

**HYPERSURFACES OF CONSTANT MEAN CURVATURE IN  
HYPERBOLIC SPACE WITH PRESCRIBED ASYMPTOTIC BOUNDARY  
AT INFINITY**

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*Abstract.* In this paper we study the problem of finding smooth complete hypersurfaces of constant mean curvature in hyperbolic space with a prescribed asymptotic boundary at infinity. Our main results are proved by deriving *a priori* global gradient estimates based on the classical maximum principle for elliptic equations.

**1. Introduction.** In this paper we are concerned with questions of existence and uniqueness of complete hypersurfaces of constant mean curvature in hyperbolic space  $\mathbb{H}^{n+1}$  with a prescribed asymptotic boundary at infinity. We will use the half-space model, i.e.,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$(1.1) \quad ds_H^2 = \frac{1}{x_{n+1}^2} ds_E^2,$$

where  $ds_E^2$  denotes the Euclidean metric on  $\mathbb{R}^{n+1}$ . We denote by  $\partial_\infty \mathbb{H}^{n+1}$  the boundary at infinity of  $\mathbb{H}^{n+1}$ , which is identified with  $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ . Given an  $(n-1)$  dimensional closed submanifold  $\Gamma$  of  $\partial_\infty \mathbb{H}^{n+1}$ , one asks whether there exist hypersurfaces of constant mean curvature in  $\mathbb{H}^{n+1}$  asymptotic to  $\Gamma$  at infinity. Using methods of geometric measure theory, Anderson [2] first proved the existence of a complete area-minimizing integrable  $n$ -current  $\Sigma$  asymptotic to  $\Gamma$ . In [4] and [6], Hardt and Lin established the boundary regularity at infinity of such area-minimizing  $\Sigma$  that says any singularity of  $\Sigma$  must remain in a bounded region of  $\mathbb{H}^{n+1}$ . These results were extended by Tonegawa [8] to the constant mean curvature case. See also [1] for related results. There arises a natural question: when does there exist a smooth constant mean curvature hypersurface asymptotic to  $\Gamma$  at infinity? In [7], Nelli and Spruck proved that if  $\Gamma$  is the boundary of a

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mean convex domain  $\Omega$  in  $\mathbb{R}^n \times \{0\}$ , then for any  $H \in (0, 1)$  there is a unique function  $u \in C^\infty(\Omega)$  whose graph is a hypersurface of constant mean curvature  $H$  with asymptotic boundary  $\Gamma$ . For minimal hypersurfaces ( $H = 0$ ), this was proved by Anderson [2] (see also [6]). In this paper we continue the study initiated in [7] of finding smooth complete hypersurfaces in  $\mathbb{H}^{n+1}$  of constant mean curvature with given asymptotic boundary at infinity. As one of the main results of this paper we will prove:

**THEOREM 1.1.** *Suppose  $\Gamma$  is the boundary of a star-shaped  $C^{1,1}$  domain in  $\mathbb{R}^n$  and let  $|H| < 1$ . Then there exists a unique hypersurface  $\Sigma$  of constant mean curvature  $H$  in  $\mathbb{H}^{n+1}$  with asymptotic boundary  $\Gamma$ . Moreover,  $\Sigma$  may be represented as the radial graph over the upper hemisphere  $\mathbb{S}_+^n \subset \mathbb{R}^{n+1}$  of a function in  $C^\infty(\mathbb{S}_+^n) \cap C^{1,1}(\overline{\mathbb{S}_+^n})$ .*

We are also interested in hypersurfaces that can be written as vertical graphs. In this case, we observe the following important fact.

**THEOREM 1.2.** *Let  $\Sigma$  be a smooth hypersurface of constant hyperbolic mean curvature in  $\mathbb{H}^{n+1}$ . Suppose  $\Sigma$  can be written as a vertical graph  $x_{n+1} = u(x)$  over  $\mathbb{R}^n$ . Then the Euclidean mean curvature  $H_E$  of  $\Sigma$  is a subharmonic function on  $\Sigma$ .*

Throughout this paper, the mean curvatures are calculated with respect to the upward normal for vertical graphs and the outward normal for radial graphs. With the aid of Theorem 1.2, we are able to prove the following existence result for vertical graphs.

**THEOREM 1.3.** *Let  $\Omega$  be a  $C^1$  domain in  $\mathbb{R}^n \times \{0\}$  satisfying the exterior sphere condition. Let  $H \in (0, 1)$  and suppose that*

$$(1.2) \quad \frac{d(\Omega)}{r_1} \leq \frac{H}{1 - H},$$

where  $d(\Omega)$  denotes the diameter of  $\Omega$  and  $r_1$  is the largest number such that for each  $x \in \partial\Omega$  there exists a ball of radius  $r_1$  externally tangent to  $\partial\Omega$  at  $x$  (both in the Euclidean metric). Then there exists a complete hypersurface  $\Sigma$  of constant mean curvature  $H$  in  $\mathbb{H}^{n+1}$  with asymptotic boundary  $\Gamma = \partial\Omega$ . Moreover,  $\Sigma$  is the graph over  $\Omega$  of a function  $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$  satisfying the gradient estimate

$$(1.3) \quad w \equiv \sqrt{1 + |Du|^2} \leq \frac{2}{H} \text{ on } \bar{\Omega}.$$

We remark that while both  $d(\Omega)$  and  $r_1$  depend on the half-space model we use, their quotient  $d(\Omega)/r_1$  may be defined intrinsically with respect to the hyperbolic metric. We also note that it is necessary to assume  $|H| < 1$  in Theorem 1.1 and  $0 < H < 1$  in Theorem 1.3, respectively, as there would be no solutions otherwise. This is a consequence of the maximum principle using horospheres as barriers.

The article is organized as follows. In Section 2 we look at the Euclidean mean curvature  $H_E$  and other quantities of a constant hyperbolic mean curvature hypersurface  $\Sigma$  in  $\mathbb{H}^{n+1}$ . We will derive an identity for  $\Delta H_E$  where  $\Delta$  is the Laplace-Beltrami operator on  $\Sigma$  (with the induced metric from  $\mathbb{H}^{n+1}$ ). As a consequence of this identity we prove Theorem 1.2. Section 3 contains the proof of Theorem 1.3. Finally, in Section 4 we prove Theorem 1.1, the central part of the proof being the global gradient estimates in subsection 4.1, which is equivalent to the global strict star-shapedness of  $\Sigma$ .

**2. The Euclidean mean curvature.** The main purpose of this section is to prove Theorem 1.2. We start by fixing some notation.

Let  $\Sigma$  be a hypersurface in  $\mathbb{H}^{n+1}$ . We use  $g, H, A$  to denote the induced hyperbolic metric on  $\Sigma$ , the mean curvature, and the second fundamental form of  $\Sigma$  with respect to the hyperbolic metric, respectively, and  $g_E, H_E, A_E$  the corresponding quantities with respect to the Euclidean metric. Similarly,  $\Delta$  and  $\Delta_E$  denote the Laplace-Beltrami operators on  $\Sigma$  with respect to the hyperbolic metric and Euclidean metric, respectively. Let  $X$  be the position vector of  $\Sigma$  in  $\mathbb{R}^{n+1}$  and set

$$u = X \cdot \mathbf{e}$$

where, throughout this paper,  $\mathbf{e}$  is the unit vector in the positive  $x_{n+1}$  direction in  $\mathbb{R}^{n+1}$ , and ‘ $\cdot$ ’ denotes the Euclidean inner product in  $\mathbb{R}^{n+1}$ . We refer  $u$  as the *height function* of  $\Sigma$ .

