ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

PAUL C. YANG
SHING-TUNG YAU

Eigenvalues of the laplacian of compact Riemann surfaces and minimal submanifolds

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 1 (1980), p. 55-63.

<http://www.numdam.org/item?id=ASNSP_1980_4_7_1_55_0>

Eigenvalues of the Laplacian of Compact Riemann Surfaces and Minimal Submanifolds.

PAUL C. YANG - SHING-TUNG YAU

Introduction.

Given a compact Riemannian manifold (M, ds^2), the spectrum of its Laplacian is an important analytic invariant. There has been a considerable amount of work devoted to estimating the first eigenvalue λ_1 in terms of other geometric quantities associated to (M, ds^2). We recall the known estimates valid for general Riemannian manifolds: The eigenvalue comparison theorem of Cheng [2] gives a computable sharp upper bound in terms of the diameter, and lower bound of the Ricci curvature; while Yau ([8]) derived a computable lower bound in terms of the diameter, volume and lower bound of the Ricci curvature. In the special case when M has dimension two, Hersch ([3]) has extended the method of Szegö to obtain an upper bound of λ_1 for an arbitrary metric on S^2 simply in terms of its area. Subsequently in [1], Berger suggested, having verified for flat metrics, that a similar estimate holds for the torus.

In the first part of this paper we give an affirmative answer:

THEOREM 1. *Let (M, ds^2) be an orientable Riemannian surface of genus g with area A. Then we have*

$$\lambda_1 \leqslant 8\pi(g+1)A^{-1}.$$

To be more precise, we give in section 2 the following lower bound for the quantity $\sum_{i=1}^{3} \lambda_i^{-1} \geqslant 3A/8\pi d$ for a metric Riemann surface, with a meromorphic function of degree d (holomorphic map $\pi: M \to S^2$).

Pervenuto alla Redazione il 9 Ottobre 1978 ed in forma definitiva il 23 Agosto 1979.

REMARKS. 1) In the case of a torus $g = 1$, Berger suggested $8\pi^2/\sqrt{3} \sim \sim 0.9 \times 16\pi$ as an upper bound for λ_1, we do not know whether Theorem 1 holds with this better constant. In this connection it is of interest to note that Berger also showed ([1], Proposition 4.22) that $\sum_{i=1}^{6} 1/\lambda_i < 6A/(8\pi^2/\sqrt{3})$ for the flat torus with lattice generated by $(1, 0)$ and $(\frac{1}{2}, \sqrt{3}/2(1 + \varepsilon))$ for ε small.

2) For manifold of dimension greater than two, an analogous upper bound for λ_1 in terms of its volume is false in general. In fact Urakawa and Tano ([7], [6] respectively) have shown that for odd dimensional spheres S^n and compact Lie groups with nontrivial commutator subgroup, the inequality $\lambda_1(g)\,\mathrm{vol}^{2/n}(g) \leqslant C_n$ does not hold for a constant c_n independent of the metric. Thus the problem remains open only for compact non-orientable surfaces.

3) We observe two immediate consequences: 1) For holomorphic curves in $\boldsymbol{CP}^n$, the area of the metric induced from the Fubini-Study metric is 4π times its degree, hence $\lambda_1 \leqslant 2$; 2) for Riemann surfaces of genus g carrying the Poincaré metric with constant curvature -1, the Gauss-Bonnet formula and Theorem 1 yield $\lambda_1 \leqslant 2((g+1)/(g-1))$.

In the second part of this paper we study the eigenvalues of compact minimal submanifolds M^m of the unit sphere $S^N(1) \subset \boldsymbol{R}^{N+1}$. Once more we exploit the fact that the coordinate functions $\{x_\alpha\}$ are eigenfunctions and satisfy the equation $\sum_{\alpha=1}^{N+1} x_\alpha^2 = 1$ to estimate the consecutive difference of eigenvalues of M^m as in Payne, Polya and Weinberger ([5]).

THEOREM 2. *If $M^m \to S^N(1)$ is a compact minimally immersed submanifold of $S^N(1)$, then we have, setting $\Lambda_k = \sum_{i=1}^{k} \lambda_i$*

$$\lambda_{k+1} - \lambda_k \leqslant m + \frac{\sqrt{\Lambda_k^2 + 2m^2\Lambda_k(k+1)} + \Lambda_k}{m(k+1)}.$$

We would like to thank Professor M. Berger for providing relevant references and suggestions clarifying our exposition.

