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MOTION OF LEVEL SETS BY MEAN CURVATURE. I

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Abstract

We construct a unique weak solution of the nonlinear PDE which asserts each level set evolves in time according to its mean curvature. This weak solution allows us then to define for any compact set Γ_0 a unique generalized motion by mean curvature, existing for all time. We investigate the various geometric properties and pathologies of this evolution.

1. Introduction

We set forth in this paper rigorous justification of a new approach for defining and then investigating the evolution of a hypersurface in $\mathbb{R}^n$ moving according to its mean curvature. This problem has been long studied using parametric methods of differential geometry (see, for instance, Gage [15], [16], Gage-Hamilton [17], Grayson [19], Huisken [23], Ecker-Huisken [10], etc.). In this classical setup, we are given at time 0 a smooth hypersurface Γ_0 which is, say, the connected boundary of a bounded open subset of $\mathbb{R}^n$. As time progresses we allow the surface to evolve, by moving each point at a velocity equal to $(n-1)$ times the mean curvature vector at that point. Assuming this evolution is smooth, we define thereby for each $t > 0$ a new hypersurface Γ_t. The primary problem is then to study geometric properties of $\{\Gamma_t\}_{t>0}$ in terms of Γ_0.

For the case $n = 2$ this program has been successfully carried out in great detail (see [17], [19]). For $n \geq 3$, however, it is fairly clear that even if Γ_0 is smooth, a smooth evolution as envisioned above cannot exist beyond some initial time interval. Imagine, for instance, Γ_0 to be the boundary of a "dumbbell" shaped region in $\mathbb{R}^3$, as illustrated in Figure 1 (next page).

In view of Grayson [20] and numerical calculations of Sethian [35], we expect that as time evolves, the surface will smoothly evolve (and shrink)

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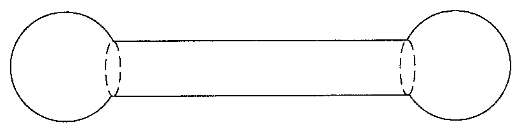

FIGURE 1

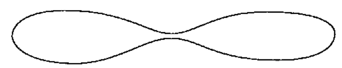

FIGURE 2

up until a critical time $t_* > 0$ when the two ends pinch off, as drawn in Figure 2.

After this time, the classical motion via mean curvature is undefined. In addition, if it were possible to define the subsequent motion in some reasonable way, we expect Γ_t for $t > t_*$ to comprise two pieces which pull apart at time t_*. If this were so, then Γ_t would have changed topological type. This possibility suggests inherent problems in the classical differential geometric approach of regarding Γ_0 as a parametrized surface: the parametrization will in general develop singularities.

What is needed is an alternative description of the evolution for all times $t > 0$, sufficiently general as to allow for the possible onset of singularities and attendant topological complications. To our knowledge there have been two different such undertakings, by Brakke [5] and by Osher-Sethian [33] (see also the note at the end of this section). Brakke [5] recasts the mean curvature motion problem (even in arbitrary codimension) into the setting of varifold theory from geometric measure theory (cf. Allard [2]). Brakke defines and then constructs an appropriate generalized varifold solution, which is defined for all time (although it may vanish after a finite time). He then deduces many geometric properties and under an additional density assumption establishes partial regularity. The principal drawback seems to be the lack of any uniqueness assertion.

A completely different viewpoint is to be found in the paper [33] by Osher and Sethian. Their approach, recast slightly, is this: given the initial hypersurface Γ_0 as above, select some continuous function $g: \mathbb{R}^n \to \mathbb{R}$ so that

$$\Gamma_0 = \{x \in \mathbb{R}^n | g(x) = 0\}. \tag{1.1}$$

Consider then the parabolic PDE

$$u_t = (\delta_{ij} - u_{x_i} u_{x_j} / |Du|^2) u_{x_i x_j} \quad \text{in } \mathbb{R}^n \times [0, \infty), \tag{1.2}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{1.3}$$

for the unknown $u = u(x, t)$, $(x \in \mathbb{R}^n,\ t \geq 0)$. Now the PDE (1.2) says that each *level set of u evolves according to its mean curvature*, at least in regions where u is smooth and its spacial gradient Du does not vanish. Consequently, focusing our attention on the set $\{u = 0\}$, it seems reasonable in view of (1.1), (1.2) to *define*

$$\Gamma_t \equiv \{x \in \mathbb{R}^n | u(x, t) = 0\} \tag{1.4}$$

for each time $t > 0$. Osher and Sethian [33] and Sethian [35] introduce various techniques to study (1.2) and related PDE's numerically, thereby to track computationally the evolution of Γ_0 into Γ_t $(t \geq 0)$. (Notice by the way that our utilizing (1.1)–(1.3) amounts in the language of fluid mechanics to adopting an Eulerian viewpoint, as opposed to the Lagrangian, parametric viewpoint of classical differential geometry.)

