

COMPLETE TRANSLATING SOLITONS TO THE MEAN CURVATURE FLOW IN \mathbb{R}^3 WITH NONNEGATIVE MEAN CURVATURE

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ABSTRACT. We prove that any complete immersed two-sided mean convex translating soliton $\Sigma \subset \mathbb{R}^3$ for the mean curvature flow is convex. As a corollary it follows that an entire mean convex graphical translating soliton in \mathbb{R}^3 is the axisymmetric “bowl soliton”. We also show that if the mean curvature of Σ tends to zero at infinity, then Σ can be represented as an entire graph and so is the “bowl soliton”. Finally we classify all locally strictly convex graphical translating solitons defined over strip regions.

1. INTRODUCTION

A complete immersed hypersurface $f : \Sigma^n \rightarrow \mathbb{R}^{n+1}$ with trivial normal bundle (two-sided for short) is called a translating soliton for the mean curvature flow, with respect to a unit direction e_{n+1} , if its mean curvature is given by $H = \langle N, e_{n+1} \rangle$ where N is a global unit normal field for Σ . Then $F(x, t) := f(x) + te_{n+1}$ satisfies

$$\Delta^\Sigma F = HN = \langle N, e_{n+1} \rangle N = (e_{n+1})^\perp = F_t^\perp,$$

thus justifying the terminology. Translating solitons form a special class of eternal solutions for the mean curvature flow that besides having their own intrinsic interest, are models of slow singularity formation. Therefore there has been a great deal of effort in trying to classify them in the case $H > 0$. In this paper, *we shall always assume our translating solitons are mean convex which by abuse of language we take to mean $H > 0$* . The abundance of glueing constructions for translating solitons with high genus and H changing sign (see [23],[24],[25], [13], [11], [27]) suggests a general classification is unlikely.

For $n = 1$ the unique solution is the grim reaper curve $\Gamma : x_2 = \log \sec x_1, |x_1| < \frac{\pi}{2}$, while for $n \geq 2$ we have the one parameter family of convex grim cylinders (see Lemma 5.1 for the $n = 2$ case)

$$(1.1) \quad x_{n+1} = \lambda^2 \log \sec \frac{x_1}{\lambda} + \sum_{k=2}^n \alpha_k x_k, \quad \sum_{k=2}^n \alpha_k^2 = \lambda^2 - 1, \quad |x_1| < \frac{\pi}{2} \lambda, \quad \lambda \geq 1,$$

which can be obtained from the standard grim cylinders $\Gamma \times \mathbb{R}^{n-1}$ by a rotation and scaling. The family of grim graphical solitons in \mathbb{R}^3 :

$$(1.2) \quad u^\lambda(x_1, x_2) = \lambda^2 \log \sec \frac{x_1}{\lambda} \pm Lx_2, \quad L = \sqrt{\lambda^2 - 1}, \quad \lambda \geq 1$$

defined over the strip $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}\}$ will play a central role in our classification of mean convex graphical translating solitons in \mathbb{R}^3 .

Similarly if $x_{m+1} = v(x_1, \dots, x_m)$ is a graphical translating soliton in \mathbb{R}^{m+1} , then for $x = (x_1, \dots, x_n) = (x', x_{m+1}, \dots, x_n)$,

$$(1.3) \quad v^\lambda(x) := \lambda^2 v\left(\frac{x'}{\lambda}\right) + \sum_{k=m+1}^n \alpha_k x_k, \quad \sum_{k=m+1}^n \alpha_k^2 = \lambda^2 - 1, \quad \lambda \geq 1,$$

is a graphical translating soliton in $\mathbb{R}^{n+1} = \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$. Conversely if $x_{n+1} = u(x_1, \dots, x_n)$ is a *convex* graphical translating soliton in \mathbb{R}^{n+1} , then D^2u has constant rank m , $1 \leq m < n$ by Corollary 1.3 of Bian and Guan [4] which implies $u(x) = v^\lambda(x)$ for an appropriate choice of x_1, \dots, x_n and $\alpha_{m+1}, \dots, \alpha_n$. Moreover by the results of Wang [32], $\Sigma = \text{graph}(v)$ is a complete graph in \mathbb{R}^{m+1} defined over a strip in \mathbb{R}^m or is an entire graph over \mathbb{R}^m .

There is as well a unique (up to horizontal translation) axisymmetric solution called the “bowl soliton” [1], [8] which has the asymptotic expansion as an entire graph $u(x) = \frac{1}{2(n-1)}|x|^2 - \log|x| + O(1)$.

White [34] tentatively conjectured that all convex translating solutions have the form

$$\{(x, y, z) \in \mathbb{R}^j \times \mathbb{R}^{n-j} \times \mathbb{R} : z = f(|x|)\}$$

for $j \geq 2$; for $j = 1$, f defines the grim reaper curves so is defined on an interval. White remarks that even if this conjecture is false, *it may be true for blow up limits of mean convex mean curvature flows*. In [32] Wang proved that in dimension $n = 2$, any entire convex graphical translating soliton must be rotationally symmetric, and hence the bowl soliton. He also showed there exist entire locally strictly convex graphical translating solitons for dimensions $n > 2$ that are not rotationally symmetric (thus disproving one conjecture of White [34]) as well as complete locally strictly convex graphical translating solitons defined over strip regions in \mathbb{R}^n . Wang also conjectured that for $n = 2$, any entire graphical translating soliton must be locally strictly convex and a similar statement in dimension $n > 2$ under the additional assumption that H tends to zero at infinity.

More recently, Haslhofer [16] proved the uniqueness of the bowl soliton in arbitrary dimension under the assumption that the translating soliton Σ is α -noncollapsed and uniformly 2-convex. The α -noncollapsed condition means that for each $P \in \Sigma$, there are closed balls B^\pm disjoint from $\Sigma - P$ of radius at least $\frac{\alpha}{H(P)}$ with $B^+ \cap B^- = \{P\}$. It figures prominently in the regularity theory for mean convex mean curvature flow [33], [34], [19], [29]. The 2-convex condition (automatic if $n = 2$) means that if $\kappa_n \leq \kappa_{n-1} \leq \dots \leq \kappa_1$ are the ordered principal curvatures of Σ , then $\kappa_n + \kappa_{n-1} \geq \beta H$ for some uniform $\beta > 0$. The α -noncollapsed condition is a deep and powerful property of weak solutions of the mean convex mean curvature flow [34],[17] which implies that any complete (α -noncollapsed) mean convex translating soliton Σ is convex with uniformly bounded second fundamental form.

The main result of this paper is a proof of a more general form of the $n = 2$ conjecture of Wang.

Theorem 1.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with nonnegative mean curvature. Then Σ is convex.*

By Sacksteder's theorem [27], the condition $H > 0$ and Corollary 2.1 of [32], we may conclude

Corollary 1.2. *Σ is the boundary of a convex region in \mathbb{R}^3 whose projection on the plane spanned by e_1, e_2 is (after rotation of coordinates) either a strip region $\{(x_1, x_2) : |x_1| < R\}$ or \mathbb{R}^2 . Moreover Σ is the graph of a function $u(x_1, x_2)$ which satisfies the equation*

$$(1.4) \quad \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla u|^2}},$$

or in nondivergence form,

$$(1.5) \quad (1 + u_{x_2}^2)u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 + u_{x_1}^2)u_{x_2x_2} = 1 + u_{x_1}^2 + u_{x_2}^2.$$

Combining Theorem 1.1 with Theorem 1.1 in [32] we have

Corollary 1.3. *Any entire solution in \mathbb{R}^2 to the equation*

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla u|^2}}$$

must be rotationally symmetric in an appropriate coordinate system and hence is the bowl soliton.

A necessary and sufficient condition for mean convex translating solitons to be graphical over \mathbb{R}^2 and thus the bowl soliton is given in the next theorem.

Theorem 1.4. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with nonnegative mean curvature and suppose that $H(P) \rightarrow 0$ as $P \in \Sigma$ tends to infinity. Then Σ is after translation the axisymmetric bowl soliton.*

The existence of the grim family u^λ (see 1.2), the convexity Theorem 1.1 and a global curvature bound (see Theorem 2.8 in section 2) are the key tools that allows us to classify locally strictly convex graphical translating solitons defined over strips.

Theorem 1.5. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}\}$. Then (after possibly relabeling the e_2 direction)*

i. *For $\lambda \leq 1$ there is no locally strictly convex solution in \mathcal{S}^λ .*

ii. $\lim_{x_2 \rightarrow +\infty} u_{x_2}(x_1, x_2) = L := \sqrt{\lambda^2 - 1}$, $\lambda > 1$.

iii. $\lim_{x_2 \rightarrow -\infty} u_{x_2}(x_1, x_2) = -L$.

iv.

$$(1.6) \quad \begin{aligned} \lim_{A \rightarrow \pm\infty} (u(x_1, x_2 + A) - u(0, A)) &= u^\lambda(x_1, x_2) \\ \lim_{A \rightarrow \pm\infty} u_{x_1}(x_1, A) &= \lambda \tan \frac{x_1}{\lambda}. \end{aligned}$$

The above limits are uniform in $|x_1| \leq R - \varepsilon$ for any $\varepsilon > 0$.

v. *For $P = (x_1, x_2, u(x_1, x_2)) \in \Sigma$, $(x_1, x_2) \in \mathcal{S}^\lambda$,*

$$(1.7) \quad H(P) \leq R - |x_1|, \quad H(P) \geq \theta(\varepsilon) \text{ if } |x_1| \leq R - \varepsilon.$$

vi. *$u(x_1, x_2) = u(-x_1, x_2)$ and $u_{x_1}(x_1, x_2) > 0$ for $x_1 > 0$.*

Remark 1.6. It was shown in [32] that locally strictly convex solutions in strips do exist. At the moment it is not clear ¹ if there exists a locally strictly convex solution in every strip \mathcal{S}^λ nor if it exists whether it is unique.