We will always assume that we have chosen a global unit normal vector field  $\mathbf{n}$  to  $\Sigma$  with respect to the hyperbolic metric. This also determines a unit normal  $\nu$  to  $\Sigma$  with respect to the Euclidean metric by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote  $\nu^{n+1} = \mathbf{e} \cdot \nu$ . Let  $\tau_1, \dots, \tau_n$  be an orthonormal frame of vector fields on  $(\Sigma, g)$ . The (hyperbolic) mean curvature of  $\Sigma$  is given by

$$H = \frac{1}{n} \sum \langle D_{\tau_i} \tau_i, \mathbf{n} \rangle = \frac{1}{n} \sum \kappa_i$$

where  $D$  denotes the Levi-Civita connection on  $\mathbb{H}^{n+1}$ , and  $\kappa_1, \dots, \kappa_n$  are the hyperbolic principal curvatures, i.e., the eigenvalues of the second fundamental form  $A$ , which are related to the Euclidean principal curvatures  $\tilde{\kappa}_1, \dots, \tilde{\kappa}_n$  by

$$\kappa_i = \nu^{n+1} + u \tilde{\kappa}_i \quad i = 1, \dots, n.$$

Therefore,

$$(2.1) \quad H = \nu^{n+1} + uH_E.$$

As usual,  $|A|$ ,  $|A_E|$  denote the norms of the second fundamental forms, i.e.,

$$|A|^2 = \sum \kappa_i^2, \quad |A_E|^2 = \sum \tilde{\kappa}_i^2.$$

We see that

$$(2.2) \quad |A|^2 = u^2|A_E|^2 + 2nuH_E\nu^{n+1} + n(\nu_{n+1})^2.$$

The Laplace-Beltrami operators,  $\Delta$  and  $\Delta_E$ , are related by the identity

$$(2.3) \quad \Delta v = u^2\Delta_E v - (n - 2)u\langle \nabla u, \nabla v \rangle_E \quad \text{for } v \in C^2(\Sigma),$$

where  $\nabla$  denotes the covariant derivative, and  $\langle \cdot, \cdot \rangle_E$  the inner product, both with respect to the induced Euclidean metric on  $\Sigma$ . This may be verified by a straightforward calculation using the relation induced from (1.1) between the metrics. In particular,

$$(2.4) \quad \Delta \frac{1}{u} = -\Delta_E u + \frac{n}{u}|\nabla u|_E^2 = -\Delta_E u + \frac{n}{u}(1 - (\nu^{n+1})^2).$$

Here we have used the fact that

$$(2.5) \quad \nabla u = \mathbf{e} - \nu^{n+1}\nu.$$

We also note the following identity

$$(2.6) \quad \Delta \frac{v}{u} = v\Delta \frac{1}{u} + u\Delta_E v - n\langle \nabla u, \nabla v \rangle_E.$$

LEMMA 2.1. *Let  $\Sigma$  be a hypersurface of constant mean curvature  $H$  in  $\mathbb{H}^{n+1}$ . Then*

$$(2.7) \quad \Delta \frac{1}{u} = \frac{n}{u}(1 - H\nu^{n+1}),$$

$$(2.8) \quad \Delta \frac{\nu^{n+1}}{u} = \frac{1}{u}(nH - |A|^2\nu^{n+1}).$$

*Proof.* The first identity, (2.7) which holds even if  $H$  is not constant, may be derived by combining (2.4) with (2.1) and the identity

$$(2.9) \quad \Delta_E X = nH_E\nu,$$

while the second one, (2.8), is the last component of the identity

$$(2.10) \quad \Delta \frac{\nu}{u} = (n - |A|^2) \frac{\nu}{u} + \frac{n}{u} (H - \nu^{n+1}) \mathbf{e}.$$

This should be compared with the corresponding identity in the Minkowski space model (see [5]). To derive (2.10), we start with the identity

$$(2.11) \quad \Delta_E \nu = -|A_E|^2 \nu - n \nabla H_E.$$

By (2.1) and (2.5), we see that

$$(2.12) \quad \nabla H_E = -\frac{1}{u} (H_E \nabla u + \nabla \nu^{n+1}) = -\frac{1}{u} H_E (\mathbf{e} - \nu^{n+1} \nu) - \frac{1}{u} \nabla \nu^{n+1}.$$

One verifies directly that

$$(2.13) \quad \nabla \nu^{n+1} = \nabla (\nu \cdot \mathbf{e}) = (\mathbf{e} \cdot \nabla) \nu.$$

By (2.6), (2.7), (2.11), (2.12), (2.13) and (2.2), we obtain

$$(2.14) \quad \begin{aligned} \Delta \frac{\nu}{u} &= \left( \Delta \frac{1}{u} \right) \nu + u \Delta_E \nu - n \langle \nabla u, \nabla \nu \rangle_E \\ &= (n - |A|^2) \frac{\nu}{u} + n H_E \mathbf{e}. \end{aligned}$$

This proves (2.10). □

Now Theorem 1.2 follows from Lemma 2.1. More precisely, we have:

**THEOREM 2.2.** *Let  $\Sigma$  be a hypersurface of constant mean curvature  $H$  in  $\mathbb{H}^{n+1}$ . Then*

$$(2.15) \quad \Delta H_E = \frac{\nu^{n+1}}{u} (|A|^2 - nH^2).$$

*In particular,  $\Delta H_E \geq 0$  if  $\Sigma$  is a graph.*

*Proof.* By (2.1) we see that (2.15) follows from (2.7) and (2.8). Next, if  $\Sigma$  is the graph of a function  $x_{n+1} = u(x)$  over a domain  $\Omega \subset \mathbb{R}^n$ , then  $\nu^{n+1} = 1/\sqrt{1 + |Du|^2} > 0$ . By Cauchy-Schwarz inequality, it follows from (2.15) that  $\Delta H_E \geq 0$ , i.e.,  $H_E$  is a subharmonic function on  $\Sigma$ . □

**COROLLARY 2.3.** *Let  $\Sigma$  be a hypersurface of constant mean curvature  $H$ ,  $0 < H < 1$ . Suppose  $\Sigma$  is the graph of a function  $x_{n+1} = u(x)$  defined in a domain*

$\Omega \subset \partial_\infty \mathbb{H}^{n+1}$ . For any  $\lambda \in (0, 1)$ ,

$$(2.16) \quad w \equiv \sqrt{1 + |Du|^2} \leq \frac{1}{(1 - \lambda)H} \quad \text{in } \Omega_\lambda$$

where  $\Omega_\lambda = \left\{ x \in \Omega: u(x) \leq \frac{\lambda H}{\sup_{\partial\Sigma} H_E} \right\}$ .

*Proof.* Since  $H_E$  is a subharmonic function on  $\Sigma$ , by the maximum principle

$$(2.17) \quad H_E = \frac{1}{u} \left( H - \frac{1}{w} \right) \leq \sup_{\partial\Sigma} H_E$$

from which (2.16) follows. □

We end this section with the following identity which seems of interest to be included here.

LEMMA 2.4. *Let  $\Sigma$  be a constant mean curvature hypersurface. Then*

$$(2.18) \quad \Delta \frac{X \cdot \nu}{u} = (n - |A|^2) \frac{X \cdot \nu}{u}.$$

*Proof.* It is straightforward to verify that

$$(2.19) \quad \Delta_E(X \cdot \nu) = X \cdot \Delta_E \nu - \nu \cdot \Delta_E X.$$

By (2.5)

$$(2.20) \quad \begin{aligned} \Delta \frac{X \cdot \nu}{u} &= X \cdot \nu \Delta \frac{1}{u} + u \Delta_E(X \cdot \nu) - n \langle \nabla u, \nabla(X \cdot \nu) \rangle_E \\ &= X \cdot \left( \nu \Delta \frac{1}{u} + u \Delta_E \nu - n \langle \nabla u, \nabla \nu \rangle_E \right) - u \nu \cdot \Delta_E X \\ &= X \cdot \Delta \frac{\nu}{u} - u \nu \cdot \Delta_E X. \end{aligned}$$

Now (2.18) follows from (2.9) and (2.10). □

**3. Proof of Theorem 1.3.** We first give some preliminary estimates which will be needed in the proof of Theorem 1.3. In Lemmas 3.1–3.5 below,  $\Sigma$  is a hypersurface of constant mean curvature  $H$ ,  $|H| < 1$ , in  $\mathbb{H}^{n+1}$  with  $\partial\Sigma \subset P(\epsilon) \equiv \{x_{n+1} = \epsilon\}$ ,  $\epsilon \geq 0$ . Let  $\Omega$  be the domain in  $\mathbb{R}^n \times \{0\}$  such that its vertical lift  $\Omega^\epsilon$  to  $P(\epsilon)$  is bounded by  $\partial\Sigma$ . The (hyperbolic and Euclidean) mean curvatures of  $\Sigma$  are calculated with respect to the outward normal direction (to  $\Sigma \cup \overline{\Omega^\epsilon}$ ).

Our estimates are all based on the following fact: Let  $B_R(a)$  be a ball of radius  $R$  centered at  $a = (a', -HR)$  in  $\mathbb{R}^{n+1}$  where  $a' \in \mathbb{R}^n$ . Then  $S = \partial B_R(a) \cap \mathbb{H}^{n+1}$  has constant hyperbolic mean curvature  $H$  with respect to its outward normal.

