The Structure of Complete Stable Minimal Surfaces in 3-Manifolds of Non-Negative Scalar Curvature*

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The purpose of this paper is to study minimal surfaces in three-dimensional manifolds which, on each compact set, minimize area up to second order. If M is a minimal surface in a Riemannian three-manifold N, then the condition that M be stable is expressed analytically by the requirement that on any compact domain of M, the first eigenvalue of the operator $\Delta + \mathrm{Ric}(\nu) + |A|^2$ be positive. Here $\mathrm{Ric}(\nu)$ is the Ricci curvature of N in the normal direction to M and $|A|^2$ is the square of the length of the second fundamental form of M.

In the case that N is the flat $\mathbb{R}^3$, we prove that any complete stable minimal surface M is a plane (Corollary 4). The earliest result of this type was due to S. Bernstein [2] who proved this in the case that M is the graph of a function (stability is automatic in this case). The Bernstein theorem was generalized by R. Osserman [10] who showed that the statement is true provided the image of the Gauss map of M omits an open set on the sphere. The relationship of the stable regions on M to the area of their Gaussian image has been studied by Barbosa and do Carmo [1] (cf. Remark 5). The methods of Schoen–Simon–Yau [11] give a proof of this result provided the area growth of a geodesic ball of radius r in M is not larger than r^6. An interesting feature of our theorem is that it does not assume that M is of finite type topologically, or that the area growth of M is suitably small.

The theorem for $\mathbb{R}^3$ is a special case of a classification theorem which we prove for stable surfaces in three-dimensional manifolds N having scalar curvature $S \geqq 0$. We use an observation of Schoen–Yau [8] to rearrange the stability operator so that S comes into play (see formula (12)). Using this, and the study of certain differential operators on the disc (Theorem 2), we are

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able to show that any complete noncompact stable surface M in a 3-manifold N of non-negative scalar curvature is conformally diffeomorphic to the complex plane $\mathbb{C}$ or the cylinder A. It was shown in [8] that if M is compact and stable, then M is the sphere or a flat totally geodesic torus. We show also that if M is a cylinder and either M has finite absolute total curvature or N has non-negative Ricci curvature, then M is flat and totally geodesic. We also observe that all of the above cases do occur. Finally, we give an intrinsic characterization (Theorem 4) for a metric on S^2 to be realized on a stable minimal immersion into a compact scalar flat three-dimensional manifold. We present a similar characterization for metrics on $\mathbb{C}$ to be realized on a stable minimal immersion into a complete scalar flat 3-manifold.

The paper is divided into three sections. In the first section we study the operator $\Delta - q$ on a complete Riemannian manifold M of arbitrary dimension. Here q is a smooth function on M. We show that the existence of a positive function f on M satisfying $\Delta f - qf = 0$ is equivalent to the condition that the first eigenvalue of $\Delta - q$ be positive on each bounded domain in M. This result is well known if M is $\mathbb{R}^n$ and was proven by Glazman [9], p. 159. It is a fairly easy generalization of the $\mathbb{R}^n$ proof to give a proof in our setting; however, since we use the result many times, and since it may be useful in other geometric problems, we give a fairly complete proof of the theorem.

In Section 2 we study the operator $\Delta - aK$ for conformal metrics on the disc, where K is the Gauss curvature function and a is a constant. We show that if $a \geqq 1$ and the metric is complete, then there is no positive solution f of $\Delta f - aKf = 0$. The operators $\Delta - aK$ are intimately connected with the stability of minimal surfaces, the case $a = 2$ for surfaces in $\mathbb{R}^3$, and the case $a = 1$ for surfaces in scalar flat 3-manifolds (see Theorem 4). We do not know the smallest value of a for which $\Delta - aK$ has a positive solution. For the Poincaré metric on the disc the value is $\frac{1}{4}$. In Section 3 we prove our results on minimal surfaces.

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We have recently learned that the special case of our results for stable surfaces in $\mathbb{R}^3$ has also been obtained by do Carmo and Peng by a different method.