The organization of the paper is as follows. In section 2 we show that any immersed two-sided mean convex translating soliton is stable. This allows us to use the method of Choi-Schoen [7] as modified by Colding-Minicozzi [9], [10] to prove a global curvature

¹The existence of a locally strictly convex translating soliton in every \mathcal{S}^λ has just been announced in the preprint [6].

bound (Theorem 2.8). This is needed for the proof of Theorem 1.1 in section 3, which is based on a delicate maximum principle argument. In section 4 we give the proof of Theorem 1.4. This result also allows us to start the proof of Theorem 1.5 which is long and detailed and contained in section 5. We wish to thank Theodora Bourni for a careful reading and helpful comments.

2. STABILITY AND CURVATURE ESTIMATES.

We will need the the following well-known identities that hold on any translating soliton in \mathbb{R}^{n+1} (see for example [22]).

Lemma 2.1. *Let Σ be a two-sided immersed translating soliton in \mathbb{R}^{n+1} with second fundamental form A . Let $A = (h_{ij})$ be the second fundamental form of Σ , $u = x_{n+1}|_{\Sigma}$ and $\Delta^f := \Delta^{\Sigma} + \langle \nabla, e_{n+1} \rangle$ be the drift Laplacian on Σ . Then*

- i. $|\nabla u|^2 = 1 - H^2$, $\Delta^{\Sigma} u = H^2$
- ii. $\Delta^f A + |A|^2 A = 0$,
- iii. $\Delta^f H + H|A|^2 = 0$,
- iv. $\Delta^f(|A|)^2 - 2|\nabla A|^2 + 2|A|^4 = 0$.

It is well known that a translating soliton Σ with respect to the direction e_{n+1} in \mathbb{R}^{n+1} is a critical point of the weighted area functional

$$\tilde{\mathcal{A}}(\Sigma) = \int_{\Sigma} e^{x_{n+1}} dv$$

and is in fact minimal with respect to the weighted Euclidean metric $e^{x_{n+1}} \delta$ on \mathbb{R}^{n+1} . The second variation of $\tilde{\mathcal{A}}$ with respect to a compactly supported normal variation ηN is easily computed to be

$$\tilde{\mathcal{A}}''(0) = \int_{\Sigma} (|\nabla \eta|^2 - |A|^2 \eta^2) e^{x_{n+1}} dv = \int_{\Sigma} -\eta L \eta e^{x_{n+1}} dv ,$$

where $L\eta = e^{-x_{n+1}} \operatorname{div}^{\Sigma}(e^{x_{n+1}} \nabla \eta) + |A|^2 \eta$ is the associated stability operator.

Proposition 2.2. *i. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a complete immersed two-sided translating soliton with respect to e_{n+1} with $H \geq 0$. Then Σ is strictly stable, that is, $\lambda_1(-L)(D) > 0$ on any compact subdomain $D \subset \Sigma$.*

ii. The following are equivalent:

- a. *There exists a positive solution v of $Lv = 0$ for every bounded D .*

- b. $\lambda_1(-L)(D) \geq 0$ for every bounded D .
c. $\lambda_1(-L)(D) > 0$ for every bounded D .

Proof. i. We may assume that $H > 0$. Then $w = \log H$ satisfies $\Delta w + \langle \nabla w, e_{n+1} \rangle + |A|^2 = -|\nabla w|^2$. Hence

$$\begin{aligned} \int_{\Sigma} (\eta^2 |A|^2 - |\nabla \eta|^2) e^{x_{n+1}} dv &= - \int_{\Sigma} (\eta^2 |\nabla w|^2 - 2\eta \langle \nabla \eta, \nabla w \rangle + |\nabla \eta|^2) e^{x_{n+1}} dv \\ &= - \int_{\Sigma} |\eta \nabla w - \nabla \eta|^2 e^{x_{n+1}} dv < 0. \end{aligned}$$

ii. The proof is a straightforward modification of that of Fischer-Colbrie and Schoen [12] and will not be given. \square

We will need the following corollary for the case at hand $n = 2$.

Corollary 2.3. *Let $\mathcal{B}_\rho(P)$ be an intrinsic ball in Σ with $\rho < 2\pi$. Then $\mathcal{B}_\rho(P)$ is disjoint from the conjugate locus of P and*

$$(2.1) \quad \int_{\Sigma} f^2 |A|^2 dv \leq e^{2\rho} \int_{\Sigma} |\nabla f|^2 dv$$

for $f \in H_0^1(\mathcal{B}_\rho(P))$.

Proof. Let $K = K^\Sigma$ denote the Gauss curvature of Σ . Since $K \leq \frac{H^2}{4} \leq \frac{1}{4}$, the first statement follows from standard comparison geometry. Also $|\nabla x_3|^2 = 1 - H^2 \leq 1$, so $|x_3(Q) - x_3(P)| \leq \rho$ for any $Q \in \mathcal{B}_\rho(P)$. Hence $e^{-\rho} e^{x_3}(P) \leq e^{x_3}(Q) \leq e^\rho e^{x_3}(P)$ and the result follows from Proposition 2.2. \square

We now follow the Colding-Minicozzi method [9, 10] with appropriate modification, to prove intrinsic area bounds and then curvature bounds. For two dimensional graphs, such curvature estimates follow immediately from the work of Leon Simon [30], see also [28].

Proposition 2.4. *Let $\Sigma \subset \mathbb{R}^3$ be a two-sided immersed translating soliton with $H \geq 0$ and let $\mathcal{B}_\rho(P)$ be a topological disk in Σ . Then $\mathcal{B}_\rho(P)$ is disjoint from the cut locus of P for $e^{2\rho} < 2$ and*

$$(2.2) \quad \begin{aligned} &i. \frac{\text{area}(\mathcal{B}_\rho(P))}{\rho^2} \leq 2\pi, \\ &ii. \int_{\mathcal{B}_{\mu^2\rho}(P)} |A|^2 dv \leq 2\pi \left\{ \left(\log \frac{1}{\mu}\right)^{-2} + 2\left(\log \frac{1}{\mu}\right)^{-1} \right\} \text{ for } \mu \in (0, 1). \end{aligned}$$

Proof. We first prove $\mathcal{B}_\rho(P)$ is disjoint from the cut locus $C(P)$ of P , that is the injectivity radius of P satisfies $r_0 := \text{inj}_p(\Sigma) > \rho$. Suppose for contradiction that $Q \in \partial\mathcal{B}_{r_0}(P)$ is a cut locus of P and $r_0 \leq \rho$. We know by Corollary 1.2 that Q is not in the conjugate locus of P so by Klingenberg's lemma (see for example Lemma 5.7.12 of [26]) there are two minimizing geodesics from P to Q which fit together smoothly at Q with a possible corner at P and bounding a domain $D \subset \mathcal{B}_{r_0}(P)$. By Gauss-Bonnet,

$$\frac{1}{4}\text{area}(D) \geq \int_D K \, dv = 2\pi - \int_{\partial D} \kappa_g \, d\sigma \geq \pi,$$

hence $\text{area}(\mathcal{B}_{r_0}(P)) > \text{area}(D) \geq 4\pi$. On the other hand by (2.2) with $\rho = r_0$ (proved below), $\text{area}(\mathcal{B}_{r_0}(P)) < 2\pi r_0^2$. Therefore $r_0 > \sqrt{2}$, contradicting $r_0 \leq \rho < \frac{1}{2} \log 2$.