1. – In this section we collect the elementary facts concerning the Laplacian operator, and recall the basic equations for minimal submanifolds of the sphere.

Given a compact m-dimensional Riemannian manifold M with metric given in local coordinates by $\sum g_{ij}\,dx_i\,dx_j$, the Laplacian operator Δ acting on functions is given by

$$g^{-\frac{1}{2}}\frac{\partial}{\partial x_i}\left(g^{\frac{1}{2}}g^{ij}\frac{\partial}{\partial x_j}\right),$$

where $(x_1, \ldots, x_m)$ is a local coordinate system, g^{ij} is the inverse of the matrix g_{ij} and $g = \det(g_{ij})$, and we denote the volume element by dv. It is well-known that Δ have discrete spectrum. We list the eigenvalues as $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, and the corresponding eigenfunctions u_k with $\Delta u_k + \lambda_k u_k = 0$ form a complete orthonormal basis for $L^2(M)$. An immediate consequence is the minimum principle:

$$\lambda_{k+1} = \inf\left(\int_M \langle\nabla\varphi, \nabla\varphi\rangle\,dv\right)\left(\int_M \varphi^2\,dv\right)^{-1} \tag{1.1}$$

where the infimum is taken over piecewise C^1 functions $\varphi \neq 0$ satisfying

$$\int_M \varphi u_i\,dv = 0 \qquad \text{for } 0 \leqslant i \leqslant k\,. \tag{1.2}$$

Similarly a simultaneous diagonalization argument involving the two quadratic forms $\int_M \langle\nabla\cdot, \nabla\cdot\rangle\,dv$ and $\int_M \langle\cdot, \cdot\rangle\,dv$ yields the following variational characterization

$$\sum_{i=1}^{k}\lambda_i^{-1} = \sup\sum_{i=1}^{k}\left(\int_M \varphi^2\,dv\right)\left(\int_M |\nabla\varphi_i|^2\,dv\right)^{-1} \tag{1.3}$$

where the supremum is taken over piecewise C^1 functions $\varphi_1, \ldots, \varphi_k$ satisfying

$$\int_M \varphi_i\,dv = 0 \qquad \text{for } 1 \leqslant i \leqslant k \tag{1.4}$$

and

$$\int_M \langle\nabla\varphi_i, \nabla\varphi_j\rangle = 0 \qquad \text{for } i \neq j\,. \tag{1.5}$$

In the special case when M has dimension two, the existence of local holomorphic coordinates $z = x + iy$ simplifies the local expression for Δ: if $ds^2 = F|dz|^2$, then $\Delta = (4/F)\big(\partial^2/(\partial z\,\partial\bar{z})\big)$. An essential feature of the surface Laplacian is the invariance of the Dirichlet integrand: if $\widetilde{ds^2} = \lambda\,ds^2$ then $|\tilde{\nabla}u|^2\,d\tilde{v} = |\nabla u|^2\,dv$.

Suppose $x\colon M^m \to S^N(1)$ is an isometric minimal immersion of a Riemannian manifold M into the standard unit sphere $S^N(1)$ in Euclidean space E^{N+1}. It is well known ([4], p. 342) that the coordinate functions $x_1, \ldots, n_{N+1}$ are eigenfunctions for M^m with eigenvalue m.

2. – Theorem 1 will be an easy consequence of the following more precise:

PROPOSITION. *Let (M, ds^2) be a metric Riemann surface, and suppose $\pi\colon M^2 \to S^2$ in a non-constant holomorphic map of degree d, then we have, denoting by A the area of M,*

$$\sum_{i=1}^{k} \lambda_i^{-1} \geqslant \frac{3A}{8\pi d}\,. \tag{2.1}$$

PROOF. From complex analysis, it is well known that π is a branched cover; that is there is a finite set $\{p_1, \ldots, p_n\} \subset S^2$ so that $\pi\colon M - \pi^{-1}\cdot\{p_1, \ldots, p_n\} \to S^2 - \{p_1, \ldots, p_n\}$ is a covering map with d sheets; and at each singular point say $q \in \pi^{-1}(p_i)$, π can be expressed relative to local coordinates z (resp. w) around q (resp. p_i) as

$$w = \pi(z) = z^j \quad \text{for some positive integer } j\,. \tag{2.2}$$

For a generic point $p \in S^2 - \{p_1, \ldots, p_n\}$, we define a conformal metric by $ds_*^2 = \sum_\alpha (\pi_\alpha^{-1})^* ds^2$, where the sum is taken over the various sheets of the covering. In other words, ds_*^2 is a finite sum of conformal metrics $ds_\alpha^2 = (\pi_\alpha^{-1})^* ds^2$ with respect to which π is a local isometry on each corresponding sheet indexed by α. In terms of local coordinates z_α around each $q_\alpha \in \pi^{-1}(p)$, if the metric near q_α is given by $G(z_\alpha)|dz|^2$, then ds_*^2 is given by

$$\sum_{\pi(z_\alpha)=w} G(z_\alpha) \left|\frac{dz_\alpha}{dw}\right|^2 |dw|^2. \tag{2.3}$$

Near the singular points $q \in \pi^{-1}\{p_1, \ldots, p_n\}$, according to the representation (2.2), the singular contribution to ds_*^2 is

$$\sum_{z_\alpha^i = w} G(z_\alpha) \left|\frac{dw^{1/j}}{dw}\right|^2 |dw|^2 = \sum_{z_\alpha^i = w} G(z_\alpha) \left(\frac{1}{j}\right)^2 |w|^{-2+2/j} |dw|^2, \tag{2.4}$$

which is clearly integrable.

LEMMA. *Let $u \in C^1(S^2)$, and let dv (resp. dv_*) denote the volume element of ds^2 (resp. ds_*^2) on M (resp. S^2), we have*

(i) $\int_{S^2} u\, dv_* = \int_M (u \circ \pi)\, dv.$

(ii) *If ds_0^2 is another conformal metric on S^2, then*

$$d\int_{S^2} |\nabla_0 u|^2\, dv_0 = d\int_{S^2} |\nabla_* u|^2\, dv_* = \int_{S^2} |\nabla(u \circ \pi)|^2\, dv\,.$$

PROOF. It sufficies to prove the identities locally, i.e., for those u supported in a trivializing neighborhood U of the covering. Then (i) is an immediate consequence of the definition of ds_*^2. The first identity in (ii) is simply the invariance of Dirichlet integrand under conformal change of metric. For the second identity, observe that for each sheet α in the local trivialization we have

$$\int_U |\nabla^* u|^2\, dv^* = \int_U |\nabla_\alpha u|^2\, dv^* = \int_{\pi_\alpha^{-1}(U)} |\nabla u|^2\, dv$$

hence the assertion follows.

Let x_1, x_2, x_3 be the standard coordinate functions on (S^2, ds_0^2), ds_0^2 is the standard metric.

$$\sum_1^3 x_1^2 = 1\,, \tag{2.5}$$

$$\int_{S^2} \langle \nabla_0 x_i, \nabla_0 x_j \rangle\, dv_0 = \delta_{ij}\, \frac{8\pi}{3}\,. \tag{2.6}$$

Consider the transplanted pseudometric ds_*^2 on S^2 introduced above. In order to apply the estimate of (1.13), we need to find a conformal self map φ of S^2 so that

$$\int (x_i \circ \varphi)\, dv_* = 0$$

for $i = 1, 2,$ and 3. The topological argument of Hersch still applies since dv_* is an integrable density on S^2. Assuming this is done, then letting $u = (x_i \circ \varphi)$, and $v = (x_i \circ \varphi \circ \pi)$, we have by (1.3)

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \geqslant \sum_{i=1}^3 \frac{\int_M v_i^2\, dv}{\int_M |\nabla v_i|^2 dv}\,.$$

In the right hand side each denominator is according to (ii) of Lemma equal to $d\cdot(8\pi/3)$, hence

$$\sum_1^3 \frac{1}{\lambda_i} \geqslant \frac{3}{8\pi d} \sum_1^3 \int_M v_i^2\, dv = \frac{3}{8\pi d}\int dv = \frac{3A}{8\pi d}\,. \qquad \text{Q.E.D.}$$

Theorem 1 follows by observing that $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ and that Riemann-Roch theorem ensures that each Riemann surface of genus g has at least one non-constant meromorphic function of degree $\leqslant g+1$.