For convenience, we sometimes call  $S$  a sphere of constant mean curvature  $H$  in  $\mathbb{H}^{n+1}$ . We observe that:

LEMMA 3.1. *Let  $B_1$  and  $B_2$  be balls in  $\mathbb{R}^{n+1}$  of radius  $R$  centered at  $a = (a', -HR)$  and  $b = (b', HR)$ , respectively.*

- (i) *If  $\partial\Sigma \subset B_1$ , then  $\Sigma \subset B_1$ .*
- (ii) *If  $B_1 \cap P(\epsilon) \subset \Omega^\epsilon$ , then  $B_1 \cap \Sigma = \emptyset$ .*
- (iii) *If  $B_2 \cap \Omega^\epsilon = \emptyset$ , then  $B_2 \cap \Sigma = \emptyset$ .*

*Proof.* This follows from the maximum principle by performing homothetic dilations (hyperbolic isometries) from  $(a', 0)$  and  $(b', 0)$ , respectively. For (i) we expand  $B_1$  continuously until it contains  $\Sigma$ ; for (ii) and (iii) we shrink  $B_1$  and  $B_2$  until they are respectively inside and outside  $\Sigma$ . Since the mean curvature of  $\Sigma$  is calculated with respect to its outward normal direction and  $\partial B_2$  has mean curvature  $H$  with respect to its inward normal, we see by the maximum principle that  $\Sigma$  cannot touch  $B_1$  or  $B_2$  when we reverse this process.  $\square$

Lemma 3.1 allows us to use spheres in  $\mathbb{H}^{n+1}$  as barriers in deriving estimates. The first is a height estimate.

LEMMA 3.2. *On  $\Sigma$  there holds the height estimate*

$$u < \frac{d(\Omega)}{2} \sqrt{\frac{1-H}{1+H}} + \epsilon,$$

where  $d(\Omega)$  is the (Euclidean) diameter of  $\Omega$ .

*Proof.* Let  $B$  be a ball of radius  $R$  with center on the plane  $x_{n+1} = -HR$  such that the  $n$ -ball  $B \cap P(\epsilon)$  has radius  $r = d(\Omega)/2$  and contains  $\Omega^\epsilon$ . By Lemma 3.1 (i),  $\Sigma$  is contained in  $B \cap \mathbb{H}^{n+1}$  and therefore

$$u < (1 - H)R \quad \text{on } \Sigma.$$

On the other hand,  $R^2 = (\epsilon + HR)^2 + r^2$ , which implies

$$(3.1) \quad \frac{r}{\sqrt{1-H^2}} + \frac{H}{1-H^2}\epsilon \leq R \leq \frac{r}{\sqrt{1-H^2}} + \frac{1+H}{1-H^2}\epsilon.$$

This completes the proof.  $\square$

Next, we estimate the Euclidean mean curvature of  $\Sigma$  on its boundary. Assume  $\Omega$  is a  $C^{1,1}$  domain. For  $x \in \partial\Omega$ , we denote by  $r_1(x)$  and  $r_2(x)$  the radius of the largest exterior and interior spheres to  $\partial\Omega$  at  $x$ , respectively, and let  $r_1 \equiv \min_{x \in \partial\Omega} r_1(x)$ ,  $r_2 \equiv \min_{x \in \partial\Omega} r_2(x)$ .

LEMMA 3.3. For  $\epsilon > 0$  sufficiently small,

$$(3.2) \quad -\frac{\sqrt{1-H^2}}{r_2} - \frac{\epsilon(1-H)}{r_2^2} < \frac{H-\nu^{n+1}}{u} < \frac{\sqrt{1-H^2}}{r_1} + \frac{\epsilon(1+H)}{r_1^2} \quad \text{on } \partial\Sigma.$$

In particular,  $\nu^{n+1} \rightarrow H$  on  $\partial\Sigma$  as  $\epsilon \rightarrow 0$ , provided that  $\partial\Omega$  is  $C^{1,1}$ .

*Proof.* We first assume  $r_1 < \infty$ . For a fixed point  $x_0 \in \partial\Omega$ , let  $\mathbf{e}_1$  be the outward unit normal vector to  $\partial\Omega$  at  $x_0$ . Let  $B_i, i = 1, 2$  be the balls in  $\mathbb{R}^{n+1}$  of radius  $R_i$  centered at  $a_i = (x_0 - (-1)^i r_i \mathbf{e}_1, -(-1)^i R_i H)$  where  $R_i$  satisfy  $R_i^2 = r_i^2 + (\epsilon + (-1)^i R_i H)^2, i = 1, 2$ . Note that  $B_1 \cap P(\epsilon)$  is an  $n$ -ball of radius  $r_1$  externally tangent to  $\partial\Omega^\epsilon$  while  $B_2 \cap P(\epsilon)$  is an  $n$ -ball of radius  $r_2$  internally tangent to  $\partial\Omega^\epsilon$ . By Lemma 3.3 (ii) and (iii),  $B_i \cap \Sigma = \emptyset, i = 1, 2$ . Thus

$$-\frac{u - HR_1}{R_1} < \nu^{n+1} < \frac{u + HR_2}{R_2} \quad \text{at } x_0.$$

That is,

$$-\frac{1}{R_2} < \frac{H - \nu^{n+1}}{u} < \frac{1}{R_1}.$$

Finally, from  $R_i^2 = r_i^2 + (\epsilon + (-1)^i R_i H)^2$  we obtain

$$R_i = \frac{r_i^2 + \epsilon^2}{\sqrt{(1-H^2)r_i^2 + \epsilon^2} - (-1)^i H\epsilon} \geq \frac{r_i^2}{\sqrt{(1-H^2)r_i + (1 - (-1)^i H)\epsilon}}, \quad i = 1, 2.$$

This gives (3.2) as desired.

If  $r_1 = \infty$  (i.e.,  $\Omega$  is convex), in the above argument one can replace  $r_1$  by any  $r > 0$  and then let  $r \rightarrow \infty$ . □

When  $\Sigma$  is a graph over  $\Omega$  that lies in a narrow slab, the following gives a better estimate of the height function.

LEMMA 3.4. Assume  $\Sigma$  is a graph with  $\partial\Sigma$  contained in a slab of width  $2R$  in  $P(\epsilon)$ . Then

$$u \leq n(1-H)R + \epsilon, \quad \text{on } \Sigma.$$

*Proof.* For simplicity we may assume that  $\partial\Sigma$  is contained in  $\{-R \leq x_1 \leq R\}$  in the plane  $x_{n+1} = \epsilon$ . For  $t > 0$  consider the function  $v_t(x) = t + \sqrt{R^2 - x_1^2}$ ; the hyperbolic mean curvature of its graph is given by

$$H_t = \frac{-t + (n-1)\sqrt{R^2 - x_1^2}}{nR}.$$



Thus  $H_t \leq H$  for  $t \geq (n - 1 - nH)R$ . It follows by the maximum principle that  $u \leq v_{t_0} \leq n(1 - H)R + \epsilon$  for  $t_0 = (n - 1 - nH)R + \epsilon$ , since  $u \leq v_t$  for  $t$  sufficiently large.  $\square$

We will also need a lower bound of the height functions for graphs.

LEMMA 3.5. *Suppose  $\Sigma$  is a graph  $x_{n+1} = u(x)$  over  $\Omega$  and  $0 \leq H < 1$ . Then*

$$(3.3) \quad u(x) \geq d(x) \sqrt{\frac{1 - H}{1 + H}} + \frac{H\epsilon}{1 + H}, \quad x \in \Omega$$

where  $d(x)$  is the distance from  $x$  to  $\partial\Omega$ .

*Proof.* For  $x \in \Omega$ , let  $r = d(x)$  and  $R > 0$  satisfy  $R^2 = (\epsilon + HR)^2 + d(x)^2$ . Note that  $B_R(x, -HR) \cap P(\epsilon) \subset \Omega^\epsilon$ . By Lemma 3.1 (ii),

$$u(x) > (1 - H)R.$$

By the first inequality in (3.1) we obtain (3.3) as desired.  $\square$

We now proceed to the proof of Theorem 1.3. Let  $\Sigma \subset \mathbb{H}^{n+1}$  be a graph over a domain  $\Omega$  in  $\mathbb{R}^n \equiv \mathbb{R}^n \times \{0\} = \partial_\infty \mathbb{H}^{n+1}$ , i.e., the height function  $u$  of  $\Sigma$  is a function defined on  $\Omega$ . Then  $u$  satisfies the hyperbolic mean curvature equation

$$(3.4) \quad \left( \delta_{ij} - \frac{u_{x_i} u_{x_j}}{w^2} \right) u_{x_i x_j} = \frac{n(wH - 1)}{u} \quad \text{in } \Omega,$$

where  $w = (1 + |Du|^2)^{\frac{1}{2}}$ . This quasilinear elliptic equation is degenerate where  $u = 0$ . In order to prove the existence of a solution, we first consider the boundary condition

$$(3.5) \quad u = \epsilon \quad \text{on } \partial\Omega$$

for fixed constant  $\epsilon > 0$  sufficiently small.