1. Some General Results for the Operator $\Delta - q$

Let (M, ds^2) be an n-dimensional complete noncompact Riemannian manifold, and let q be a smooth function on M. Given any bounded domain $D \subset M$, we let $\lambda_1(D) < \lambda_2(D) \leqq \lambda_3(D) \leqq \cdots$ be the sequence of eigenvalues of $\Delta - q$ acting on functions vanishing on ∂D. The usual variational characterization of $\lambda_1(D)$ is

$$\lambda_1(D) = \inf\left\{\int_D (|\nabla f|^2 + qf^2)\,dv : \operatorname{spt} f \subset D, \int_D f^2\,dv = 1\right\}, \tag{1}$$

where $|\nabla f|$ denotes the magnitude of the gradient of f taken with respect to ds^2, and dv denotes the volume element of M. The following lemma is a well-known consequence of (1) and the unique continuation property for solutions of $\Delta - q + \lambda_1$.

LEMMA 1. *If D, D' are connected domains in M with $D \subset D'$, then $\lambda_1(D) \geqq \lambda_1(D')$. If $D' \sim \bar{D} \neq \varnothing$, then we have $\lambda_1(D) > \lambda_1(D')$.*

We now state the main result of this section.

THEOREM 1. *The following condition are equivalent:*
(i) $\lambda_1(D) \geqq 0$ *for every bounded domain* $D \subset M$;
(ii) $\lambda_1(D) > 0$ *for every bounded domain* $D \subset M$;
(iii) *there exists a positive function g satisfying the equation* $\Delta g - qg = 0$ *on* M.

Proof: (i)$\Rightarrow$(ii). This is a consequence of Lemma 1 since, for any bounded domain $D \subset M$ and any point $x_0 \in M$, we can choose R large enough so that the ball $B_R(x_0) = \{x \in M : \operatorname{dist}(x, x_0) < R\}$ satisfies $B_R(x_0) \sim \bar{D} \neq \varnothing$ and we have $\lambda_1(B_R(x_0)) \geqq 0$ by hypothesis.

(ii)$\Rightarrow$(iii). To prove the existence of a positive solution g of $\Delta g - qg = 0$ we fix a point $x_0 \in M$. For each $R > 0$ we consider the problem

$$\begin{aligned} \Delta u - qu &= 0 \quad \text{on} \quad B_R(x_0), \\ u &= 1 \quad \text{on} \quad \partial B_R(x_0). \end{aligned} \tag{2}$$

Since $\lambda_1(B_R(x_0)) > 0$, there is no nonzero solution of $\Delta u - qu = 0$ on $B_R(x_0)$ with $u = 0$ on $\partial B_R(x_0)$. The Fredholm alternative ([6], Theorem 6.15, p. 102) thus implies the existence of a unique solution v on $B_R(x_0)$ of

$$\begin{aligned} \Delta v - qv &= q \quad \text{on} \quad B_R(x_0), \\ v &= 0 \quad \text{on} \quad \partial B_R(x_0). \end{aligned}$$

It follows that $u = v + 1$ is the unique solution of (2).

We now prove that $u > 0$ on $B_R(x_0)$. It follows from the strong maximum principle ([6], pp. 33–34) that if $u \geqq 0$ on $B_R(x_0)$, then $u > 0$ on $B_R(x_0)$. Suppose now that $\Omega = \{x \in B_R(x_0) : u(x) < 0\} \neq \varnothing$. Hence $\Omega \subset B_R(x_0)$ is a bounded domain and thus, by Lemma 1, $\lambda_1(\Omega) > 0$. Since $\Delta u - qu = 0$ on Ω

and $u=0$ on $\partial\Omega$, we would have $u\equiv 0$ in Ω, contradicting the unique continuation property. We have shown that $u>0$ on $B_R(x_0)$.

We now set $g_R(x)=u(x_0)^{-1}u(x)$ for $x\in M$. We have seen that g_R satisfies

$$\Delta g_R - q g_R = 0 \quad \text{on} \quad B_R(x_0),$$
$$g_R(x_0)=1, \quad g_R>0 \quad \text{on} \quad B_R(x_0).$$