We now prove the stated inequalities. Let $l(s)$ be the length of $\partial\mathcal{B}_s(P)$ and $K(s) = \int_{\mathcal{B}_s(P)} K \, dv$. By Gauss-Bonnet,

$$(2.3) \quad l'(s) = \int_{\partial\mathcal{B}_s(P)} \kappa_g \, d\sigma = 2\pi - K(s).$$

For $r = d(P, x)$, $f(x) = \eta(r)$, $\eta, -\eta' \geq 0$, $\eta(\rho) = 0$, we use the stability inequality (2.1) and write

$$|A|^2 = H^2 - 2K \geq -2K,$$

to obtain (for $\rho \leq r_0$)

$$(2.4) \quad -2 \int_{\mathcal{B}_\rho(P)} K f^2 \leq 2 \int_{\mathcal{B}_\rho(P)} |\nabla f|^2.$$

By the coarea formula,

$$(2.5) \quad \begin{aligned} \int_{\mathcal{B}_\rho(P)} K f^2 &= \int_0^\rho \eta^2(s) \int_{\partial\mathcal{B}_s(P)} K \, d\sigma \, ds = \int_0^\rho \eta^2(s) K'(s) \, ds, \\ \int_{\mathcal{B}_\rho(P)} |\nabla f|^2 &= \int_0^\rho \int_{\partial\mathcal{B}_s(P)} |\nabla f|^2 \, d\sigma \, ds = \int_0^\rho (\eta')^2(s) l(s) \, ds. \end{aligned}$$

For $\eta(r) = 1 - \frac{r}{\rho}$, using (2.3)-(2.5) this gives

$$(2.6) \quad -\frac{4}{\rho} \int_0^\rho (2\pi - l'(s)) \left(1 - \frac{s}{\rho}\right) ds \leq 2 \frac{\text{area}(\mathcal{B}_\rho(P))}{\rho^2}.$$

By integration by parts,

$$(2.7) \quad \int_0^\rho (2\pi - l'(s)) \left(1 - \frac{s}{\rho}\right) ds = \pi\rho - \frac{\text{area}(\mathcal{B}_\rho(P))}{\rho}.$$

Finally combining (2.6), (2.7) and simplifying gives inequality i. in (2.2). For part ii. we use the logarithmic cutoff

$$(2.8) \quad \eta(s) = \begin{cases} 1 & \text{if } s \leq \mu^2 \rho \\ \frac{\log \frac{s}{\rho}}{\log \mu} - 1 & \text{if } \mu^2 \rho < s < \mu \rho \\ 0 & \text{if } s > \mu \rho \end{cases}$$

in (2.1). Then

$$(2.9) \quad \begin{aligned} \int_{\mathcal{B}_{\mu^2 \rho}(P)} |A|^2 &\leq \frac{2}{(\log \mu)^2} \int_{\mu^2 \rho}^{\mu \rho} \frac{l(s)}{s^2} ds \\ &\leq \frac{2}{(\log \mu)^2} \left\{ \frac{\text{area}(\mathcal{B}_s(P))}{s^2} \Big|_{\mu^2 \rho}^{\mu \rho} + 2 \int_{\mu^2 \rho}^{\mu \rho} \frac{\text{area}(\mathcal{B}_s(P))}{s^3} ds \right\} \\ &\leq 4\pi \left\{ \frac{1}{(\log \mu)^2} + \frac{2}{\log \frac{1}{\mu}} \right\} \end{aligned}$$

by (2.2) part i. □

For later use we will need two well known lemmas; the first is an extrinsic mean value inequality (for a proof see [10] p. 26-27) and the second one says that curvature bounds implies graphical with intrinsic balls \mathcal{B}_s and extrinsic balls B_s related (see [10] Lemma 2.4).

Lemma 2.5. *Let $\Sigma \subset \mathbb{R}^3$ be an embedded surface with $x_0 \in \Sigma$, $B_s(x_0) \cap \partial \Sigma = \emptyset$. Suppose the mean curvature of Σ satisfies $|H| \leq C$ and $f \geq 0$ is a smooth function on Σ satisfying $\Delta^\Sigma f \geq -\lambda s^{-2} f$. Then*

$$f(x_0) \leq \frac{e^{(\frac{\lambda}{4} + Cs)}}{\pi s^2} \int_{B_s(x_0) \cap \Sigma} f dv .$$

Lemma 2.6. *Let $\Sigma \subset \mathbb{R}^3$ be an immersed surface with $16s^2 \sup_\Sigma |A|^2 \leq 1$. If $P \in \Sigma$ and $\text{dist}^\Sigma(P, \partial \Sigma) \geq 2s$, then*

- i. $\mathcal{B}_{2s}(P)$ can be written as a graph of a function u over $T_P \Sigma$ with $|\nabla u| \leq 1$ and $\sqrt{2}s |Hess_u| \leq 1$;
- ii. The connected component of $B_s(P) \cap \Sigma$ containing P is contained in $\mathcal{B}_{2s}(P)$.

Proposition 2.7. *(Choi-Schoen type curvature bound) Let $\Sigma \subset \mathbb{R}^3$ be a two-sided immersed translating soliton and let $\mathcal{B}_\rho(P)$ be disjoint from the cut locus of P . Then there exists $\varepsilon, \tau < \frac{\sqrt{\varepsilon}}{2\pi} < \rho$ such that if for all $r_0 \leq \tau$, there holds $\int_{\mathcal{B}_{r_0}(P)} |A|^2 \leq \varepsilon$, then for all $0 < \sigma \leq r_0$, $y \in \mathcal{B}_{r_0 - \sigma}(P)$ we have $|A|^2(y) \leq \sigma^{-2}$.*

Proof. Define $F(x) = (r_0 - r(x))^2 |A|^2(x)$ on $\mathcal{B}_{r_0}(P)$ and suppose F assumes its maximum at x_0 . Note that if $F(x_0) \leq 1$, then $\sigma^2 |A|^2(y) \leq (r_0 - r(y))^2 |A|^2(y) \leq F(x_0) \leq 1$ and we are done. If not, define σ by $4\sigma^2 |A|^2(x_0) = 1$. Then by the triangle inequality on $\mathcal{B}_\sigma(x_0)$,

$$\frac{1}{2} \leq \frac{r_0 - r(x)}{r_0 - r(x_0)} \leq 2,$$

which implies $\sup_{\mathcal{B}_\sigma(x_0)} |A|^2 \leq 4|A|^2(x_0) = \sigma^{-2}$. On Σ we have the Simons' type equation $L(|A|^2) - 2|\nabla A|^2 + 2|A|^4 = 0$ which implies $\Delta(|A|^2 + \frac{1}{2}) \geq -2|A|^2(|A|^2 + \frac{1}{2})$. Hence for $f(x) := |A|^2 + \frac{1}{2}$, $\Delta f \geq -2\sigma^{-2}f$ on $\mathcal{B}_\sigma(x_0)$. Using Lemmas 2.6 and 2.5 with $s = \frac{\sigma}{4}$, $\lambda = \frac{1}{6}$, $C = 1$ and Proposition 2.4 part i., we find

$$(2.10) \quad \left(\frac{1}{4\sigma^2} + \frac{1}{2}\right) = |A|^2(x_0) + \frac{1}{2} \leq \frac{16}{\pi\sigma^2} e^{(\frac{1}{24} + \frac{\sigma}{4})} \{\varepsilon + 2\pi r_0^2\}.$$

Multiplying (2.10) by σ^2 we find

$$(2.11) \quad \frac{1}{4} < \frac{1}{4} + \frac{\sigma^2}{2} \leq \frac{16}{\pi} e^{(\frac{1}{24} + \frac{\sigma}{4})} \{\varepsilon + 2\pi r_0^2\} < \frac{32e}{\pi} \varepsilon$$

which is impossible for $\varepsilon \leq \frac{\pi}{128e}$. \square

We are now in a position to prove the curvature estimate we will need in the next section.

Theorem 2.8. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton with respect to e_3 with $H \geq 0$. Then there is a universal constant C such that $|A|^2(P) \leq C$ for $P \in \Sigma$.*

Proof. For any $P \in \Sigma$, we fix $\rho > 0$ such that $e^\rho < 2$ as in Proposition 2.4 so that $\mathcal{B}_\rho(P)$ is disjoint from the cut locus of P . We may assume $\mathcal{B}_\rho(P)$ is a topological disk since by Proposition 2.2, the universal cover of $\mathcal{B}_\rho(P)$ endowed with pull-back metric is also a stable translating soliton with nonnegative mean curvature. Thus using Proposition 2.4 part ii. with $\mu = e^{-\frac{6\pi}{\varepsilon}}$, the conditions of Proposition 2.7 are satisfied for $\tau = \mu^2 \rho$. We can choose $\sigma = \tau$ and obtain $|A|^2(P) \leq \tau^{-2}$. \square

3. PROOF OF THEOREM 1.1.

We restate for the readers convenience our main result.

Theorem 3.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with nonnegative mean curvature. Then Σ is convex.*

Proof. Without loss of generality, we assume Σ satisfies

$$H = \langle N, e_3 \rangle > 0.$$

Let

$$(3.1) \quad f(x_1, x_2) = \frac{x_1 + x_2}{2} + \left[\left(\frac{x_1 - x_2}{2} \right)^2 \right]^{1/2},$$

$$(3.2) \quad \phi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0 \\ 0 & \text{if } r \geq 0, \end{cases}$$

and

$$(3.3) \quad g(z) = f(z) \sum_{i=1}^2 \phi\left(\frac{z_i}{f(z)}\right).$$

It's easy to see that f and g are smooth when $z_1 \neq z_2$. Now denote

$$G(A) = g(\kappa(A)) \text{ and } F(A) = f(\kappa(A)),$$

where A is a 2×2 symmetric matrix and $\kappa(A)$ are the eigenvalues of A . Now let $A = (h_{ij})$ be the second fundamental form of Σ . If we order the principle curvatures $\kappa_1 \geq \kappa_2$ of Σ , then $\frac{G}{F} = \phi\left(\frac{\kappa_2}{\kappa_1}\right) \geq 0$ is smooth when $\kappa_1 > 0 > \kappa_2$. Since $G/F \leq 1$, G/F achieves its maximum either at an interior point or "at infinity".