Our purpose here is to provide theoretical justification for this approach. The undertaking is analytically subtle, principally because the mean curvature evolution equation (1.2) is nonlinear, degenerate, and indeed even undefined at points where $Du = 0$. In addition, it is not so clear that our definition (1.3) is independent of the choice of initial function g verifying (1.1). We will resolve these problems by introducing an appropriate definition of a weak solution for (1.2), inspired by the notion of so-called "viscosity solutions" of nonlinear PDE as in Evans [12], Crandall-Lions [9], Crandall-Evans-Lions [8], Lions [32], Jensen [25], and Ishii [24]. We then prove that there exists a unique weak solution of (1.2), and, further, that definition (1.3) is then independent of the choice of initial function g satisfying (1.1). We additionally check that $\{\Gamma_t\}_{t\geq 0}$ so defined agrees with the classical notion of motion via mean curvature, over any time interval for which the latter exists. Finally we employ the PDE (1.2) to deduce assorted geometric properties of $\{\Gamma_t\}_{t\geq 0}$.

The main theoretical advantage of (1.1)–(1.3) as compared with Brakke's varifold methods seems to us to be the following uniqueness assertion: the set Γ_t is unambiguously defined by (1.3) once we have a uniqueness assertion for the PDE (1.2). The primary disadvantage is that our techniques work only in codimension one.

In a companion paper [14] we give a new proof of short time existence for classical motion by mean curvature by studying the PDE solved by the distance function. We also hope to establish in a forthcoming paper a partial regularity theorem for $\{\Gamma_t\}_{t\geq 0}$.

Our paper is organized as follows. In §2 we motivate and introduce our definition of weak solution for (1.2) and in §3 we prove the uniqueness of a weak solution. §4 establishes the existence of a weak solution to (1.2). In §5 we verify the independence of the definition (1.3) on the choice of g. §6 contains a consistency check that the definition (1.3) agrees with the classical motion by mean curvature, if and so long as the latter exists. §§7 and 8 contain various geometric assertions, examples of pathologies, and conjectures.

After this work was completed, we learned of the recent paper of Chen, Giga, and Goto [7], which announces results very similar to ours, especially the existence of a unique weak solution of the PDE (1.2), (1.3). Their work includes as well generalizations to other geometric problems.

Another new paper concerning curvature and viscosity solutions is Trudinger [36].

2. Definition and elementary properties of weak solutions

2.1. Heuristics. We start with a formal derivation of the mean curvature evolution PDE (1.2). For this, suppose temporarily $u = u(x, t)$ is a smooth function whose spatial gradient $Du = (u_{x_1}, \cdots, u_{x_n})$ does not vanish in some open region O of $\mathbb{R}^n \times (0, \infty)$. Assume further that each level set of u smoothly evolves according to its mean curvature, as described in §1. We focus our attention onto any one such level set, and for definiteness consider the zero sets

$$\Gamma_t \equiv \{x \in \mathbb{R}^n | u(x, t) = 0\} \qquad (t \geq 0). \tag{2.1}$$

Let $\nu = \nu(x, t)$ be a smooth unit normal vector field to $\{\Gamma_t\}_{t\geq 0}$ in O. Then

$$-\frac{1}{n-1}\operatorname{div}(\nu)\nu$$

is the mean curvature vector field. Thus if we fix $t \geq 0, x \in \Gamma_t \cap O$, the point x evolves according to the nonautonomous ODE

$$\begin{cases} \dot{x}(s) = -[\operatorname{div}(\nu)\nu](x(s), s) & (s > t), \\ x(t) = x. \end{cases} \tag{2.2}$$

As $x(s) \in \Gamma_s$ $(s \geq t)$, we have $u(x(s), s) = 0$ $(s > t)$; and so

$$0 = \frac{d}{ds}u(x(s), s) = -[(Du \cdot \nu)\operatorname{div}(\nu)](x(s), s) + u_t(x(s), s).$$

Setting $s = t$, we discover

$$u_t = (Du \cdot \nu)\operatorname{div}(\nu) \quad \text{at } (x, t).$$

Choosing then

$$\nu \equiv \frac{Du}{|Du|} \tag{2.3}$$

it follows that

$$u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right) u_{x_i x_j} \quad \text{at } (x, t). \tag{2.4}$$

Similar reasoning demonstrates this PDE to hold throughout the region O.