From the Harnack inequality (see([6], Theorem 8.20, p. 189) it follows that on any ball $B_\sigma(x_0)$, there is a constant C depending only on σ and M (independent of R) such that, for $R>2\sigma$,

$$g_R \leqq C \quad \text{on} \quad B_\sigma(x_0).$$

It now follows from standard elliptic theory ([6], Theorem 6.2, p. 85) that all derivatives of g_R are bounded uniformly (independent of R) on compact subsets of M. We may therefore choose a sequence $R_i\to\infty$ so that g_{R_i} converges along with its derivatives on any compact subset of M, and by taking a diagonal sequence we can arrange that g_{R_i}, along with its derivatives, converges uniformly on compact subsets of M to a function g satisfying $\Delta g - qg = 0$ and $g(x_0)=1$. Since g is not identically zero and $g\geqq 0$, the strict maximum principle implies that $g>0$. This finishes the proof that (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i). If $g>0$ satisfies $\Delta g - qg=0$ on M, we define a new function $w=\log g$. We now calculate

$$\Delta w = q - |\nabla w|^2. \tag{3}$$

Let f be any function with compact support on M. Multiplying (3) by f^2 and integrating by parts, we obtain

$$-\int_M qf^2\,dv + \int_M |\nabla w|^2 f^2\,dv = 2\int_M f\langle\nabla f, \nabla w\rangle\,dv.$$

Applying the Schwarz inequality and the arithmetic-geometric mean inequality we have

$$2|f|\langle\nabla f, \nabla w\rangle \leqq 2|f|\,|\nabla f|\,|\nabla w| \leqq f^2|\nabla w|^2 + |\nabla f|^2.$$

Putting this into the above equation and canceling the terms $\int_M f^2|\nabla w|^2\,dv$ we obtain

$$-\int_M qf^2\,dv \leqq \int_M |\nabla f|^2\,dv.$$

If D is any bounded domain and f is any function with support in D, we have shown that

$$\int_D (|\nabla f|^2 + qf^2)\, dv \geqq 0.$$

It now follows from (1) that $\lambda_1(D) \geqq 0$. This finishes the proof of Theorem 1.

The last part of the proof actually yields

COROLLARY 1. *If $D \subset M$ is any bounded domain, and if there is a function $g > 0$ in D satisfying $\Delta g - qg = 0$, then $\lambda_1(D) \geqq 0$.*

2. The Operator $\Delta - aK$ for Conformal Metrics on the Disc

Let M be the unit disc in the complex plane endowed with the metric $ds^2 = \mu(z)|dz|^2$. We assume ds^2 is a complete metric. Let K denote the Gaussian curvature of M and Δ the metric Laplacian, i.e., $\Delta f = \mu^{-1}(f_{xx} + f_{yy})$, where $z = x + iy$. The well-known formula for K is $K = -\frac{1}{2}\Delta \log \mu$. We shall prove the following theorem.

THEOREM 2. *Assume ds^2 is complete. For $a \geqq 1$, there is no positive solution g of $\Delta g - aKg$ on M.*

Proof: We first note that it suffices to prove the theorem for $a = 1$. This follows since for any function f with compact support on M and any $a > 1$ we have

$$\int_M |\nabla f|^2 + Kf^2\, dv \geqq \int_M \left(\frac{1}{a}|\nabla f|^2 + Kf^2\right) dv = \frac{1}{a}\int_M (|\nabla f|^2 + aKf^2)\, dv\,.$$

Thus (1) shows that the positivity of the first eigenvalue on any bounded domain D for $\Delta - aK$ implies the positivity of λ_1 for $\Delta - K$. Thus by Theorem 1 the existence of a positive solution of $\Delta - aK$ for $a > 1$ implies the existence of a positive solution for $\Delta - K$.

To prove Theorem 2 for $a = 1$, we define a function h by $h = \mu^{-1/2}$. We see from the definition of K that $\Delta \log h = K$, i.e.,

$$\frac{\Delta h}{h} - \frac{|\nabla h|^2}{h^2} = K.$$

In particular, h satisfies

$$\Delta h = Kh + \frac{|\nabla h|^2}{h}. \tag{4}$$

Let $D \subset M$ be a bounded domain, and let ζ be a smooth function on M with compact support in D. We now calculate

$$
\begin{aligned}
\int_M (|\nabla \zeta h|^2 + K(\zeta h)^2)\, dv &= \int_M (-\zeta h(\Delta \zeta h) + K(\zeta h)^2)\, dv \\
&= \int_M [(-\zeta\Delta\zeta)h^2 - 2\zeta h\langle \nabla\zeta, \nabla h\rangle - \zeta^2 h\Delta h + K(\zeta h)^2]\, dv.
\end{aligned} \tag{5}
$$

It follows from (1) that

$$\lambda_1(D)\int_M (\zeta h)^2\, dv \leqq \int_M (|\nabla \zeta h|^2 + K(\zeta h)^2)\, dv. \tag{6}$$