Lemma 3.2. *On $\Sigma \cap \{\kappa_1 > 0 > \kappa_2\}$,*

$$(3.4) \quad \begin{aligned} & \Delta^\Sigma \left(\frac{G}{F} \right) + 2 \left\langle \frac{\nabla F}{F}, \nabla \left(\frac{G}{F} \right) \right\rangle + \left\langle \nabla \left(\frac{G}{F} \right), e_3 \right\rangle \\ &= -\frac{G^{ij} h_{ij} |A|^2}{F} + \frac{GF^{ij} h_{ij} |A|^2}{F^2} + \left(\frac{G^{pq,rs}}{F} - \frac{GF^{pq,rs}}{F^2} \right) h_{pqk} h_{rsk}, \end{aligned}$$

$$\text{where } G^{ij} = \frac{\partial G}{\partial a_{ij}}, \quad F^{ij} = \frac{\partial F}{\partial a_{ij}}, \quad G^{pq,rs} = \frac{\partial^2 G}{\partial a_{pq} \partial a_{rs}}, \quad F^{pq,rs} = \frac{\partial^2 F}{\partial a_{pq} \partial a_{rs}}.$$

Proof. Let τ_1, τ_2 be a local orthonormal frame on Σ . Then using Lemma 2.1

$$(3.5) \quad \begin{aligned} \nabla_k \left(\frac{G}{F} \right) &= \left(\frac{G}{F} \right)^{pq} h_{pqk} = \left(\frac{FG^{pq} - GF^{pq}}{F^2} \right) h_{pqk} \\ \Delta^\Sigma \left(\frac{G}{F} \right) &= \left(\frac{G}{F} \right)^{pq} \Delta h_{pq} + \left(\frac{G}{F} \right)^{pq,rs} h_{pqk} h_{rsk} \\ &= -|A|^2 \left(\frac{FG^{ij} h_{ij} - GF^{ij} h_{ij}}{F^2} \right) - \langle \nabla \left(\frac{G}{F} \right), e_3 \rangle + \left(\frac{G}{F} \right)^{pq,rs} h_{pqk} h_{rsk}. \end{aligned}$$

But

$$(3.6) \quad \left(\frac{G}{F}\right)^{pq,rs} h_{pqk} h_{rsk} = \left(\frac{G^{pq,rs}}{F} - \frac{GF^{pq,rs}}{F^2}\right) h_{pqk} h_{rsk} - 2 \langle \nabla\left(\frac{G}{F}\right), \frac{\nabla F}{F} \rangle,$$

and together (3.5), (3.6) give (3.4). \square

We next compute the last term in (3.4). We use the notation $g^p = \frac{\partial g}{\partial \kappa_p}$, $g^{pq} = \frac{\partial^2 g}{\partial \kappa_p \partial \kappa_q}$ and similarly for f . As is now well-known (see for example [2])

$$(3.7) \quad G^{pq,rs} h_{pqk} h_{rsk} = g^{pq} h_{ppk} h_{qqk} + 2 \frac{g^2 - g^1}{\kappa_2 - \kappa_1} h_{12k}^2,$$

where

$$(3.8) \quad \begin{aligned} g^{pq} &= f^{pq} \sum_{i=1}^2 \left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f} \dot{\phi}\left(\frac{z_i}{f}\right) \right] \\ &+ \frac{1}{f} \sum_{i=1}^2 \ddot{\phi}\left(\frac{z_i}{f}\right) \left(\delta_i^p - \frac{z_i}{f} f^p \right) \left(\delta_i^q - \frac{z_i}{f} f^q \right). \end{aligned}$$

It follows that

$$(3.9) \quad \begin{aligned} &FG^{pq,rs} - GF^{pq,rs} \\ &= (fg^{pq} - gf^{pq}) + 2 \left\{ f \frac{g^2 - g^1}{\kappa_2 - \kappa_1} - g \frac{f^2 - f^1}{\kappa_2 - \kappa_1} \right\} \\ &= I + II. \end{aligned}$$

We proceed to calculate the terms I and II.

$$(3.10) \quad \begin{aligned} I &= f^{pq} \left[g - \sum_{i=1}^2 z_i \dot{\phi}\left(\frac{z_i}{f}\right) \right] + \sum_{i=1}^2 \ddot{\phi}\left(\frac{z_i}{f}\right) \left(\delta_i^p - \frac{z_i}{f} f^p \right) \left(\delta_i^q - \frac{z_i}{f} f^q \right) - gf^{pq} \\ &= -f^{pq} \sum_{i=1}^2 z_i \dot{\phi}\left(\frac{z_i}{f}\right) + \sum_{i=1}^2 \ddot{\phi}\left(\frac{z_i}{f}\right) \left(\delta_i^p - \frac{z_i}{f} f^p \right) \left(\delta_i^q - \frac{z_i}{f} f^q \right), \end{aligned}$$

$$(3.11) \quad \begin{aligned} g^2 - g^1 &= \dot{\phi}\left(\frac{z_2}{f}\right) + f^2 \sum_{i=1}^2 \left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f} \dot{\phi}\left(\frac{z_i}{f}\right) \right] \\ &- \dot{\phi}\left(\frac{z_1}{f}\right) - f^1 \sum_{i=1}^2 \left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f} \dot{\phi}\left(\frac{z_i}{f}\right) \right] \end{aligned}$$

and

$$\begin{aligned}
(3.12) \quad & \frac{\kappa_2 - \kappa_1}{2} II = f(g^2 - g^1) - g(f^2 - f^1) \\
& = f \left[\dot{\phi} \left(\frac{z_2}{f} \right) - \dot{\phi} \left(\frac{z_1}{f} \right) \right] + f^2 \left[g - \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) \right] \\
& - f^1 \left[g - \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) \right] - g(f^2 - f^1) \\
& = f \left[\dot{\phi} \left(\frac{z_2}{f} \right) - \dot{\phi} \left(\frac{z_1}{f} \right) \right] + \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) (f^1 - f^2).
\end{aligned}$$

Assume $\kappa_1 > 0 > \kappa_2$; then

$$\begin{aligned}
(3.13) \quad & \Delta^\Sigma \left(\frac{G}{F} \right) + 2 \left\langle \frac{\nabla F}{F}, \nabla \left(\frac{G}{F} \right) \right\rangle + \left\langle \nabla \left(\frac{G}{F} \right), e_3 \right\rangle \\
& = \left(\frac{G^{pq,rs}}{F} - \frac{GF^{pq,rs}}{F^2} \right) h_{pqk} h_{rsk} = \frac{1}{F^2} (FG^{pq,rs} - GF^{pq,rs}) h_{pqk} h_{rsk} \\
& = \frac{1}{\kappa_1^2} \ddot{\phi} \left(\frac{\kappa_2}{\kappa_1} \right) \left(\delta_2^p - \frac{\kappa_2}{\kappa_1} f^p \right) \left(\delta_2^q - \frac{\kappa_2}{\kappa_1} f^q \right) h_{ppk} h_{qqk} + \frac{2}{\kappa_1^2} \dot{\phi} \left(\frac{\kappa_2}{\kappa_1} \right) \left(\frac{\kappa_1 + \kappa_2}{\kappa_2 - \kappa_1} \right) h_{12k}^2 \geq 0.
\end{aligned}$$

Here we used $f^{pq} = 0$, $\dot{\phi} \left(\frac{\kappa_1}{f} \right) = 0$, and $f^2 = 0$ when $\kappa_1 > 0 > \kappa_2$. It follows that if G/F achieves its maximum at an interior point, then by the strong maximum principle we have $\frac{G}{F} \equiv \text{constant}$. If $\kappa_2/\kappa_1 \neq 0$, then $\varphi := \log \frac{|A|^2}{H^2}$ is constant and so

$$\begin{aligned}
(3.14) \quad & 0 = \nabla \varphi = \frac{\nabla |A|^2}{|A|^2} - 2 \frac{\nabla H}{H} \\
& 0 = \Delta^\Sigma \varphi + \langle \nabla \varphi, e_3 \rangle \\
& = 2 \frac{|\nabla A|^2}{|A|^2} - \frac{|\nabla |A|^2|^2}{|A|^4} + 2 \frac{|\nabla H|^2}{H^2} \\
& = \frac{2}{|A|^2} (|\nabla A|^2 - |\nabla |A|^2|^2).
\end{aligned}$$

Hence $|\nabla A|^2 = |\nabla |A|^2|^2$ and so $h_{12k} = 0$, $k = 1, 2$. By the Codazzi equations, $h_{112} = h_{221} = 0$. Since $h_{22} = r_0 h_{11}$, we deduce $\nabla A = 0$. Thus M is a complete surface with constant mean curvature. Since M satisfies $H = \langle N, e_3 \rangle$, we conclude that M is a plane, a contradiction. Therefore in this case we must have $\frac{G}{F} \equiv 0$, and thus $\kappa_2 \geq 0$.

