0.3704$.

Now assume that $2\varepsilon_0 := \sigma-\sigma_0 > 0$ and choose $a=\sigma_0+\varepsilon_0$. Then $\phi_\theta(y)>0$ on $(a,1)$ if $\theta>0$ is chosen sufficiently small. By Proposition 4.1, $y-a\ge\sigma-a\ge\varepsilon_0$ at $\mathbf{z}_0$, so by (6.14) (assuming $\kappa_1>\frac{2}{a\theta}$) we obtain $\varepsilon_0\kappa_1^2\le 2\kappa_1$. Hence

$$\kappa_1 \le 2\max\Big\{\frac{1}{a\theta},\frac{1}{\varepsilon_0}\Big\} = 4\max\Big\{\frac{1}{\theta(\sigma+\sigma_0)},\frac{1}{\sigma-\sigma_0}\Big\}$$

completing the proof of Theorem 6.1. □

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