Combining (5) and (6), using (4) and a slight rearrangement we have

$$\lambda_1(D)\int_M (\zeta h)^2\, dv \leqq \int_M [(-\zeta\Delta\zeta)h^2 - \tfrac{1}{2}\langle \nabla\zeta^2, \nabla h^2\rangle]\, dv - \int_M |\nabla h|^2 \zeta^2\, dv.$$

Integration by parts gives

$$\lambda_1(D)\int_M (\zeta h)^2\, dv \leqq \int_M |\nabla\zeta|^2 h^2\, dv - \int_M |\nabla h|^2\zeta^2\, dv. \tag{7}$$

Now define a smooth function $\zeta(r)$ for $r \in \mathbb{R}$ which satisfies

$$
\begin{aligned}
&\zeta(r) = 1 \quad \text{for} \quad r \leqq \tfrac{1}{2}R, \qquad \zeta(r) = 0 \quad \text{for} \quad r \geqq R, \\
&\zeta \geqq 0 \quad \text{for all} \quad r, \qquad |\zeta'| \leqq \frac{3}{R} \quad \text{for all} \quad r.
\end{aligned} \tag{8}
$$

If r measures the metric distance to 0, and R is any positive number, then $\zeta(r)$ defines a Lipschitz function on M with support in $B_R(0)$. A standard approximation argument justifies this choice of ζ in (7). Using (8) and the fact that $dv = \mu\, dx\, dy$ we have

$$\int_M |\nabla\zeta|^2 h^2\, dv \leqq \frac{9}{R^2}\int_M dx\, dy = \frac{9\pi}{R^2}.$$

Putting this into (7) we have

$$\lambda_1(B_R(0))\int_M (\zeta h)^2\, dv \leqq \frac{9\pi}{R^2} - \int_M |\nabla h|^2\zeta^2\, dv. \tag{9}$$

Since $\mu(z)\,|dz|^2$ is a complete metric on the disc, μ cannot be a constant function. Therefore, $|\nabla h|^2$ is not identically zero on M. Thus, by choosing R sufficiently large in (9), we conclude that $\lambda_1(B_R(0))<0$. By Theorem 1 this implies that there is no positive solution of $\Delta - K$ on M. This completes the proof of Theorem 2.

The following corollary is a slight strengthening of Theorem 2.

COROLLARY 2. *Let $ds^2=\mu(z)\,|dz|^2$ be a complete metric on the disc. There exists a number a_0 depending on ds^2 satisfying $0\leqq a_0<1$ so that for $a\leqq a_0$ there is a positive solution of $\Delta - a$, and for $a>a_0$ there is no positive solution.*

Proof: Let $S=\{a : \text{there is a } g>0 \text{ satisfying } \Delta g - aKg=0\}$. From Theorem 2 and the remarks at the beginning of its proof it follows that there is an a_0 satisfying $0\leqq a_0\leqq 1$ so that S is either $(-\infty, a_0)$ or $(-\infty, a_0]$. The corollary will follow if we can show that S is a closed set. To see this, suppose $\{a_n\}$ is a sequence in S such that $\lim_{n\to\infty} a_n = a_0$. Let g_n be a positive solution of $\Delta g_n - a_n K g_n = 0$ normalized so that $g_n(0)=1$. Using the Harnack inequality ([6], Theorem 8.20, p. 189) and elliptic estimates as in the proof of Theorem 1 we can assert the existence of a subsequence of $\{g_n\}$ converging uniformly along with its derivatives on compact subsets of M to a positive solution of $\Delta - a_0 K$. This shows that $S=(-\infty, a_0]$ and, since $1\notin S$, this completes the proof of Corollary 2.

Remark 1. For the Poincaré metric on the disc we have $K\equiv -1$, and we are studying the operator $\Delta + a$. In this case it can be shown (cf. [7]) that the value of a_0 in Corollary 2 is $\frac{1}{4}$. For metrics of variable curvature we do not know the possible values of a_0 which can occur.

The next result is an extension of Theorem 2 which will be used later in our study of stable minimal surfaces. The proof follows directly from Theorem 1, Theorem 2, and formula (1).