If G/F achieves its maximum at infinity, by Theorem 2.8 we may apply the Omori-Yau maximum principle and conclude that there exists a sequence P_n tending to infinity with

$r_n := \kappa_2/\kappa_1(P_n) \rightarrow r_0$, where $-1 \leq r_0 < 0$. Introduce a local orthonormal frame τ_1, τ_2 in a neighborhood of P_n which diagonalizes $(h_{ij}(P_n))$. Then we have at P_n :

$$(3.15) \quad \begin{aligned} \frac{1}{n} &\geq \ddot{\phi}(r_n) \left[\frac{r_n h_{11k}}{\kappa_1} - \frac{h_{22k}}{\kappa_1} \right]^2 + \frac{2}{\kappa_1^2} \dot{\phi}(r_n) \frac{1+r_n}{r_n-1} h_{12k}^2 - 2 \left\langle \frac{\nabla F}{F}, \nabla \left(\frac{G}{F} \right) \right\rangle \\ &= \ddot{\phi}(r_n) \left[\frac{r_n h_{11k}}{\kappa_1} - \frac{h_{22k}}{\kappa_1} \right]^2 + \frac{2}{\kappa_1^2} \dot{\phi}(r_n) \frac{1+r_n}{r_n-1} h_{12k}^2 - 2 \dot{\phi}(r_n) \left\langle \frac{h_{11k}}{h_{11}}, \frac{h_{22k}}{h_{11}} - r_n \frac{h_{11k}}{h_{11}} \right\rangle \end{aligned}$$

and

$$(3.16) \quad C_{n,k} := \frac{h_{22k}}{h_{11}} - r_n \frac{h_{11k}}{h_{11}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Note that

$$(3.17) \quad \begin{aligned} \tilde{C}_{n,k} &:= \frac{\nabla_k H}{\kappa_1} = \frac{-\kappa_k \langle \tau_k, e_3 \rangle}{\kappa_1} \\ &= \frac{h_{22k}}{h_{11}} - r_n \frac{h_{11k}}{h_{11}} + (1+r_n) \frac{h_{11k}}{h_{11}} = C_{n,k} + (1+r_n) \frac{h_{11k}}{h_{11}} \end{aligned}$$

From (3.16) we have,

$$(3.18) \quad \frac{h_{22k}}{\kappa_1} = -|r_n| \frac{h_{11k}}{\kappa_1} + C_{n,k}, \quad k = 1, 2$$

and

$$(3.19) \quad \frac{h_{11k} h_{22k}}{\kappa_1^2} = -|r_n| \left(\frac{h_{11k}}{\kappa_1} \right)^2 + C_{n,k} \frac{h_{11k}}{\kappa_1}.$$

Claim: $(1+r_n) \left| \frac{h_{11k}}{h_{11}} \right| \rightarrow 0$, $k = 1, 2$.

If not, we can choose a subsequence, still denoted by $\{P_n\}$, so that for $n \geq N_0$

$$(3.20) \quad (1+r_n) \left| \frac{h_{112}}{\kappa_1} \right| (P_n) \geq \varepsilon_0, \quad (1+r_n) \left| \frac{h_{221}}{\kappa_1} \right| (P_n) \geq \varepsilon_0.$$

Then from (3.15) we have

$$\begin{aligned}
(3.21) \quad \frac{1}{n} &\geq \ddot{\phi} \sum C_{n,k}^2 - 2\dot{\phi} \sum C_{n,k} \frac{h_{11k}}{\kappa_1} - \frac{2\dot{\phi}}{1+|r_n|} (1+r_n) \left(\frac{h_{112}^2}{\kappa_1^2} + \frac{h_{221}^2}{\kappa_1^2} \right) \\
&= \ddot{\phi} \sum C_{n,k}^2 - 2\dot{\phi} \left(C_{n,1} \left(-\frac{1}{|r_n|} \frac{h_{221}}{\kappa_1} + \frac{C_{n,1}}{|r_n|} \right) \right. \\
&\quad \left. + C_{n,2} \frac{h_{112}}{\kappa_1} + \frac{1+r_n}{1+|r_n|} \left(\frac{h_{112}^2}{\kappa_1^2} + \frac{h_{221}^2}{\kappa_1^2} \right) \right) \\
&= \ddot{\phi} \sum C_{n,k}^2 - 2\dot{\phi} \frac{C_{n,1}^2}{|r_n|} \\
&\quad - 2\dot{\phi} \left\{ \left(\frac{1+r_n}{1+|r_n|} \frac{h_{112}^2}{\kappa_1^2} + C_{n,2} \frac{h_{112}}{\kappa_1} \right) + \left(\frac{1+r_n}{1+|r_n|} \frac{h_{221}^2}{\kappa_1^2} - \frac{C_{n,1}}{|r_n|} \frac{h_{221}}{\kappa_1} \right) \right\} \\
&\geq -\frac{2\dot{\phi}}{1+|r_n|} \left(\left| \frac{h_{112}}{\kappa_1} \right| \left((1+r_n) \left| \frac{h_{112}}{\kappa_1} \right| - C_{n,2}(1+|r_n|) \right) \right. \\
&\quad \left. + \left| \frac{h_{221}}{\kappa_1} \right| \left((1+r_n) \left| \frac{h_{221}}{\kappa_1} \right| - \frac{C_{n,1}}{|r_n|} (1+|r_n|) \right) \right) \\
&\geq -2 \frac{\dot{\phi}}{1+|r_n|} \frac{\varepsilon_0}{2} \left(\left| \frac{h_{112}}{\kappa_1} \right| + \left| \frac{h_{221}}{\kappa_1} \right| \right),
\end{aligned}$$

which leads to a contradiction for $n \geq N_0$ by (3.20). Thus the claim is proven and so $\tilde{C}_{n,k} \rightarrow 0$. Therefore $N(P_n)$ converges to e_3 and so $H(P_n) \rightarrow 1$.

Now let $\Sigma_n = \Sigma - P_n$ be the surface obtained from Σ by translation of P_n to the origin. Since Σ has bounded principle curvatures, so do the Σ_n . Choosing a subsequence which we still denote by Σ_n , the Σ_n converge smoothly to Σ_∞ . Thus Σ_∞ is a translating soliton which satisfies $H = \langle N, e_3 \rangle$, and $H(0) = 1$. Moreover, we have

$$\inf_{x \in \Sigma_\infty} \frac{\kappa_2}{\kappa_1} = \frac{\kappa_2}{\kappa_1}(0).$$

As before we conclude that $G/F = \text{constant}$, and Σ_∞ has constant mean curvature one, which is impossible. Therefore Σ is convex. \square

4. PROOF OF THEOREM 1.4.

In this section we give the proof of

Theorem 4.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with positive mean curvature and suppose that $H(P) \rightarrow 0$ as $P \in \Sigma$ tends to infinity. Then Σ is after translation the axisymmetric bowl soliton.*

Proof. By Corollaries 1.2, 1.3 we may suppose for contradiction that $\Sigma = \text{graph}(u)$ projects onto the strip $|x_1| < R$. Consider for any A , the convex curve $\gamma^A(x_1) = (x_1, A, u(x_1, A)) : -R \leq x_1 \leq R$. Since $u(x_1, A) \rightarrow +\infty$ as $x_1 \rightarrow \pm R$, there is a smallest value $x_1 = x_1(A)$ so that $u_{x_1}(x_1(A), A) = 0$. Since Σ is not a grim cylinder we may assume by a suitable change of coordinates that $u_{x_2}(x_1(0), 0) \geq \delta > 0$. We normalize $u(x_1(0), 0) = 0$; then by convexity of u ,

$$(4.1) \quad \begin{aligned} u(x_1(A), A) &\geq u(x_1(0), 0) + Au_{x_2}(x_1(0), 0) \geq A\delta, \\ 0 = u(x_1(0), 0) &\geq u(x_1(A), A) - Au_{x_2}(x_1(A), A) \geq A(\delta - u_{x_2}(x_1(A), A)), \end{aligned}$$

which implies $u_{x_2}(x_1(A), A) \geq \delta$. By the assumption that $H(P) \rightarrow 0$ as $P \in \Sigma$ tends to infinity, we have that $|Du(x_1(A), A)| = |u_{x_2}(x_1(A), A)| = u_{x_2}(x_1(A), A) \rightarrow \infty$ as $A \rightarrow \infty$. Therefore

$$(4.2) \quad u(x_1, x_2) \geq u(x_1(A), A) + (x_2 - A)u_{x_2}(x_1(A), A) \geq A\delta + (x_2 - A)u_{x_2}(x_1(A), A),$$

and so choosing $x_2 = B$, $A = \frac{B}{2}$,

$$(4.3) \quad \lim_{B \rightarrow \infty} \frac{u(x_1, B)}{B} \geq \frac{\delta}{2} + \frac{1}{2} \lim_{B \rightarrow \infty} u_{x_2}(x_1(\frac{B}{2}), \frac{B}{2}) = +\infty.$$

Hence u grows superlinearly as $x_2 \rightarrow \infty$.