THEOREM 3.6. *Let  $\Omega$  be a  $C^1$  domain satisfying condition (1.2) and  $0 < H < 1$ . Then problem (3.4)–(3.5) admits a solution  $u^\epsilon \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$  for every positive constant  $\epsilon < r_1 H \sqrt{1 - H}/4$ . Moreover, for any positive  $\delta < 1$ , there exists  $\epsilon_0 > 0$  such that  $u^\epsilon$  satisfies the gradient estimate*

$$(3.6) \quad w^\epsilon \equiv \sqrt{1 + |Du^\epsilon|^2} \leq \frac{2}{H(1 - \delta)} \quad \text{on } \bar{\Omega} \text{ for all } \epsilon \leq \epsilon_0.$$

*Proof.* We first assume  $u^\epsilon \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$  is a solution of (3.4)–(3.5). Note that, since the constant function  $\underline{u}^\epsilon \equiv \epsilon$  is a subsolution, by the comparison

principle and Lemma 3.2 we have

$$(3.7) \quad \epsilon \leq u^\epsilon \leq \frac{d(\Omega)}{2} \sqrt{\frac{1-H}{1+H}} + \epsilon \quad \text{on } \bar{\Omega}.$$

Thus, for  $0 < \delta < 1$ , by (2.17) and Lemma 3.3

$$(3.8) \quad \begin{aligned} H - \frac{1}{w^\epsilon} &\leq u^\epsilon \left( \frac{\sqrt{1-H^2}}{r_1} + \frac{(1+H)\epsilon}{r_1^2} \right) \\ &\leq \frac{(1-H)d(\Omega)}{2r_1} + \frac{\epsilon\sqrt{1-H^2}}{r_1} \left( 1 + \frac{d(\Omega)}{2r_1} \right) + \frac{(1+H)\epsilon^2}{r_1^2} \\ &\leq \frac{H}{2} + \frac{(2-H)\epsilon}{r_1\sqrt{1-H}} + \frac{(1+H)\epsilon^2}{r_1^2} \\ &\leq \frac{H}{2} + \frac{\delta H}{2}, \end{aligned}$$

by (1.2), when  $\epsilon \leq \epsilon_0 = \delta r_1 H \sqrt{1-H}/4$ . This proves (3.6). By the classical elliptic PDE theory (see, e.g., [3]), the solvability of (3.4)–(3.5) in  $C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$  follows from the *a priori* estimates (3.6) and (3.7).  $\square$

*Proof of Theorem 1.3.* Let  $K$  be a compact subset of  $\Omega$ . By Lemma 3.5, there exists  $c_0 > 0$ , independent of  $\epsilon$ , such that

$$(3.9) \quad u^\epsilon \geq c_0 \quad \text{on } K.$$

By Schauder theory, it follows from (3.6), (3.7) and (3.9) that for all  $k \geq 2$ ,

$$(3.10) \quad \|u^\epsilon\|_{C^k(K)} \leq C, \quad \text{independent of } \epsilon.$$

Thus there is a sequence  $\epsilon_k \rightarrow 0$  such that  $u^{\epsilon_k}$  converges to a solution  $u \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$  of (3.4) with  $u = 0$  on  $\partial\Omega$ . By the regularity results of Tonegawa [8],  $u \in C^1(\bar{\Omega})$ . Finally, by (3.6) we see that  $u$  satisfies (1.3).  $\square$

Using Lemma 3.4 in place of Lemma 3.2 in the proof of Theorem 3.6, the above argument yields the following variation of Theorem 1.3.

**THEOREM 3.7.** *Let  $\Omega$  be as in Theorem 1.3 and let  $0 < H < 1$ . Suppose in addition that  $\Omega$  is contained in a slab of width  $d$  in  $\mathbb{R}^n$  such that*

$$(3.11) \quad \frac{d}{r_1} \leq \frac{H}{n(1-H)^{\frac{3}{2}}}.$$

*Then there exists a function  $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$  whose graph is a hypersurface of constant mean curvature  $H$  in  $\mathbb{H}^{n+1}$  with asymptotic boundary  $\Gamma = \partial\Omega$ .*

**4. Starshaped constant mean curvature hypersurfaces.** The main purpose of this section is to prove Theorem 1.1. Throughout this section, let  $\nabla$  denote the covariant derivative on the standard unit sphere  $\mathbb{S}^n$  in  $\mathbb{R}^{n+1}$  and  $y = \mathbf{e} \cdot \mathbf{x}$  for  $\mathbf{x} \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ . Let  $\tau_1, \dots, \tau_n$  be a local frame of smooth vector fields on the upper hemisphere  $\mathbb{S}^n$ . We denote by  $\sigma_{ij} = \tau_i \cdot \tau_j$  the standard metric of  $\mathbb{S}^n$  and  $\sigma^{ij}$  its inverse. For a function  $v$  on  $\mathbb{S}^n$ , we denote  $v_i = \nabla_i v = \nabla_{\tau_i} v$ ,  $v_{ij} = \nabla_j \nabla_i v$ , etc.

Suppose  $\Sigma$  is a radial graph over the upper hemisphere  $\mathbb{S}_+^n \subset \mathbb{R}^{n+1}$ , i.e., it can be represented as

$$X = e^v \mathbf{x}, \quad \mathbf{x} \in \mathbb{S}_+^n \subset \mathbb{R}^{n+1}.$$

The Euclidean outward unit normal vector and mean curvature of  $\Sigma$  are

$$\nu = \frac{\mathbf{x} - \nabla v}{w}$$

and

$$H_E = \frac{a^{ij} v_{ij} - n}{ne^v w}$$

respectively, where  $a^{ij} = \sigma^{ij} - \frac{\sigma^{ik} v_k v_j}{w^2}$  and  $w = (1 + |\nabla v|^2)^{1/2}$ . The hyperbolic mean curvature of  $\Sigma$  is therefore given by

$$(4.1) \quad H = ye^v H_E + \frac{y - \mathbf{e} \cdot \nabla v}{w} = \frac{ya^{ij} v_{ij}}{nw} - \frac{\mathbf{e} \cdot \nabla v}{w}.$$

Consequently,  $v$  satisfies the elliptic equation

$$(4.2) \quad a^{ij} v_{ij} = \frac{n}{y}(Hw + \mathbf{e} \cdot \nabla v).$$

The existence part of Theorem 1.1 is a consequence of the following theorem.

**THEOREM 4.1.** *Let  $H$  be a constant,  $|H| < 1$ , and  $\varphi \in C^{1,1}(\partial\mathbb{S}_+^n)$ . Then there exists a unique solution  $v \in C^\infty(\mathbb{S}_+^n) \cap C^{1,1}(\overline{\mathbb{S}_+^n})$  of (4.2) satisfying the boundary condition*

$$(4.3) \quad v = \varphi \quad \text{on } \partial\mathbb{S}_+^n.$$

The uniqueness in Theorem 4.1 is a direct consequence of the maximum principle. We will prove the existence in three steps. In subsection 4.1, we construct an auxiliary function satisfying the maximum principle from which we derive global gradient bounds for solutions of equation (4.2). In subsection 4.2, we show the existence of a solution in  $C^\infty(\mathbb{S}_+^n) \cap C^{0,1}(\overline{\mathbb{S}_+^n})$  by an approximation approach. In subsection 4.3, we present a proof of the  $C^{1,1}$  boundary regularity,

therefore completing the proof of Theorem 4.1. Note that the  $C^{1,1}$  boundary regularity could be derived from the results of Tonegawa [8]. For completeness we include a proof here which is a bit more natural in our setting. In subsection 4.4, we derive an interior gradient estimate that allows us to prove the existence of a unique solution to (4.2)–(4.3) belonging to  $C^\infty(\mathbb{S}_+^n) \cap C^0(\overline{\mathbb{S}_+^n})$  when  $\varphi$  is only continuous. We conclude the paper with a remark about the uniqueness in subsection 4.5.