COROLLARY 3. *Let $ds^2=\mu(z)\,|dz|^2$ be a complete metric on the disc. If $a\geqq 1$ and P is a non-negative function, then there is no positive solution g of $\Delta g - aKg + Pg = 0$ on M.*

3. Complete Stable Minimal Surfaces in 3-Manifolds

Let N be a complete oriented three-dimensional Riemannian manifold. Let M be a two-dimensional complete oriented submanifold minimally immersed in N. We say that M is stable if the second variation of the area is non-negative on any compact subset of M. More precisely, let e_1, e_2, e_3 be a positively oriented orthonormal frame defined locally on M with e_1, e_2 tangential, and e_3 the globally defined unit normal. The second fundamental form of M is the symmetric quadratic tensor on M defined by $h_{ij} = \langle \bar{\nabla}_{e_i} e_3, e_j \rangle$ for $1 \leqq i, j \leqq 2$, where $\bar{\nabla}$ is the Riemannian connection of N. The condition that M be a minimal surface is

$$h_{11} + h_{22} = 0 .$$

The stability of M is given by the following inequality (see [4]):

$$\int_M \left[|\nabla f|^2 - \left(\operatorname{Ric}(e_3) + \sum_{i,j=1}^{2} h_{ij}^2 \right) f^2 \right] dv \geqq 0, \tag{10}$$

where f is any function having compact support on M and $\operatorname{Ric}(e_3)$ is the Ricci curvature of N in the direction of e_3. We now do the rearrangement described in Schoen–Yau [8]. The Gauss curvature equation says that $K = K_{12} + h_{11}h_{22} - h_{12}^2$, where K is the intrinsic Gaussian curvature of M and K_{ij} is the sectional curvature of N for the section determined by e_i and e_j. Using minimality and symmetry of h_{ij} we have

$$K = K_{12} - \tfrac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 .$$

Inequality (10) may then be written in the form

$$\int_M \left[|\nabla f|^2 - \left(S - K + \tfrac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \right) f^2 \right] dv \geqq 0 , \tag{11}$$

where S is the scalar curvature of N given by

$$S = K_{12} + K_{23} + K_{13} = K_{12} + \operatorname{Ric}(e_3) .$$

According to (1), this inequality is equivalent to $\lambda_1(D) \geqq 0$ for every bounded domain $D \subset M$, where λ_1 is the first eigenvalue of the operator

$$\Delta + \left(S - K + \tfrac{1}{2} \sum_{i,j=1} h_{ij}^2 \right) . \tag{12}$$

We now classify the stable minimal surfaces in three-manifolds of non-negative scalar curvature.

THEOREM 3. *Let N be a complete oriented 3-manifold of non-negative scalar curvature. Let M be an oriented complete stable minimal surface in N.*

There are two possibilities:

(i) *If M is compact, then M is conformally equivalent to the sphere S^2 or M is a totally geodesic flat torus T^2. If $S>0$ on N, then M is conformally equivalent to S^2.*

(ii) *If M is not compact, then M is conformally equivalent to the complex plane $\mathbb{C}$, or the cylinder A. If M is a cylinder and the absolute total curvature of M is finite, then M is flat and totally geodesic. If the scalar curvature of N is everywhere positive, then M cannot be a cylinder with finite total curvature.*

If the Ricci curvature of N is non-negative, then the assumption of finite total curvature in (ii) *can be removed.*

Remark 2. We feel that the assumption of finite total curvature should not be essential in proving that the cylinder is flat and totally geodesic in (ii), but so far we have not been able to remove it.

Before giving the proof of Theorem 2 we state the following corollary for the case when N is $\mathbb{R}^3$. This implies the classical Bernstein theorem [2] for complete minimal graphs in $\mathbb{R}^3$.

COROLLARY 4. *The only complete stable oriented minimal surface in $\mathbb{R}^3$ is the plane.*

Proof: In this case the stability operator (12) becomes $\Delta-2K$ and by Theorem 3 we know that M is conformally either $\mathbb{C}$ or A. By Theorem 1, there is a positive function g on M satisfying $\Delta g-2Kg=0$. If M is conformal to A, we may lift g to the universal covering $\mathbb{C}$ of A. In either case we have a metric on $\mathbb{C}$ with $K\leqq 0$ and a positive g satisfying $\Delta g-2Kg=0$. Thus $\Delta g\leqq 0$, and g is a positive superharmonic function on $\mathbb{C}$ which must be constant. Therefore K is identically zero and hence $\sum_{i,j} h_{ij}^2=-2K$ is identically zero. Consequently M is a plane.