We now compare Σ with a tilted cylinder of radius R . Consider the parametrized family of graphs

$$x_3 = v^t(x_1, x_2) := -\sqrt{1+t^2}\sqrt{R^2 - x_1^2} + t(x_2 - A), \quad |x_1| \leq R, \quad x_2 \geq A$$

of constant mean curvature $H = \frac{1}{R}$ with respect to upward normal direction. Since $v^t(x_1, x_2) \leq t(x_2 - A)$, for any choice of $t \geq 0$, $u > v^t$ for x_2 sufficiently large by (4.3). Also, $u > v^t$ for $|x_1| = R$ and for $x_2 = A$. For $t \leq \delta$, $u > v^t$ in $x_2 \geq A$, $|x_1| \leq R$ by (4.2). Note that $\lim_{t \rightarrow \infty} v^t(x_1, 3A) \geq \lim_{t \rightarrow \infty} (2tA - \sqrt{1+t^2}R) = +\infty$ for $A > R$. We can therefore increase t until there is a first contact of $\Sigma = \text{graph}(u)$ and $\text{graph}(v^t)$, which must occur over an interior point of the half-strip $\{(x_1, x_2), |x_1| < R, x_2 > A\}$. This gives a $P := (x_1, x_2, u(x_1, x_2)) \in \Sigma$, $x_2 > A$ with $H(P) \geq \frac{1}{R}$. Since A is arbitrary we have a contradiction. \square

5. ASYMPTOTIC BEHAVIOR OF COMPLETE LOCALLY STRICTLY CONVEX
TRANSLATING SOLITONS

In this section we study the asymptotic behavior of complete locally strictly convex translating solitons.

Lemma 5.1. *Let $\Sigma = \text{graph}(u)$ be a complete mean convex translating soliton in \mathbb{R}^3 and suppose that the smallest principle curvature κ_2 vanishes at a point of Σ . Then $\kappa_2 \equiv 0$ everywhere and after translation, Σ is grim cylinder of the form $\Sigma = \text{graph}(u^\lambda)$ defined in a strip $\{(x_1, x_2) : |x_1| < R\}$ where*

$$u^\lambda(x_1, x_2) := \lambda^2 \log \sec\left(\frac{x_1}{\lambda}\right) \pm \sqrt{\lambda^2 - 1} x_2, \quad R = \frac{\pi}{2} \lambda, \quad \lambda \geq 1.$$

In particular if Σ contains a line, then Σ is a grim cylinder of the above form.

Proof. If we choose an orthonormal frame τ_1, τ_2 so that $\kappa_2(P) = h_{22}(P)$, then $\kappa_2 \equiv 0 < \kappa_1$ on Σ by Lemma 2.1 part ii. and the maximum principle. Thus the Gauss curvature $K^\Sigma \equiv 0$ and so Σ has a representation (see for example [15]) $\Sigma = \text{graph}(z)$ where $z(x_1, x_2) = \eta(x_1) + \alpha x_2$ for a constant α and a scalar function η defined in a simply connected region containing the projection of P on the x_1, x_2 plane. Therefore

$$(5.1) \quad (1 + \alpha^2)\eta'' = 1 + \alpha^2 + \eta'^2.$$

Set $\tilde{\eta}(x_1) = \lambda^{-2}\eta(\lambda x_1)$, where $\lambda^2 = 1 + \alpha^2$. Then

$$(5.2) \quad \tilde{\eta}'' = 1 + \tilde{\eta}'^2$$

and so (up to translation of coordinates and an additive constant)

$$(5.3) \quad \tilde{\eta}(x_1) = \log \sec x_1,$$

which proves the lemma. □

We have the following Harnack inequality for the mean curvature H on Σ (compare Hamilton [14], Corollary 1.2).

Lemma 5.2. *For any two points $P_1, P_2 \in \Sigma$,*

$$(5.4) \quad H(P_2) \geq e^{-d^\Sigma(P_1, P_2)} H(P_1).$$

Proof. Since $H = \langle N, e_3 \rangle$, $\nabla_k H = -\kappa_k \langle \tau_k, e_3 \rangle$ so that

$$|\nabla H|^2 \leq |A|^2 = H^2 - 2K \leq H^2.$$

Therefore, $|\nabla \log H| \leq 1$ and (5.4) follows. □

Lemma 5.3. *Let $\Sigma = \text{graph}(u)$ be a complete mean convex translating soliton in \mathbb{R}^3 defined in the strip $\{(x_1, x_2) : |x_1| < R\}$. Then*

$$H(x_1, x_2) := H((x_1, x_2, u(x_1, x_2))) \leq R - |x_1|.$$

Proof. We use that $\frac{\sum u_{ij}^2}{W^6} \leq |A|^2 \leq H^2 = \frac{1}{W^2}$ where $W^2 = 1 + |\nabla u|^2$. Then $|DW| \leq W^2$ or $|DH| = |D(\frac{1}{W})| \leq 1$. By the convexity of u and the fact that $u(x_1, x_2) \rightarrow \infty$ as $R - |x_1| \rightarrow 0$, x_2 fixed, we see that $H(x_1, x_2) \rightarrow 0$ as $R - |x_1| \rightarrow 0$, x_2 fixed. Hence $H(x_1, x_2) \leq \min(R - x_1, x_1 + R) = R - |x_1|$. \square

Lemma 5.4. *Let $\Sigma = \text{graph}(u)$ be a convex translating soliton in \mathbb{R}^3 defined in the strip $\{(x_1, x_2) : |x_1| < R\}$. Then*

$$(5.5) \quad \begin{aligned} \left| \frac{d}{dx_i} \arctan u_{x_i} \right| &\leq 1, \quad i = 1, 2, \\ \left| \frac{d}{dx_2} \sqrt{1 + (u_{x_1})^2} \right| &\leq |u_{x_1}| \sqrt{1 + (u_{x_2})^2}. \end{aligned}$$

Proof. Since u is a convex graphical solution to the translating soliton equation,

$$(5.6) \quad \begin{aligned} (1 + u_{x_2}^2)u_{x_1x_1} + (1 + u_{x_1}^2)u_{x_2x_2} &= 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 + u_{x_1}^2 + u_{x_2}^2) \\ &\leq 2|u_{x_1}||u_{x_2}|\sqrt{u_{x_1x_1}u_{x_2x_2}} + (1 + u_{x_1}^2 + u_{x_2}^2) \\ &\leq (1 + u_{x_1}^2)u_{x_2x_2} + \frac{u_{x_1}^2u_{x_2}^2}{(1 + u_{x_1}^2)}u_{x_1x_1} + (1 + u_{x_1}^2 + u_{x_2}^2). \end{aligned}$$

This implies

$$(5.7) \quad \begin{aligned} u_{x_1x_1} &\leq 1 + u_{x_1}^2 \quad \text{and by symmetry} \quad u_{x_2x_2} \leq 1 + u_{x_2}^2, \\ |u_{x_1x_2}| &\leq \sqrt{u_{x_1x_1}u_{x_2x_2}} \leq \sqrt{1 + u_{x_1}^2} \sqrt{1 + u_{x_2}^2}, \end{aligned}$$

and (5.5) follows. \square

Lemma 5.5. *Let $\Sigma = \text{graph}(u)$ be a complete mean convex translating soliton in \mathbb{R}^3 defined over \mathbb{R}^2 . Then $H(P) \rightarrow 0$ for $P \in \Sigma$ tending to infinity.*

Proof. We slightly modify the argument of Haslhofer [16]. Fix $P_0 \in \Sigma$ and suppose there is a sequence $P_n \in \Sigma$ tending to infinity with $\liminf_{n \rightarrow \infty} H(P_n) > 0$. Passing to a subsequence, we may assume $\frac{P_n - P_0}{|P_n - P_0|}$ converges to a unit direction ω . Let $\Sigma_n = \Sigma - P_n$ be the surface obtained from Σ by translation of P_n to the origin. Since Σ has bounded principle curvatures, so do the Σ_n . Choosing a subsequence which we still denote by Σ_n , the Σ_n converge smoothly to Σ_∞ , a convex complete translating soliton. Moreover, Σ_∞

is not a vertical plane since $H(0) > 0$, so must be a graph. Since the region K above Σ is convex and $\frac{P_n - P_0}{|P_n - P_0|} \rightarrow \omega$, the limit Σ_∞ contains a line (see Lemma 3.1 of [5] for more details). Therefore by Lemma 5.1, Σ_∞ is a grim cylinder and thus is a graph over a strip, a contradiction. \square

Lemma 5.6. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}\}$. Then there exist sequences $P_n = (x_1^n, x_2^n, u(x_1^n, x_2^n))$, $\bar{P}_n = (\bar{x}_1^n, \bar{x}_2^n, u(\bar{x}_1^n, \bar{x}_2^n)) \in \Sigma$ with $x_2^n \rightarrow \infty$, $\bar{x}_2^n \rightarrow -\infty$ and $H(P_n), H(\bar{P}_n) \geq \theta > 0$.*