**4.1. Global gradient bounds.** Here we prove the following result that is important to our proof of Theorem 4.1.

**THEOREM 4.2.** *Let  $v$  be a  $C^3$  function satisfying equation (4.2) in a domain  $\Omega \subset \mathbb{S}_+^n$ . Then*

$$(4.4) \quad \mathcal{L}(e^v(w + H(y + \mathbf{e} \cdot \nabla v))) \geq 0 \quad \text{in } \Omega,$$

where  $\mathcal{L}$  is the linear elliptic operator

$$\mathcal{L} \equiv a^{ij} \nabla_{ij} - \frac{2}{w} a^{ij} w_i \nabla_j - \frac{n}{wy} (H \nabla v + w \mathbf{e}) \cdot \nabla.$$

Consequently, by the maximum principle,

$$\max_{\Omega} e^v(w + H(y + \mathbf{e} \cdot \nabla v)) = \max_{\partial\Omega} e^v(w + H(y + \mathbf{e} \cdot \nabla v)).$$

*Proof.* We assume the local vector fields  $\tau_1, \dots, \tau_n$  to be orthonormal so that  $\sigma_{ij} = \delta_{ij}$ . We start with the calculation of  $\mathcal{L}y$  and  $\mathcal{L}v$ . The covariant derivatives of  $y$  are

$$y_i = \nabla_i y = (\mathbf{e} \cdot \mathbf{x})_i = \mathbf{e} \cdot \tau_i,$$

$$y_{ij} = \nabla_i \nabla_j y = \mathbf{e} \cdot \nabla_i \nabla_j \mathbf{x} = \mathbf{e} \cdot \nabla_i \tau_j = -\mathbf{e} \cdot \mathbf{x} \delta_{ij} = -y \delta_{ij}.$$

Therefore

$$(4.5) \quad \mathbf{e} \cdot \nabla y = \sum (\mathbf{e} \cdot \tau_i)^2 = 1 - y^2,$$

$$(4.6) \quad \nabla v \cdot \nabla y = \mathbf{e} \cdot \nabla v.$$

We also note the identities

$$(4.7) \quad a^{ij} v_i = \frac{v_j}{w^2}, \quad a^{ij} v_i v_j = 1 - \frac{1}{w^2}, \quad \sum a^{ii} = n - 1 + \frac{1}{w^2}.$$

One sees that

$$(4.8) \quad \begin{aligned} \mathcal{L}y &= -y \sum a^{ii} - \frac{2}{w} a^{ij} w_i y_j - \frac{n}{wy} (H \nabla v + w \mathbf{e}) \cdot \nabla y \\ &= -\frac{2}{w} a^{ij} w_i (\mathbf{e} \cdot \tau_j) - \frac{n}{wy} (H \mathbf{e} \cdot \nabla v + w) + y - \frac{y}{w^2}, \end{aligned}$$

and, by (4.2)

$$(4.9) \quad \mathcal{L}v = -\frac{2 \nabla v \cdot \nabla w}{w^3} + \frac{nH}{wy}.$$

In order to compute  $\mathcal{L}w$  and  $\mathcal{L}(\mathbf{e} \cdot \nabla v)$ , we differentiate equation (4.2) with respect to  $\tau_k$  and use the identity

$$(4.10) \quad (\nabla_k a^{ij}) v_{ij} = -\frac{2}{w} a^{ij} w_i v_{kj}$$

to obtain

$$(4.11) \quad a^{ij} v_{ijk} - \frac{2}{w} a^{ij} w_i v_{kj} = \frac{n}{y} (H w_k + (\mathbf{e} \cdot \nabla v)_k) - \frac{n}{y^2} (H w + \mathbf{e} \cdot \nabla v) \nabla_{ky}.$$

Multiplying (4.11) by  $v_k$  and  $y_k$ , respectively, and taking sum over  $k$ , we see that

$$a^{ij} v_k v_{ijk} - \frac{2}{w} a^{ij} w_i v_k v_{kj} = \frac{n}{y} \nabla v \cdot (H \nabla w + \nabla(\mathbf{e} \cdot \nabla v)) - \frac{n \mathbf{e} \cdot \nabla v}{y^2} (H w + \mathbf{e} \cdot \nabla v),$$

and

$$a^{ij} y_k v_{ijk} - \frac{2}{w} a^{ij} w_i y_k v_{kj} = \frac{n}{y} \mathbf{e} \cdot (H \nabla w + \nabla(\mathbf{e} \cdot \nabla v)) - \frac{n(1-y^2)}{y^2} (H w + \mathbf{e} \cdot \nabla v).$$

Now,

$$w_i = \frac{v_k v_{ki}}{w}, \quad w_{ij} = \frac{v_k v_{kij}}{w} + \frac{1}{w} a^{kl} v_{ki} v_{lj},$$

$$(\mathbf{e} \cdot \nabla v)_i = (\mathbf{e} \cdot \tau_k v_k)_i = \mathbf{e} \cdot \tau_k v_{ki} - y v_i = y_k v_{ki} - y v_i,$$

$$(\mathbf{e} \cdot \nabla v)_{ij} = \mathbf{e} \cdot \tau_k v_{kij} - 2y v_{ij} - \mathbf{e} \cdot \tau_j v_i = y_k v_{kij} - 2y v_{ij} - y_j v_i,$$

and

$$\nabla v \cdot \nabla(\mathbf{e} \cdot \nabla v) = v_i (\mathbf{e} \cdot \tau_k v_{ki} - y v_i) = w \mathbf{e} \cdot \nabla w - y(w^2 - 1).$$

By the formula for commuting the covariant derivatives

$$(4.12) \quad v_{ijk} = v_{kij} + v_j \delta_{ik} - v_k \delta_{ij}$$

we have

$$(4.13) \quad \begin{aligned} \mathcal{L}w &= \frac{1}{w} a^{ij} v_k v_{kij} + \frac{1}{w} a^{ij} a^{kl} v_{ki} v_{lj} - \frac{2}{w^2} a^{ij} w_i v_k v_{kj} \\ &\quad - \frac{n}{wy} (H \nabla v + w \mathbf{e}) \cdot \nabla w \\ &= \frac{1}{w} a^{ij} a^{kl} v_{ik} v_{jl} - \frac{1}{w} a^{ij} v_i v_j + \frac{|\nabla v|^2}{w} \sum a^{ii} \\ &\quad + \frac{n}{wy} (\nabla v \cdot \nabla (\mathbf{e} \cdot \nabla v) - w \mathbf{e} \cdot \nabla w) - \frac{n \mathbf{e} \cdot \nabla v}{wy^2} (Hw + \mathbf{e} \cdot \nabla v) \\ &= \frac{1}{w} a^{ij} a^{kl} v_{ik} v_{jl} - \frac{n \mathbf{e} \cdot \nabla v}{wy^2} (Hw + \mathbf{e} \cdot \nabla v) - \frac{w^2 - 1}{w} \end{aligned}$$

and

$$(4.14) \quad \begin{aligned} \mathcal{L}(\mathbf{e} \cdot \nabla v) &= a^{ij} (y_k v_{kij} - 2y v_{ij} - v_i y_j) \\ &\quad - \frac{2}{w} a^{ij} w_i (v_{kj} y_k - y v_j) \\ &\quad - \frac{n}{wy} (H \nabla v + w \mathbf{e}) \cdot \nabla (\mathbf{e} \cdot \nabla v) \\ &= -2a^{ij} v_i y_j + y_k v_k \sum a^{ii} - 2n(Hw + \mathbf{e} \cdot \nabla v) \\ &\quad + \frac{2y}{w^3} \nabla v \cdot \nabla w \\ &\quad - \frac{n}{wy} (\nabla v \cdot \nabla (\mathbf{e} \cdot \nabla v) - w \mathbf{e} \cdot \nabla w) - \frac{n(1 - y^2)}{y^2} \\ &\quad \times (Hw + \mathbf{e} \cdot \nabla v) \\ &= \frac{2y}{w^3} \nabla v \cdot \nabla w - \frac{n}{y^2} (Hw + \mathbf{e} \cdot \nabla v) - \frac{nH}{w} \\ &\quad - \frac{w^2 + 1}{w^2} \mathbf{e} \cdot \nabla v. \end{aligned}$$

By Cauchy-Schwarz inequality

$$(4.15) \quad a^{ij} a^{kl} v_{ik} v_{jl} \geq \frac{1}{n} (a^{ij} v_{ij})^2 = \frac{n}{y^2} (Hw + \mathbf{e} \cdot \nabla v)^2.$$

It follows from (4.13) that

$$(4.16) \quad \mathcal{L}w \geq \frac{nH}{y^2} (Hw + \mathbf{e} \cdot \nabla v) - \frac{w^2 - 1}{w}.$$