Now, conversely, assume u is a smooth solution of (2.4) in some region O with Du nonvanishing. Fix $t > 0, x \in \Gamma_t \cap O$ and solve then the ODE (2.2), (2.3). Since u solves (2.4), we deduce as above

$$u(x(s), s) = 0 \qquad (s > t).$$

Consequently the zero sets, and similarly all the level sets, of u evolve in O according to their mean curvatures.

Since the motion of any level set thus depends only upon its own geometry, and not that of any other level set, our PDE (2.4) should be invariant under an arbitrary relabelling of these sets. Thus if $\Psi: \mathbb{R} \to \mathbb{R}$ is smooth, we expect that $v \equiv \Psi(u)$ will also be a solution of (2.4) in the region O. A direct calculation verifies this in the regions where $Dv \neq 0$. Hence we see that an *arbitrary function of a solution is still a solution*; this is in strong contrast to the situation for uniformly parabolic PDE's. Indeed, we may informally interpret (2.4) as being somehow "uniformly parabolic along each level set", but as being also "totally degenerate across different level sets".

2.2. Weak solutions. The foregoing heuristics done with, we turn now to the full mean curvature evolution equation:

$$u_t = (\delta_{ij} - u_{x_i}u_{x_j}/|Du|^2)u_{x_i x_j} \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{2.5}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{2.6}$$

the function $g: \mathbb{R}^n \to \mathbb{R}$ being given. We want to define a notion of weak solution to (2.5). Since, however, the right-hand side of the PDE cannot be put into divergence form, we are not able to define a weak solution by means of formal integration by parts of derivatives onto a smooth test function (as for instance in Bombieri, De Giorgi, Giusti [4, §1]). We will instead follow Evans [12], Lions [32], Jensen [25], etc. and define our weak solution in terms of pointwise behavior with respect to a smooth test function. The primary difficulty will be to modify extant theory to cover the possibility that Du may vanish.

Definition 2.1. A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a *weak subsolution* of (2.5) provided that if

(2.7) $u - \phi$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$

for each $\phi \in C^\infty(\mathbb{R}^{n+1})$, then

$$(2.8) \quad \begin{cases} \phi_t \leq (\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2)\phi_{x_ix_j} & \text{at } (x_0, t_0) \\ \text{if } D\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$(2.9) \quad \begin{cases} \phi_t \leq (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j} \quad \text{at } (x_0, t_0) \\ \text{for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \leq 1, \text{ if } D\phi(x_0, t_0) = 0. \end{cases}$$

Definition 2.2. A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a *weak supersolution* of (2.5) provided that if

(2.10) $u - \phi$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$

for each $\phi \in C^\infty(\mathbb{R}^{n+1})$, then

$$(2.11) \quad \begin{cases} \phi_t \geq (\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2)\phi_{x_ix_j} & \text{at } (x_0, t_0) \\ \text{if } D\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$(2.12) \quad \begin{cases} \phi_t \geq (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j} \quad \text{at } (x_0, t_0) \\ \text{for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \neq 1, \text{ if } D\phi(x_0, t_0) = 0. \end{cases}$$

Definition 2.3. A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a *weak solution* of (2.5) provided u is both a weak subsolution and a weak supersolution.

As preliminary motivation for these definitions, suppose u is a smooth function on $\mathbb{R}^n \times (0, \infty)$ satisfying

$$u_t \leq (\delta_{ij} - u_{x_i}u_{x_j}/|Du|^2)u_{x_ix_j}$$

wherever $Du \neq 0$. Our function u is thus a classical subsolution of (2.5) on $\{Du \neq 0\}$. Suppose now $Du(x_0, t_0) = 0$. Assume additionally there are points $(x_k, t_k) \to (x_0, t_0)$ for which $Du(x_k, t_k) \neq 0$ $(k = 1, 2, \ldots)$. Then

$$u_t \leq (\delta_{ij} - \eta_j^k\eta_j^k)u_{x_ix_j} \quad \text{at } (x_k, t_k)$$