Proof. By Theorem 1.4 there is a sequence of points $P_n = (x_1^n, x_2^n, u(x_1^n, x_2^n)) \in \Sigma$ where (after relabeling axes if necessary) $\liminf_{n \rightarrow \infty} H(P_n) \geq \theta > 0$ and $x_2^n \rightarrow +\infty$. Note that by Lemma 5.3, $|x_1^n| \leq R - \theta$. We claim there is also a sequence $\bar{P}_n = (\bar{x}_1^n, \bar{x}_2^n, u(\bar{x}_1^n, \bar{x}_2^n)) \in \Sigma$ with $\bar{x}_2^n \rightarrow -\infty$ and $H(\bar{P}_n) \geq \theta > 0$. If not, then for $x_2 \rightarrow -\infty$, $H(x_1, x_2) := H(x_1, x_2, u(x_1, x_2)) \rightarrow 0$. For any $B \leq 0$ let $x_1(B)$ be the unique value so that $u_{x_1}(x_1(B), B) = 0$. Since Σ is not a grim cylinder, we may assume by a translation of coordinates that $u_{x_2}(x_1(0), 0) = \delta > 0$. We normalize $u(x_1(0), 0) = 0$; then

$$u(x_1(B), B) \geq u(x_1(0), 0) + B u_{x_2}(x_1(0), 0) = B\delta,$$

and

$$0 = u(x_1(0), 0) \geq u(x_1(B), B) - B u_{x_2}(x_1(B), B) \geq B(\delta - u_{x_2}(x_1(B), B)).$$

Hence $u_{x_2}(x_1(B), B) \leq \delta$. Since $H(x_1, x_2) \rightarrow 0$ as $x_2 \rightarrow -\infty$, we conclude

$$(5.8) \quad u_{x_2}(x_1(B), B) \rightarrow -\infty \text{ as } B \rightarrow -\infty.$$

Therefore

$$(5.9) \quad u(x_1, x_2) \geq u(x_1(B), B) + (x_2 - B) u_{x_2}(x_1(B), B) \geq B\delta + (x_2 - B) u_{x_2}(x_1(B), B)$$

Now choose $x_2 = \Lambda < 0$ and $B = \frac{\Lambda}{2}$. Then

$$u(x_1, \Lambda) \geq \frac{\delta\Lambda}{2} + \frac{\Lambda}{2} u_{x_2}(x_1(\frac{\Lambda}{2}), \frac{\Lambda}{2}),$$

hence

$$(5.10) \quad \lim_{\Lambda \rightarrow -\infty} \frac{u(x_1, \Lambda)}{\Lambda} \leq \frac{\delta}{2} + \frac{1}{2} u_{x_2}(x_1(\frac{\Lambda}{2}), \frac{\Lambda}{2}) \rightarrow -\infty$$

by (5.8). Thus $u(x_1, \Lambda) \rightarrow \infty$ superlinearly as $\Lambda \rightarrow -\infty$. We now compare Σ with a tilted cylinder of radius R . Consider $x_3 = v^t(x_1, x_2) := -\sqrt{1 + t^2} \sqrt{R^2 - x_1^2} + t(x_2 - B)$ in the half-strip $\mathcal{S}^B := \{(x_1, x_2) : |x_1| \leq R, x_2 \leq B < 0\}$. Since $v^t \leq t(x_2 - B)$, for any choice

of $t \leq 0$, $u > v^t$ for x_2 sufficiently small (i.e. x_2 large negative) by (5.10). Also $u > v^t$ for $|x_1| = R$ and $x_2 = B < 0$, as soon as $u(x_1, B) > 0$. For $t \geq -\delta$, $u > v^t$ in \mathcal{S}^B and since

$$\lim_{t \rightarrow -\infty} v^t(x_1, 3B) \geq \lim_{t \rightarrow -\infty} (2Bt - \sqrt{1+t^2} R) \rightarrow \infty,$$

we can decrease t until there is a first contact point $P \in \mathcal{S}^B$ of $\text{graph}(v^t)$ and Σ . At $P \in \Sigma$, $H(P) \geq \frac{1}{R}$, a contradiction. \square

Lemma 5.7. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip $\mathcal{S}_R := \{(x_1, x_2) : |x_1| < R\}$ with $\max_{|x_1| \leq R-\delta} W(x_1, 0) \leq C_0$. If $W := \sqrt{1 + |\nabla u|^2}$ satisfies $\sup_{x_2 \geq 0} W(0, x_2) \leq C_1$, then*

$$W(x_1, x_2) \leq 2 \frac{R - \delta}{\delta} \max(C_0, C_1) \text{ in the half-strip } \mathcal{S}_{R-\frac{3}{2}\delta}^+ := \mathcal{S}_{R-\frac{3}{2}\delta} \cap \{x_2 \geq 0\}.$$

Proof. We give the proof for the right half of $\mathcal{S}_{R-\frac{3}{2}\delta}^+$ (where $0 \leq x_1 \leq R - \frac{3}{2}\delta$); the proof for the left half of $\mathcal{S}_{R-\frac{3}{2}\delta}$ is analogous.

For an n -dimensional hypersurface $\Sigma = \text{graph}(u)$,

$$\Delta^f H := \Delta^\Sigma H + \langle \nabla^\Sigma H, e_{n+1} \rangle = a^{ij} H_{x_i x_j},$$

where $a^{ij} := \delta_{ij} - \frac{u_i u_j}{W^2}$. Since $H = \frac{1}{W}$, we obtain from Lemma 2.1 part iii.,

$$(5.11) \quad \mathcal{L}W := (a^{ij} \partial_i \partial_j - \frac{2}{W} a^{ij} W_i \partial_j) W = \frac{|A|^2}{W^3} \geq \frac{1}{2W}$$

for $n = 2$. For $N > 1$ fixed and large, set $\eta = \eta^N := (1 - \frac{x_1}{R-\delta} - \frac{x_2}{N})_+$ in the domain $D = \{0 < x_1 < R - \delta, x_2 > 0\}$. Since η is linear in $D \cap \{\eta > 0\}$ and

$$\mathcal{L}(\eta W) = \eta \mathcal{L}W + W a^{ij} \eta_{ij} \geq \frac{\eta}{2W},$$

ηW cannot have an interior maximum in $D \cap \{\eta > 0\}$. Thus ηW achieves its maximum when $x_1 = 0$ or $x_2 = 0$. Restricting to $D' := \{0 \leq x_1 \leq R - \frac{3\delta}{2}, x_2 \geq 0\}$ and letting $N \rightarrow \infty$ gives

$$\frac{\delta}{2(R-\delta)} W \leq \eta W \leq \max(C_0, C_1) \text{ in } D'.$$

This completes the proof. \square

We now analyze the asymptotic behavior of complete locally strictly convex translating solitons $\Sigma = \text{graph}(u)$ in \mathbb{R}^3 that are defined over the strip region \mathcal{S}^λ .

Theorem 5.8. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}, \lambda \geq 1\}$. Then (after possibly relabeling the e_2 direction)*

- i. *For $\lambda \leq 1$ there is no locally strictly convex solution in \mathcal{S}^λ .*
- ii. $\lim_{x_2 \rightarrow +\infty} u_{x_2}(x_1, x_2) = L := \sqrt{\lambda^2 - 1}$, $\lambda > 1$.
- iii. $\lim_{x_2 \rightarrow -\infty} u_{x_2}(x_1, x_2) = -L$.
- iv.

$$(5.12) \quad \begin{aligned} \lim_{A \rightarrow \pm\infty} (u(x_1, x_2 + A) - u(0, A)) &= u^\lambda(x_1, x_2) \\ \lim_{A \rightarrow \pm\infty} u_{x_1}(x_1, A) &= \lambda \tan \frac{x_1}{\lambda}. \end{aligned}$$

The above limits hold uniformly on $\mathcal{S}^\varepsilon = \{(x_1, x_2) \in \mathcal{S}^\lambda : |x_1| \leq R - \varepsilon\}$.

- v. *For $P = (x_1, x_2, u(x_1, x_2)) \in \Sigma$, $(x_1, x_2) \in \mathcal{S}^\varepsilon$,*

$$(5.13) \quad H(P) \geq \theta(\varepsilon).$$

Proof. By Lemma 5.6 there is a sequence of points $P_n \in \Sigma$ where (after relabeling axes if necessary) $\liminf_{n \rightarrow \infty} H(P_n) \geq \theta > 0$ and $x_2^n \rightarrow +\infty$. Note that by Lemma 5.3, $|x_1^n| \leq R - \theta$. Arguing as in Lemma 5.5, $\Sigma_n := \Sigma - P_n$ converge smoothly to Σ_∞ , a complete convex translating soliton defined over a strip $\mathcal{S}^{\lambda'} - (x_1^\infty, 0)$ passing through the origin. Note that to begin with we can only assert that $\lambda' \leq \lambda$. By Lemma 5.1, $\lambda' > 1$ and $u(y_1 + x_1^n, y_2 + x_2^n) - u(x_1^n, x_2^n)$ converges locally smoothly to

$$(\lambda')^2 \log \sec \left(\frac{y_1 + x_1^\infty}{\lambda'} \right) + \sqrt{(\lambda')^2 - 1} y_2 - (\lambda')^2 \log \sec \frac{x_1^\infty}{\lambda'}$$

as $n \rightarrow \infty$ for $|y_1 + x_1^\infty| < R' := \frac{\pi}{2} \lambda'$. In particular,

$$(5.14) \quad \begin{aligned} i. \lim_{n \rightarrow \infty} u_{x_1}(y_1 + x_1^n, y_2 + x_2^n) &= \lambda' \tan \frac{y_1 + x_1^\infty}{\lambda'}, \\ ii. \lim_{n \rightarrow \infty} u_{x_2}(y_1 + x_1^n, y_2 + x_2^n) &= L' := \sqrt{(\lambda')^2 - 1}. \end{aligned}$$