Next, we note that for a function  $\eta$  defined on  $\Omega$ ,

$$(4.17) \quad \begin{aligned} e^{-v} \mathcal{L}(e^v \eta) &= \mathcal{L}\eta + \eta(\mathcal{L}v + a^{ij} v_i v_j) + 2a^{ij} v_i \eta_j \\ &= \mathcal{L}\eta + \frac{2}{w^3} \nabla v \cdot (w \nabla \eta - \eta \nabla w) + \eta \left( 1 - \frac{1}{w^2} + \frac{nH}{wy} \right) \end{aligned}$$

by (4.9). In particular,

$$(4.18) \quad e^{-v} \mathcal{L}(e^v w) \geq \frac{nH}{y^2} (Hw + \mathbf{e} \cdot \nabla v) + \frac{nH}{y},$$

by (4.16), and

$$(4.19) \quad e^{-v} \mathcal{L}(e^v y) = \mathcal{L}y + \frac{2\mathbf{e} \cdot \nabla v}{w^2} - \frac{2y}{w^3} \nabla v \cdot \nabla w + y \left( 1 - \frac{1}{w^2} \right) + \frac{nH}{w},$$

$$(4.20) \quad \begin{aligned} e^{-v} \mathcal{L}(e^v \mathbf{e} \cdot \nabla v) &= \mathcal{L}(\mathbf{e} \cdot \nabla v) + \frac{2 \nabla v \cdot \nabla(\mathbf{e} \cdot \nabla v)}{w^2} \\ &\quad - \frac{2\mathbf{e} \cdot \nabla v}{w^3} \nabla v \cdot \nabla w + \mathbf{e} \cdot \nabla v \left( 1 - \frac{1}{w^2} + \frac{nH}{wy} \right). \end{aligned}$$

Combining (4.19) and (4.20),

$$(4.21) \quad \begin{aligned} e^{-v} \mathcal{L}(e^v (y + \mathbf{e} \cdot \nabla v)) &= \mathcal{L}y + \mathcal{L}(\mathbf{e} \cdot \nabla v) + \frac{2}{w^3} (w \nabla v \cdot \nabla(\mathbf{e} \cdot \nabla v) \\ &\quad - (\mathbf{e} \cdot \nabla v) \nabla v \cdot \nabla w) \\ &\quad - \frac{2y}{w^3} \nabla v \cdot \nabla w + y \left( 1 - \frac{1}{w^2} \right) + \frac{nH}{wy} (y + \mathbf{e} \cdot \nabla v) \\ &\quad + \mathbf{e} \cdot \nabla v \left( 1 + \frac{1}{w^2} \right) \\ &= \mathcal{L}y + \frac{2}{w^3} (w^2 \mathbf{e} \cdot \nabla w - (\mathbf{e} \cdot \nabla v) \nabla v \cdot \nabla w) \\ &\quad - y \left( 1 - \frac{1}{w^2} \right) \\ &\quad + \mathcal{L}(\mathbf{e} \cdot \nabla v) - \frac{2y}{w^3} \nabla v \cdot \nabla w + \frac{nH}{w} + \mathbf{e} \cdot \nabla v \\ &\quad \times \left( 1 + \frac{1}{w^2} + \frac{nH}{wy} \right). \end{aligned}$$

Note that

$$a^{ij} w_i (\mathbf{e} \cdot \tau_j) = \mathbf{e} \cdot \nabla w - \frac{(\mathbf{e} \cdot \nabla v) \nabla v \cdot \nabla w}{w^2}.$$

We obtain from (4.21) that

$$(4.22) \quad e^{-v} \mathcal{L}(e^v(y + \mathbf{e} \cdot \nabla v)) = -\frac{n}{y^2}(Hw + \mathbf{e} \cdot \nabla v) - \frac{n}{y}.$$

by (4.8) and (4.14). Finally, (4.4) follows from (4.18) and (4.22). □

*Remark 4.3.* By the maximum principle, from (4.2) we see that

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v \text{ if } H \geq 0, \text{ and } \min_{\overline{\Omega}} v = \min_{\partial\Omega} v \text{ if } H \leq 0.$$

Using spherical caps of hyperbolic mean curvature  $H$  as barriers, we also have the following inequalities

$$\min_{\overline{\Omega}} v \geq \min_{\partial\Omega} v + \frac{1}{2} \log \frac{1-H}{1+H} \text{ if } H \geq 0$$

and

$$\max_{\overline{\Omega}} v \leq \max_{\partial\Omega} v + \frac{1}{2} \log \frac{1-H}{1+H} \text{ if } H \leq 0.$$

**4.2. The existence.** Next, we make use of Theorem 4.2 to prove the existence of a solution to (4.2)–(4.3). Since equation (4.2) is singular near  $\partial\mathbb{S}_+^n$ , we need to take an approximation approach. For fixed  $\epsilon > 0$  sufficiently small, let  $\Gamma^\epsilon$  be the vertical translation of  $\Gamma$  to the plane  $x_{n+1} = \epsilon$  and let  $\Omega^\epsilon$  be the subdomain of  $\mathbb{S}_+^n$  such that  $\Gamma^\epsilon$  is the radial graph of a function  $e^{\varphi^\epsilon}$  over  $\partial\Omega^\epsilon$ , i.e.,  $\Gamma^\epsilon$  is represented by

$$X = e^{\varphi^\epsilon} \mathbf{x}, \quad \mathbf{x} \in \partial\Omega^\epsilon.$$

We consider the approximation problem

$$(4.23) \quad \begin{aligned} a^{ij} v_{ij} &= \frac{n}{y} (Hw + \mathbf{e} \cdot \nabla v) \quad \text{on } \Omega^\epsilon \\ v &= \varphi^\epsilon \quad \text{on } \partial\Omega^\epsilon. \end{aligned}$$

Let  $v^\epsilon \in C^\infty(\Omega^\epsilon) \cap C^{1,\alpha}(\overline{\Omega^\epsilon})$  be a solution to (4.23). Since the corresponding hypersurface

$$\Sigma^\epsilon = \{e^{v^\epsilon} \mathbf{x} : \mathbf{x} \in \overline{\Omega^\epsilon}\} \subset \mathbb{H}^{n+1}$$

has mean curvature  $H$  with boundary  $\Gamma^\epsilon$  contained in the horosphere  $x_{n+1} = \epsilon$ , by



Remark 4.1 and Lemma 3.3 there exists  $\epsilon_0 > 0$  such that

$$(4.24) \quad \max_{\overline{\Omega^\epsilon}} |v^\epsilon| + \max_{\partial\Omega^\epsilon} |\nabla v^\epsilon| \leq C_0, \quad \text{independent of } \epsilon$$

for all  $\epsilon \leq \epsilon_0$ . Moreover, by Theorem 4.2 and (4.24) we have the global gradient bounds

$$(4.25) \quad |\nabla v^\epsilon| \leq \frac{C_1}{1 - |H|} \quad \text{on } \overline{\Omega^\epsilon}$$

where  $C_1$  independent of  $\epsilon$ .

It then follows from the standard theory for quasilinear elliptic equations that (4.23) admits a unique solution  $v^\epsilon \in C^\infty(\Omega^\epsilon) \cap C^{1,\alpha}(\overline{\Omega^\epsilon})$  satisfying the *a priori* estimate

$$(4.26) \quad \|v^\epsilon\|_{C^1(\overline{\Omega^\epsilon})} \leq C, \quad \text{independent of } \epsilon.$$

Moreover, there exists a sequence  $\epsilon_k$  such that there exists

$$v(x) \equiv \lim_{k \rightarrow \infty} v^{\epsilon_k}(x), \quad x \in \mathbb{S}_+^n.$$