3. – For general compact minimal submanifolds $M^m \to S^N(1)$ into the Euclidean unit sphere, the coordinate functions $x_1, \ldots, x_{N+1}$ are eigenfunctions M^m and with the behavior of eigenfunctions of the spherical harmonics of the standard S^N as a model we might expect suitable products of x_i with the first k eigenfunctions of M^m to give reasonable trial functions for λ_{n+1}. The proof given below is based on this idea.

PROOF OF THEOREM 2. Let $x = (x_1, \ldots, x_{N+1})$: $M_m \to S^N \subset \boldsymbol{R}^{N+1}$ be the minimal immersion. Let $\{(u_i, \lambda_i)\}_{i=1,\ldots,n}$ be the normalized first $k+1$ eigenfunctions and their corresponding eigenvalues (we set $\lambda_0 = 0$, $u_0 \equiv 1/\sqrt{v}$ where $v =$ volume (M)). Then

$$\Delta x_\alpha = - m x_\alpha\,, \tag{3.1}$$

$$\Delta u_i = - \lambda_i u_i\,. \tag{3.2}$$

Let

$$u_{\alpha i} = x_\alpha u_i - \sum a_{\alpha i k} u_k \tag{3.3}$$

with

$$a_{\alpha i k} = \int_M x_\alpha u_i u_k = a_{\alpha k i} \tag{3.4}$$

and we set

$$A = \sum_{\alpha, i, k} a^2_{\alpha k i}\,. \tag{3.5}$$

Then $u_{\alpha i}$ are orthogonal to $u_0, \ldots, u_k$. To estimate the Rayleigh-Ritz quotient we compute

$$-\Delta u_{\alpha i} = (m + \lambda_i) x_\alpha u_i - 2\langle \nabla x_\alpha, \nabla u_i\rangle + \sum_k a_{\alpha i k}\lambda_k u_k$$

and then integrating against $u_{\alpha i}$,

$$\int|\nabla u_{\alpha i}|^2 = -\int(\Delta u_{\alpha i})u_{\alpha i} = (m+\lambda_i)\int u_{\alpha i}^2 - 2\int\langle\nabla x_\alpha, \nabla u_i\rangle u_{\alpha i}\,.$$

Hence

$$\lambda_{k+1} \leqslant \frac{\int|\nabla u_{\alpha i}|^2}{\int u_{\alpha i}^2} \leqslant (m+\lambda_i) - 2\,\frac{\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle}{\int u_{\alpha i}^2}\,.$$

Obviously

$$\lambda_{k+1} - \lambda_k - m \leqslant \lambda_{k+1} - \lambda_i - m \leqslant -2\,\frac{\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle}{\int u_{\alpha i}^2}$$

hence

$$\lambda_{k+1} - \lambda_k - m \leqslant \frac{\sum_{\alpha,i}(-2)\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle}{\sum_{\alpha,i}\int u_{\alpha i}^2}\,. \tag{3.6}$$

Since $a_{\alpha ik}$ is symmetric in i and k we have

$$\sum a_{\alpha ik}\langle\nabla x_\alpha, \nabla(u_k u_i)\rangle = \sum 2a_{\alpha ik}u_k\langle\nabla x_\alpha, \nabla u_i\rangle\,,$$