Remark 3. Observe that each of the four possibilities of Theorem 3 does occur. For example, $S^2\times\mathbb{R}$ has positive scalar curvature and has a stable S^2, $T^2\times\mathbb{R}$ is flat and has a stable T^2. We can choose a metric on $\mathbb{C}$ of positive Gaussian curvature and by crossing with $\mathbb{R}$ construct a metric of positive scalar curvature on $\mathbb{R}^3$ having a stable $\mathbb{C}$. Similarly $A\times\mathbb{R}$ has a flat metric with a stable A.

Remark 4. It is shown by Schoen–Yau [8] that if N is compact with $S\geqq 0$, and M is a stable incompressible T^2 (i.e., the fundamental group of M injects into that of N), then N is flat. We believe it is true that if N has non-negative scalar curvature and admits a complete stable T^2 or A, then N is flat. This would be an interesting analogue of the splitting theorem of Cheeger–Gromoll [3].

Remark 5. We point out that Corollary 1 can be used to simplify the proof of Theorem 1.2 of Barbosa–do Carmo [1]. The theorem states that if D is a domain in a minimal surface $M \hookrightarrow \mathbb{R}^3$, and if the area of the Gauss image $g(D) \subset S^2$ is less than 2π, then D is stable. If λ_1 is the first eigenvalue of $g(D)$ for the spherical Laplacian and f_1 is the first eigenfunction, then the function $h = f_1 \circ g$ is a positive function on D satisfying $\Delta h - \lambda_1 K h = 0$. By Corollary 1 we conclude that the first eigenvalue for $\Delta - \lambda_1 K$ is non-negative on D. Since $\lambda_1 > 2$ and $K \leqq 0$, it follows that D is stable.

PROOF OF THEOREM 3: Case (i) was observed by Schoen–Yau [8] and follows by choosing f identically equal to one in inequality (11) to obtain

$$\int_M \left(S + \tfrac{1}{2} \sum h_{ij}^2\right) dv \leqq \int_M K\, dv .$$

The Gauss-Bonnet theorem now implies that M is the sphere or the torus. In the case of the torus, M is totally geodesic and $S \equiv 0$ on M. The stability operator reduces to $\Delta - K$ and its first eigenvalue is

$$\lambda_1 \equiv \inf \left\{ \int_M (|\nabla f|^2 + K f^2)\, dv : \int_M f^2 = 1 \right\}.$$

Since $\lambda_1 \geqq 0$ by stability and $\int_M K\, dv = 0$, we conclude that $\lambda_1 = 0$ and the constant function $f \equiv 1$ satisfies $\Delta f - K f = 0$ which implies that $K \equiv 0$.

To prove case (ii), we first show that the universal covering of M is conformally equivalent to $\mathbb{C}$. If this is not true, then M is covered by the disc. Using stability and Theorem 1 we have a positive function g on M satisfying

$$\Delta g - K g + \left(S + \tfrac{1}{2} \sum_{i,j} h_{ij}^2\right) g = 0 .$$

Lifting g to the disc we obtain a positive solution of this equation on the disc endowed with a complete metric. Since $S + \frac{1}{2} \sum_{i,j} h_{ij}^2 \geqq 0$, this yields a contradiction to Corollary 3.

Thus we have shown that M is conformally covered by $\mathbb{C}$ and hence M is either conformally equivalent to $\mathbb{C}$ or M is conformal to a cylinder A. If the latter holds we must show that M is flat and totally geodesic. Let $z = x + iy$ be a complex coordinate for M so that $|dz|^{2'}$ is the flat metric on M, and the given metric on M is $ds^2 = \mu(z)\,|dz|^2$. Fix a point $z_0 \in M$, and let r be the distance from z_0 taken with respect to the flat metric. For any $R > 0$, choose $\zeta(r)$ satisfying (8). Substituting ζ for f in (11) and using (8) and the conformal invariance of the Dirichlet integral we have

$$\frac{9}{R^2}\int_{B_R(z_0)} dx\,dy - \int_M \left(S-K+\tfrac{1}{2}\sum h_{ij}^2\right)f^2\,dv \geqq 0\,,$$

where $B_R(z_0)$ is the ball taken with respect to the flat metric. Since $\int_{B_R(z_0)} dx\,dy$ has growth bounded by a constant times R, and we are assuming $\int_M |K|\,dv<\infty$, we can use the dominated convergence theorem to let R tend to infinity to achieve

$$\int_M \left(S+\tfrac{1}{2}\sum_{i,j} h_{ij}^2\right)dv \leqq \int_M K\,dv\,.$$