Using Lemma 5.4 we see that

$$(5.15) \quad \begin{aligned} &|\arctan u_{x_1}(y_1 + o(1) + x_1^\infty, y_2 + x_2^n) - \arctan u_{x_1}(y_1 + x_1^n, y_2 + x_2^n)| \\ &\leq |x_1^n - x_1^\infty + o(1)| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ hence by (5.14) i.} \\ &\lim_{n \rightarrow \infty} u_{x_1}(y_1 + o(1) + x_1^\infty, y_2 + x_2^n) = \lambda' \tan \frac{y_1 + x_1^\infty}{\lambda'}. \end{aligned}$$

Similarly using Lemma 5.4 and (5.15),

$$\begin{aligned}
& |\arctan u_{x_2}(y_1 + x_1^\infty, y_2 + x_2^n) - \arctan u_{x_2}(y_1 + x_1^n, y_2 + x_2^n)| \\
(5.16) \quad & \leq \frac{\sqrt{1 + u_{x_1}^2}}{\sqrt{1 + u_{x_2}^2}}(y_1 + o(1) + x_1^\infty, y_2 + x_2^n)|x_1^n - x_1^\infty| \rightarrow 0 \text{ as } n \rightarrow \infty, \\
& \text{(so that by (5.14) ii.) } \lim_{n \rightarrow \infty} u_{x_2}(y_1 + x_1^n, y_2 + x_2^n) = L' := \sqrt{(\lambda')^2 - 1}.
\end{aligned}$$

Setting $y_1 = x_1 - x_1^\infty$, $y_2 = x_2$, it follows that

$$\begin{aligned}
(5.17) \quad & i. \lim_{n \rightarrow \infty} (u(x_1, x_2 + x_2^n) - u(x_1^\infty, x_2^n)) = u^{\lambda'}(x_1, x_2) - (\lambda')^2 \log \sec \frac{x_1^\infty}{\lambda'} \\
& ii. \lim_{n \rightarrow \infty} u_{x_1}(x_1, x_2 + x_2^n) = \lambda' \tan \frac{x_1}{\lambda'} \\
& iii. \lim_{n \rightarrow \infty} u_{x_2}(x_1, x_2 + x_2^n) = L'
\end{aligned}$$

for $|x_1| < \lambda'$.

From (5.17) iii. we conclude that $\limsup_{x_2 \rightarrow \infty} u_{x_2}(x_1, x_2) \geq L'$. But $u_{x_2}(x_1, x_2) \leq u_{x_2}(x_1, x_2 + x_2^n)$ for n large so

$$(5.18) \quad \lim_{x_2 \rightarrow \infty} u_{x_2}(x_1, x_2) = L' \leq L, \quad |x_1| < \lambda'.$$

In exactly the same way, we may prove

$$(5.19) \quad \lim_{x_2 \rightarrow -\infty} u_{x_2}(x_1, x_2) = L'' \leq L, \quad |x_1| < \lambda''.$$

Note that by (5.18), (5.19) for $x_2 = A$, $x_1 = x_1(A)$ (recall $u_{x_1}(x_1(A), A) = 0$), $H((x_1(A), A), u(x_1(A), A)) \geq \frac{1}{\sqrt{1+L^2}}$. Hence we may choose $x_2^n \rightarrow \infty$ arbitrary and $x_1^n = x_1(x_2^n)$. From (5.14) i. with $y_1 = y_2 = 0$, we conclude $x_1^\infty = 0$. Since the choice of $x_2^n \rightarrow \infty$ is arbitrary, it follows from (5.17) ii. (with $x_1 = x_2 = 0$) that

$$(5.20) \quad \lim_{x_2 \rightarrow \infty} u_{x_1}(0, x_2) = 0,$$

and therefore (by (5.18))

$$(5.21) \quad \lim_{x_2 \rightarrow \infty} W(0, x_2) = \lambda',$$

where $W := \sqrt{1 + |Du|^2}$.

We can now prove that there is no drop of width in the strip of convergence, i.e $\lambda' = \lambda$ (and similarly for $\lambda'' = \lambda$). Suppose for contradiction that $R' = R - 2\delta < R$ and set $C_0 := \sup_{|x_1| < R-\delta} W(x_1, 0)$. By (5.21), $C_1 := \sup_{x_2 \geq 0} W(0, x_2) < \infty$. Applying Lemma 5.7, we conclude $W(x_1, x_2) \leq 2\frac{R-\delta}{\delta} \max(C_0, C_1)$ in the half-strip $\{(x_1, x_2) : |x_1| \leq$

$R - \frac{3}{2}\delta, x_2 \geq 0\}$, contradicting the completeness of Σ_∞ . This completes the proof of parts i.ii., iii. and part iv. follows immediately from (5.17) i., ii. since $x_1^n = x_1(x_2^n)$, $x_1^\infty = 0$ and x_2^n is arbitrary.

To prove part v. we will use Lemma 5.2 with

$$P_2 = (x_1, x_2, u(x_1, x_2)), P_1 = (x_1(x_2), x_2, u(x_1(x_2), x_2)), H(P_1) \geq \frac{1}{\sqrt{1+L^2}}.$$

Normalize u by $|\nabla u(x_1(0), 0)| = 0$. We observe that

$$d^\Sigma(P_2, P_1) \leq L(x_2) := \int_{-R+\varepsilon}^{R-\varepsilon} \sqrt{1 + u_{x_1}^2(t, x_2)} dt$$

and by (5.12), $L(x_2) \leq M(\varepsilon)$ for $|x_2| \geq A(\varepsilon)$ sufficiently large. Therefore using Lemma 5.4, $|L'(x_2)| \leq \sqrt{1+L^2} L(x_2)$ and so

$$(5.22) \quad d^\Sigma(P_2, P_1) \leq L(x_2) \leq e^{\sqrt{1+L^2}(A(\varepsilon)-|x_2|)} M(\varepsilon) \leq e^{\sqrt{1+L^2}A(\varepsilon)} M(\varepsilon) =: \overline{M}(\varepsilon)$$

for $|x_2| \leq A(\varepsilon)$. Therefore $H(P_1) \geq \theta(\varepsilon) := \frac{e^{-\overline{M}(\varepsilon)}}{\sqrt{1+L^2}}$, completing the proof. \square

An immediate application of Theorem 5.8 is the following symmetry result.

Theorem 5.9. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region \mathcal{S}^λ . Then $u(x_1, x_2) = u(-x_1, x_2)$ and $u_{x_1}(x_1, x_2) > 0$ for $x_1 > 0$.*

Proof. We employ the method of moving planes; see for example [3]. Set $\mathcal{S}_t = \{(x_1, x_2) \in \mathcal{S}^\lambda : t < x_1 < R\}$ for $t \in (0, R)$. We want to show that the function

$$(5.23) \quad v^t(x_1, x_2) := u(x_1, x_2) - u(2t - x_1, x_2) > 0 \text{ in } \mathcal{S}_t \text{ for all } t \in (0, R).$$

Once (5.23) is proven, the conclusion of Theorem 5.9 follows easily. Indeed letting $t \rightarrow 0$ in (5.23) implies by continuity

$$(5.24) \quad u(x_1, x_2) \geq u(-x_1, x_2) \text{ for } 0 \leq x_1 \leq R.$$

Since we may replace x_1 by $-x_1$ in (5.24), we have equality and thus we have symmetry in x_1 . From (5.23) we may also conclude that $u_{x_1} \geq 0$ for $0 < x_1 < R$. Since u_{x_1} satisfies an elliptic equation without zeroth order term, we have from the maximum principle either $u_{x_1} > 0$ or $u_{x_1} \equiv 0$. Since the latter possibility is impossible, Theorem 5.9 follows.

Set $u^t = u(2t - x_1, x_2)$. Since u and u^t both satisfy the same elliptic equation (1.4) in \mathcal{S}_t ,

the difference $v^t = u - u^t$ satisfies a linear elliptic equation

$$(5.25) \quad \sum_{i,j=1}^2 A^{ij} v_{ij}^t + \sum_{i=1}^2 B^i v_i^t = 0 \text{ in } \mathcal{S}_t,$$

and so v^t cannot have an interior minimum in \mathcal{S}_t . Also $v^t = 0$ for $x_1 = t$, $\lim_{x_1 \rightarrow R} v^t = +\infty$, and by Theorem 5.8 part iv.,

$$(5.26) \quad \lim_{x_2 \rightarrow \pm\infty} v^t = \lambda^2 \log \left\{ \frac{\sec \frac{x_1}{\lambda}}{\sec \frac{(2t-x_1)}{\lambda}} \right\} \geq 0.$$

Therefore $v_t \geq 0$ and so by the maximum principle (5.23) holds completing the proof. \square

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