so that

$$\begin{aligned}
\sum(-2)\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle
&= \sum\int(-2)x_\alpha u_i\langle\nabla x_\alpha, \nabla u_i\rangle + \sum\int 2a_{\alpha ik}u_k\langle\nabla x_\alpha, \nabla u_i\rangle \\
&= \sum\int(-1)\langle\nabla x_\alpha^2, u_i\nabla u_i\rangle + \sum\int a_{\alpha ik}\langle\nabla x_\alpha, \nabla(u_k u_i)\rangle \\
&= \sum a_{\alpha ik}\int(-\Delta x_\alpha)u_i u_k \qquad \Big(\sum_\alpha x_\alpha^2 \equiv 1\Big) \\
&= m\sum a_{\alpha ik}\int x_\alpha u_i u_k \qquad (\Delta x_\alpha = -m x_\alpha) \\
&= mA
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\sum\int u_{\alpha i}^2 &= \sum_{\alpha,i}\int\Big(x_\alpha^2 u_i^2 - 2x_\alpha u_i\sum_k a_{\alpha ik}u_k + \sum_{k,l} a_{\alpha ik}a_{\alpha il}u_k u_l\Big) \\
&= \sum_i\int u_i^2 - 2A + A \\
&= (k+1) - A\,.
\end{aligned} \tag{3.8}$$

The Schwarz inequality gives

$$(3.9)\qquad \sum\left|\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle\right| \leqslant \sum\left[\int(u_{\alpha i})^2\right]^{\frac12}\left[\int\langle\nabla x_\alpha, \nabla u_i\rangle^2\right]^{\frac12}$$

$$\leqslant\left[\sum\int u_{\alpha i}^2\right]^{\frac12}\left[\sum\int\langle\nabla x_\alpha, \nabla u_i\rangle^2\right]^{\frac12} \leqslant \left[\sum\int u_{\alpha i}^2\right]^{\frac12}\left[\sum_i\int|\nabla u_i|^2\right]^{\frac12}$$

$\left(\text{since } \sum_\alpha \langle\nabla x_\alpha, \nabla u_i\rangle^2 \equiv |\nabla u_i|^2\right)$

$$\leqslant\left[\sum\int u_{\alpha i}^2\right]^{\frac12}\left[\sum_1^k \lambda_i\right]^{\frac12}.$$

Applying (3.9) to (3.6) we obtain, setting $\Lambda_k = \sum_1^k \lambda_i$

$$(3.10)\qquad \lambda_{k+1} - \lambda_k - m \leqslant \frac{\sum(-2)\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle}{\left[\sum\left|\int u_{\alpha i}\langle\nabla x_\alpha, \nabla u_i\rangle\right|\right]^2}\left[\sum_1^k \lambda_i\right] \leqslant \frac{2\Lambda_k}{mA}.$$

Combining (3.6), (3.7), (3.8) and (3.10) we have

$$\lambda_{k+1} - \lambda_k - m \leqslant \min\left[\frac{2\Lambda_k}{mA}, \frac{mA}{(k+1) - A}\right].$$

Consider the expression in the bracket as functions of A, and observe that the first is decreasing while the second is increasing in A for $A \in (0, k+1)$. Then both must be less than their common value which occurs at

$$A = \frac{\sqrt{\Lambda_k^2 + 2m^2\Lambda_k(k+1)} - \Lambda_k}{m^2} < k+1$$

and for this A the common value is

$$\frac{\sqrt{\Lambda_k^2 + 2m^2\Lambda_k(k+1)} + \Lambda_k}{m(k+1)}$$

as claimed.

REFERENCES

[1] M. Berger, *Sur les premières valeurs propres des variétés Riemanniennes*, Compositio Math., **26** (1973), pp. 129-149.

[2] S.-Y. Cheng, *Eigenvalues comparison theorems and its geometric applications*, Math. Z., **143** (1975), pp. 289-297.

[3] J. HERSCH, *Quatre properiétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris, **270** (1970), pp. 1645-1648.
[4] S. KOBAYASHI - K. NOMIZU, *Foundations of Differential Geometry*, Volume II.
[5] L. E. PAYNE - G. POLYA - H. F. WEINBERGER, *On the ratio of consecutive eigenvalues*, J. Mathematical Phys., **35** (1956), pp. 289-298.
[6] S. TANNO, *The first eigenvalues of the laplacian on spheres*, to appear in Tôhoku Math. J.
[7] H. URAKAWA: *On the least positive eigenvalue of the laplacian for riemannian manifolds II*, preprint.
[8] S.-T. YAU, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup., **8** (1975), pp. 487-507.

University of Maryland
Department of Mathematics
College Park, Maryland 20742

University of California
Department of Mathematics
Berkeley, California 94720