By Lemma 4.4,  $v$  extends continuously to  $\overline{\mathbb{S}_+^n}$  and  $v = \varphi$  on  $\partial\mathbb{S}_+^n$ . By the regularity theory and (4.26), we see that  $v \in C^\infty(\mathbb{S}_+^n) \cap C^{0,1}(\overline{\mathbb{S}_+^n})$  satisfying (4.2)–(4.3) and

$$(4.27) \quad \|v\|_{C^{0,1}(\overline{\mathbb{S}_+^n})} \leq C.$$

**4.3. The  $C^{1,1}$  boundary regularity.** To complete our proof of Theorem 4.1, we next show  $v \in C^{1,1}(\overline{\mathbb{S}_+^n})$  by modifying the argument of [7] for vertical graphs. The first step is to obtain a good asymptotic for  $v$  near a point  $\mathbf{x}^0 \in \partial\mathbb{S}_+^n$  corresponding to  $P_0 = e^{\varphi(\mathbf{x}^0)}\mathbf{x}^0 \in \Gamma$ . For convenience, we choose coordinates so that the exterior normal to  $\Gamma$  at  $P_0$  is  $\mathbf{e}_1$ . Let  $\delta_1$  (respectively  $\delta_2$ ) be such that for each point  $P$  on  $\Gamma$ , a ball of radius  $\delta_1$  (respectively  $\delta_2$ ) is internally (respectively externally) tangent to  $\Gamma$  at  $P$ . Let  $B_i$ ,  $i = 1, 2$  be the (Euclidean) balls of radius  $R_i$  centered at  $C_i = P_0 + (-1)^i \delta_i \mathbf{e}_1 + a_i \mathbf{e}$  where  $R_i = \frac{\delta_i}{\sqrt{1-H^2}}$ ,  $a_i = (-1)^i R_i H$ . Recall that  $S_1 = \partial B_1 \cap \mathbb{H}^{n+1}$  has constant (hyperbolic) mean curvature  $H$  with respect to its outward normal while  $S_2 = \partial B_2 \cap \mathbb{H}^{n+1}$  has constant (hyperbolic) mean curvature  $H$  with respect to its inward normal. By our construction,  $B_1$  and  $B_2$  are tangent at  $P_0$  and  $B_1 \cap \partial_\infty \mathbb{H}^{n+1}$  is internally tangent to  $\Gamma$  at  $P_0$  while  $B_2 \cap \partial_\infty \mathbb{H}^{n+1}$  is externally tangent to  $\Gamma$  at  $P_0$ .

LEMMA 4.4.  $S_1$  is interior to  $\Sigma$  and  $S_2$  is exterior to  $\Sigma$ .

*Proof.* This follows from Lemma 3.1 (ii) and (iii). □

Since  $\Gamma$  is uniformly starshaped about the origin,  $\mathbf{x}^0 \cdot \mathbf{e}_1 \geq c_0 > 0$  for some fixed constant  $c_0$  independent of  $P_0$ . Therefore, we can represent  $S_1$  and  $S_2$  near  $P_0$  as radial graphs  $X_i = e^{\varphi_i} \mathbf{x}$ ,  $i = 1, 2$  for  $\mathbf{x} \in \mathbb{S}_+^n \cap B_{\varepsilon_0}(\mathbf{x}^0)$  where  $\varepsilon_0$  depends only on  $\delta_1, \delta_2$  and  $c_0$ .

By Lemma 4.4,

$$(4.28) \quad \varphi_1(\mathbf{x}) \leq v(\mathbf{x}) \leq \varphi_2(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}_+^n \cap B_{\varepsilon_0}(\mathbf{x}^0).$$

Note also that the tangent plane  $T$  to  $S_1$  at  $P_0$  is a radial graph  $T = e^\eta \mathbf{x}$  in  $\mathbb{S}_+^n \cap \{\mathbf{x} \cdot \nu_0 > 0\}$  with

$$\eta(\mathbf{x}) = \log \frac{P_0 \cdot \mathbf{e}_1}{\lambda y + \mathbf{x} \cdot \mathbf{e}_1}$$

where  $\lambda = \frac{H}{\sqrt{1-H^2}}$  and  $\nu_0 = H\mathbf{e} + \sqrt{1-H^2}\mathbf{e}_1$  is the unit normal vector to  $S_1$  at  $P_0$ . We also have

$$(4.29) \quad \varphi_1(\mathbf{x}) \leq \eta(\mathbf{x}) \leq \varphi_2(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}_+^n \cap B_{\varepsilon_0}(\mathbf{x}^0).$$

LEMMA 4.5.  $v(\mathbf{x}) = \eta(\mathbf{x}) + O(|\mathbf{x} - \mathbf{x}^0|^2)$  in  $\mathbb{S}_+^n \cap B_{\varepsilon_0}(\mathbf{x}^0)$ .

*Proof.* By (4.28) and (4.29), it suffices to show

$$\varphi_i(\mathbf{x}) = \eta(\mathbf{x}) + O(|\mathbf{x} - \mathbf{x}^0|^2), \quad i = 1, 2, \quad \text{in } \mathbb{S}_+^n \cap B_{\varepsilon_0}(\mathbf{x}^0).$$

We sketch the argument for  $\varphi_1$ ; the argument for  $\varphi_2$  is the same. Let  $P = e^{\eta(\mathbf{x})} \mathbf{x} \in T$ ,  $P_1 = e^{\varphi_1(\mathbf{x})} \mathbf{x} \in S_1$  and  $\delta = |\mathbf{x} - \mathbf{x}^0|$ . Let  $Q \in T$  be the orthogonal projection of  $P_1$  onto  $T$ . Then

$$(4.30) \quad \begin{aligned} |P_1 - Q| &= (P - P_1) \cdot \nu_0 \\ &= |P - P_1| \mathbf{x} \cdot \nu_0 \\ &= |P - P_1| (\mathbf{x}^0 + (\mathbf{x} - \mathbf{x}^0)) \cdot \nu_0 \\ &\geq |P - P_1| (\mathbf{x}^0 \cdot \nu_0 - \delta) \\ &\geq |P - P_1| (c_0 \sqrt{1-H^2} - \delta) \\ &\geq \frac{c_2}{2} \sqrt{1-H^2} |P - P_1| \end{aligned}$$

by requiring  $\varepsilon_0 \leq \frac{c_2}{2} \sqrt{1-H^2}$ . On the other hand, since  $T$  is the tangent plane to  $S_1$  at  $P_0$ , we have

$$(4.31) \quad |P_1 - Q| = O(|Q - P_0|^2).$$

Now,

$$|Q - P_0| \leq |P - P_0| + |P - Q| \leq |P - P_0| + |P - P_1|$$

and

$$|P - P_0| = |e^{\eta(\mathbf{x})}\mathbf{x} - e^{\eta(\mathbf{x}^0)}\mathbf{x}^0| \leq |e^{\eta(\mathbf{x})} - e^{\eta(\mathbf{x}^0)}| + e^{\eta(\mathbf{x}^0)}|\mathbf{x} - \mathbf{x}^0| \leq C\delta.$$

Combining with (4.30) and (4.31), this gives

$$|e^{\eta(\mathbf{x})} - e^{\varphi_1(\mathbf{x})}| = |(P - P_1)| = O(\delta^2)$$

and therefore

$$|\eta(\mathbf{x}) - \varphi_1(\mathbf{x})| = O(\delta^2)$$

as claimed. □

Now let  $p \in \mathbb{S}_+^n$  and  $\delta$  be the geodesic distance of  $p$  to  $\partial\mathbb{S}_+^n$ . Let  $q \in \partial\mathbb{S}_+^n$  be the closest point to  $p$ . Introduce normal coordinates  $x = (x_1, \dots, x_n)$  in  $T_q\mathbb{S}^n$  with  $x(q) = (0, \dots, 0, \delta)$ . We observe that equation (4.2) may be written in divergence form:

$$\nabla_i \left( \frac{\nabla^i v}{w} \right) = \frac{n}{wy} (Hw + \nabla y \cdot \nabla v)$$

or in local coordinates:

$$(4.32) \quad \frac{1}{\sqrt{\sigma}} \frac{\partial}{\partial x_i} \left( \frac{\sqrt{\sigma} \sigma^{ij}}{w} \frac{\partial v}{\partial x_j} \right) = \frac{n}{y(x)} \left( H + \frac{\sigma^{kl}}{w} \frac{\partial y}{\partial x_k} \frac{\partial v}{\partial x_l} \right)$$

where  $\sigma = \det(\sigma_{ij})$  and  $w^2 = 1 + \sigma^{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}$ . In  $x_n > 0$ , both  $v$  and  $\eta$  satisfy (4.32).

Set  $\tilde{v}(x) = \frac{1}{\delta} v(\delta x)$  and  $\tilde{\eta}(x) = \frac{1}{\delta} \eta(\delta x)$ . Then (4.32) transforms to

$$(4.33) \quad \frac{1}{\sqrt{\tilde{\sigma}}} \frac{\partial}{\partial x_i} \left( \frac{\sqrt{\tilde{\sigma}} \tilde{\sigma}^{ij}}{\tilde{w}} \frac{\partial \tilde{v}}{\partial x_j} \right) = \frac{n}{\tilde{y}(x)} \left( H + \frac{\tilde{\sigma}^{kl}}{\tilde{w}} \frac{\partial \tilde{y}}{\partial x_k} \frac{\partial \tilde{v}}{\partial x_l} \right)$$

where  $\tilde{y}(x) = \frac{1}{\delta} y(\delta x)$ ,  $\tilde{\sigma}_{ij}(x) = \sigma_{ij}(\delta x)$ ,  $\tilde{\sigma} = \det(\tilde{\sigma}_{ij})$  and  $\tilde{w}^2 = 1 + \tilde{\sigma}^{ij} \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j}$ .