Since M is topologically the cylinder, the Cohn–Vossen inequality (cf. [5]) gives $\int_M K\,dv\leqq 0$. We thus conclude that M is totally geodesic and $S\equiv 0$ on M. Hence the stability operator reduces to $\Delta-K$. By Theorem 1 there is a positive function g on M satisfying $\Delta g-Kg=0$. Set $w=\log g$. Computing we have

$$\Delta w = K-|\nabla w|^2\,.$$

Choosing ζ as above, we multiply by ζ^2 and integrate by parts to get

$$\int_M |\nabla w|^2\zeta^2\,dv = \int_M \zeta^2 K\,dv + 2\int_M \zeta\langle\nabla\zeta,\nabla w\rangle\,dv\,.$$

The Schwarz inequality and the arithmetic-geometric mean inequality imply

$$2\,|\zeta|\,|\langle\nabla\zeta,\nabla w\rangle|\leqq \tfrac{1}{4}\zeta^2\,|\nabla w|^2+4|\nabla\zeta|^2\,.$$

Therefore,

$$\frac{3}{4}\int_M |\nabla w|^2\zeta^2\,dv\leqq \int_M \zeta^2 K\,dv+4\int_M |\nabla\zeta|^2\,dv\,.$$

Letting $R\to\infty$ as above, we conclude that

$$\frac{3}{4}\int_M |\nabla w|^2\,dv\leqq \int_M K\,dv\,.$$

Thus w is constant, so g is constant and we have $K \equiv 0$.

In case N has non-negative Ricci curvature, we write the stability operator as $\Delta + \mathrm{Ric}(e_3) + \sum_{i,j} h_{ij}^2$, and note that the proof that M is totally geodesic now follows as in the previous paragraph (without the assumption of finite total curvature). From the previous proof we also get that

$$\mathrm{Ric}(e_3) = K_{13} + K_{23} = 0 \quad \text{on} \quad M.$$

Since

$$\mathrm{Ric}(e_1) = K_{12} + K_{13} \geqq 0 \quad \text{and} \quad \mathrm{Ric}(e_2) = K_{12} + K_{23} \geqq 0,$$

we have

$$\mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) = 2K_{12} = 2K \geqq 0.$$

Thus the Gauss curvature of M is non-negative and, since M is a cylinder, we have $K \equiv 0$. This completes the proof of Theorem 3.

As a final result we note that one can intrinsically characterize the metrics on the plane and the sphere which can occur as complete stable surfaces in complete three-manifolds of zero scalar curvature.

THEOREM 4. (i) *A metric on S^2 can be realized on a stable minimal immersion of S^2 into a compact scalar flat three-manifold if and only if the first eigenvalue of $\Delta - K$ is non-negative.*

(ii) *A complete conformal metric on $\mathbb{C}$ can be realized on a stable minimal immersion of $\mathbb{C}$ into a complete scalar flat three-manifold if and only if there is a function $g > 0$ satisfying $\Delta g - Kg = 0$ on $\mathbb{C}$.*

Proof: The proof of (i) is similar to that of (ii) except that it is made easier by the compactness of S^2; hence we give only the proof of (ii). If a metric on $\mathbb{C}$ is realized on a stable immersion into a scalar flat three-manifold, then the stability operator $\Delta - K + \frac{1}{2}\sum_{i,j} h_{ij}^2$ has positive first eigenvalue on any bounded domain in $\mathbb{C}$. By (1) we see that this implies that the first eigenvalue of $\Delta - K$ is also positive on each bounded domain. Thus by Theorem 1 there is a positive g satisfying $\Delta g - Kg = 0$.

Conversely, if we have a complete metric ds^2 on $\mathbb{C}$ and a $g > 0$ satisfying $\Delta g - Kg = 0$ on $\mathbb{C}$, we define the three-dimensional manifold $N = \mathbb{C} \times S^1$, and we give N the metric

$$\widetilde{ds}^2 = ds^2 + g^2\, dt^2,$$

where t is a coordinate on S^1. Direct calculation shows that the scalar curvature of $\widetilde{ds}^2$ is

$$S = K - \frac{\Delta g}{g},$$

where Δ is taken with respect to ds^2. We conclude that $S \equiv 0$ on N. The completeness of the metric $\widetilde{ds}^2$ follows directly from that of ds^2. This completes the proof of Theorem 4.

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