In  $B_{\frac{3}{4}}(\tilde{p})$  where  $\tilde{p} = (0, \dots, 0, 1)$ , we observe that since  $\sup |\nabla \tilde{v}| = \sup |\nabla v| \leq C$ , (4.33) is uniformly elliptic and  $\tilde{y} = O(1)$ . Hence we may differentiate (4.33) with respect to  $x_k$  to obtain that  $\frac{\partial \tilde{v}}{\partial x_k}$  also satisfies a uniformly elliptic divergence form equation with an  $L^\infty$  right-hand side. By DeGiorgi-Nash theory,  $\tilde{v}$  is uniformly  $C^{1,\alpha}$  in  $B_{\frac{1}{2}}(\tilde{p})$ .

Since  $\tilde{\eta}$  satisfies the same equation, we see that the difference  $\tilde{v} - \tilde{\eta}$  satisfies a linear elliptic equation  $L(\tilde{v} - \tilde{\eta}) = 0$  with uniformly Hölder continuous coefficients.

This gives by Schauder theory

$$\sup_{B_{\frac{1}{4}}(\bar{p})} |\nabla(\tilde{v} - \tilde{\eta})| \leq \sup_{B_{\frac{1}{2}}(\bar{p})} |\tilde{v} - \tilde{\eta}| \leq C\delta$$

and

$$\sup_{B_{\frac{1}{4}}(\bar{p})} |\nabla^2(\tilde{v} - \tilde{\eta})| \leq \sup_{B_{\frac{1}{2}}(\bar{p})} |\tilde{v} - \tilde{\eta}| \leq C\delta$$

by Lemma 4.5. Returning to the original variables, we obtain

$$|\nabla v| + |\nabla^2 v| \leq C$$

and

$$|\nu^{n+1}(p) - H| \leq C\delta.$$

Thus we have proved:

**THEOREM 4.6.** *Let  $v \in C^\infty(\mathbb{S}_+^n) \cap C^{0,1}(\overline{\mathbb{S}_+^n})$  be a solution of (4.2)–(4.3) and  $\varphi \in C^{1,1}(\overline{\mathbb{S}_+^n})$ . Then  $v \in C^\infty(\mathbb{S}_+^n) \cap C^{1,1}(\overline{\mathbb{S}_+^n})$  and  $\nu^{n+1} = -\frac{\mathbf{e} \cdot \nabla v}{w} = H$  on  $\partial\mathbb{S}_+^n$ .*

It follows from the regularity results of Tonegawa [8] that if  $\varphi \in C^{2,\alpha}(\overline{\mathbb{S}_+^n})$ , then  $v \in C^\infty(\mathbb{S}_+^n) \cap C^{2,\alpha}(\overline{\mathbb{S}_+^n})$ .

**4.4. Interior gradient bounds.** We next want to prove the existence of a solution to (4.2)–(4.3) in  $C^\infty(\mathbb{S}_+^n) \cap C^0(\overline{\mathbb{S}_+^n})$  when  $\varphi$  is only continuous. For this purpose, we need the following *a priori* interior gradient estimate.

**LEMMA 4.7.** *Let  $v \in C^\infty(\mathbb{S}_+^n)$  be a solution of (4.2). Then, for any  $c \in (0, 1)$ ,*

$$w = \sqrt{1 + |\nabla v|^2} \leq C_1 e^{C_2/(y-c)^2} \quad \text{on } \mathbb{S}_+^n \cap \{y > c\}.$$

*Proof.* By adding a constant to  $v$ , we may assume  $c_0 \leq v \leq C_0$  on  $\mathbb{S}_+^n \cap \{y \geq c\}$  for some  $C_0 > c_0 > 0$ . Let  $\mathcal{L}$  be the linear operator as in Theorem 4.2. Note that for a function  $\eta$  defined on  $\mathbb{S}_+^n$ ,

$$(4.34) \quad e^{-K\eta} \mathcal{L}(we^{K\eta}) = \mathcal{L}w + w(K\mathcal{L}\eta + K^2 a^{ij} \eta_i \eta_j) + 2Ka^{ij} w_i \eta_j,$$

where  $K > 0$  is a constant. Taking  $\eta = -v/(y - c)^2$ , we calculate

$$(4.35) \quad \begin{aligned} a^{ij} \eta_i \eta_j &= \frac{a^{ij} v_i v_j}{(y - c)^4} - \frac{4va^{ij} v_i v_j}{(y - c)^5} + \frac{4v^2 a^{ij} y_i y_j}{(y - c)^6} \\ &= \frac{w^2 - 1}{w^2(y - c)^4} - \frac{4\mathbf{v}\mathbf{e} \cdot \nabla v}{w^2(y - c)^5} + \frac{4v^2 a^{ij} y_i y_j}{(y - c)^6} \end{aligned}$$

by (4.6) and (4.7), and

$$\begin{aligned}
 (4.36) \quad \mathcal{L}\eta &= -\frac{\mathcal{L}v}{(y-c)^2} + \frac{2v\mathcal{L}y}{(y-c)^3} + \frac{4a^{ij}v_i y_j}{(y-c)^3} - \frac{6va^{ij}y_i y_j}{(y-c)^4} \\
 &= -\frac{nH}{wy(y-c)^2} - \frac{2nv(\mathbf{He} \cdot \nabla v + w)}{wy(y-c)^3} + \frac{2vy(w^2-1)}{w^2(y-c)^3} \\
 &\quad - \frac{2}{w}a^{ij}w_i \eta_j + \frac{4\mathbf{e} \cdot \nabla v}{w^2(y-c)^3} - \frac{6va^{ij}y_i y_j}{(y-c)^4}
 \end{aligned}$$

by (4.8) and (4.9). Thus, by (4.16),

$$\begin{aligned}
 (4.37) \quad e^{-K\eta}\mathcal{L}(we^{K\eta}) &= \mathcal{L}w + \frac{K^2(w^2-1)}{w(y-c)^4} - \frac{4K(Kv - (y-c)^2)\mathbf{e} \cdot \nabla v}{w(y-c)^5} \\
 &\quad + \frac{2Kvw(2Kv - 3(y-c)^2)a^{ij}y_i y_j}{(y-c)^6} - \frac{nHK}{y(y-c)^2} \\
 &\quad - \frac{2nvK(\mathbf{He} \cdot \nabla v + w)}{y(y-c)^3} + \frac{2vyK(w^2-1)}{w(y-c)^3} \\
 &\geq \frac{w}{(y-c)^4} \left( K^2 - CK - C - \frac{CK^2}{w^2(y-c)} \right)
 \end{aligned}$$

when  $K \geq 3/2c_0$ . Now, since  $\eta$  goes to negative infinity as  $y \rightarrow c^+$ ,  $we^{K\eta}$  achieves its maximum value on  $\{y > c\}$  at an interior point  $\mathbf{x}_0$  where  $\mathcal{L}(we^{K\eta}) \leq 0$ . Note that if  $\lambda > 0$ ,

$$\frac{e^{-\lambda/x}}{x} \leq \frac{1}{\lambda e} \quad \text{for all } x > 0.$$

By (4.37),

$$(K^2 - CK - C)w^2e^{K\eta} - CK \leq 0 \quad \text{at } \mathbf{x}_0.$$

Choosing  $K$  sufficiently large, this gives

$$we^{K\eta} \leq C_1 \quad \text{at } \mathbf{x}_0.$$

Thus

$$w \leq C_1e^{-K\eta} \leq C_1e^{C_0K/(y-c)^2} \quad \text{on } \mathbb{S}_+^n \cap \{y > c\}$$

as desired. □

By the classical elliptic theory with the aid of Theorem 4.1, Lemma 4.7, one can prove by an approximation argument:

THEOREM 4.8. *Let  $\varphi \in C^0(\partial\mathbb{S}_+^n)$  and  $|H| < 1$  Then (4.2)–(4.3) has a unique solution  $v \in C^\infty(\mathbb{S}_+^n) \cap C^0(\overline{\mathbb{S}_+^n})$ .*

**4.5. A final remark on the uniqueness.** As we have seen the existence part of Theorem 1.1 follows from Theorem 4.1, while the uniqueness is a consequence of the comparison principle for surfaces of same mean curvature and the fact that homothetic dilations from points on  $x_{n+1} = 0$  are hyperbolic isometries. We should remark here that the uniqueness in Theorem 1.1 is meant among weak solutions in the sense of Tonegawa [8].

The proof of Theorem 1.1 is complete.

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