for $\eta^k \equiv Du(x_k, t_k)/|Du(x_k, t_k)|$. Since $|\eta^k| = 1$ $(k = 1, 2, \ldots)$ we may as necessary pass to a subsequence so that $\eta^k \to \eta$ in $\mathbb{R}^n$, $|\eta| = 1$. Passing to limits above, we find

$$u_t \leq (\delta_{ij} - \eta_i\eta_j)u_{x_ix_j} \quad \text{at } (x_0, t_0).$$

If, on the other hand, there do not exist such points $\{(x_k, t_k)\}_{k=1}^\infty$, then $Du = 0$ near (x_0, t_0), and so $D^2u = 0$ and u is a function of t only, near (x_0, t_0). Moving to the edge of the set $\{Du = 0\}$, we see that u is a nonincreasing function of t. Thus

$$u_t \leq (\delta_{ij} - \eta_i\eta_j)u_{x_ix_j} \quad \text{at } (x_0, t_0)$$

for any $\eta \in \mathbb{R}^n$.

Further motivation for our definition of weak solution, and, in particular, an explanation as to why we assume only $|\eta| \leq 1$ in (2.9), (2.12), will be found in §2.4.

2.3. An equivalent definition. It will be convenient to have at hand certain alternative definitions. We write $z = (x, t)$, $z_0 = (x_0, t_0)$ and below implicitly sum i, j from 1 to n.

Definition 2.4. A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a weak subsolution of (2.5) if whenever $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and

$$(2.13) \quad \begin{aligned} u(x, t) \leq {} & u(x_0, t_0) + p \cdot (x - x_0) + q(t - t_0) \\ & + \tfrac{1}{2}(z - z_0)^T R(z - z_0) + o(|z - z_0|^2) \quad \text{as } z \to z_0 \end{aligned}$$

for some $p \in \mathbb{R}^n$, $q \in \mathbb{R}$, $R = ((r_{ij})) \in S^{n+1 \times n+1}$, then

$$(2.14) \quad q \leq (\delta_{ij} - p_ip_j/|p|^2)r_{ij} \quad \text{if } p \neq 0$$

and

$$(2.15) \quad q \leq (\delta_{ij} - \eta_i\eta_j)r_{ij} \quad \text{for some } |\eta| \leq 1, \text{ if } p = 0.$$

Definition 2.5. A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a weak supersolution of (2.5) if whenever $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and

$$(2.16) \quad \begin{aligned} u(x, t) \geq {} & u(x_0, t_0) + p \cdot (x - x_0) + q(t - t_0) \\ & + \tfrac{1}{2}(z - z_0)^T R(z - z_0) + o(|z - z_0|^2) \quad \text{as } z \to z_0 \end{aligned}$$

for some $p \in \mathbb{R}^n$, $q \in \mathbb{R}$, $R = ((r_{ij})) \in S^{n+1 \times n+1}$, then

$$(2.17) \quad q \geq (\delta_{ij} - p_ip_j/|p|^2)r_{ij} \quad \text{if } p \neq 0$$

and

$$q \geq (\delta_{ij} - \eta_i \eta_j) r_{ij} \quad \text{for some } |\eta| \leq 1, \text{ if } p = 0. \tag{2.18}$$

Theorem 2.6. *Definitions* 2.1 *and* 2.4 *are equivalent, and Definitions* 2.2 *and* 2.5 *are equivalent.*

This follows as in, for instance, Jensen [25], Ishii [24].

2.4. Properties of weak solutions.

Theorem 2.7. (i) *Assume* u_k *is a weak solution of* (2.5) *for* $k = 1, 2, \ldots$ *and* $u_k \to u$ *boundedly and locally uniformly on* $\mathbb{R}^n \times [0, \infty)$. *Then* u *is a weak solution.*

(ii) *An analogous assertion holds for weak subsolutions and supersolutions.*

Proof. 1. Choose $\phi \in C^\infty(\mathbb{R}^{n+1})$ and suppose first $u - \phi$ has a *strict* local maximum at some point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. As $u^k \to u$ uniformly near (x_0, t_0),

(2.19) $u_k - \phi$ has a local maximum at a point (x_k, t_k) $(k = 1, 2, \ldots)$

with

$$(x_k, t_k) \to (x_0, t_0) \quad \text{as } k \to \infty. \tag{2.20}$$

Since u_k is a weak solution, we have either

$$\phi_t \leq (\delta_{ij} - \phi_{x_i}\phi_{x_i}/|D\phi|^2)\phi_{x_i x_j} \quad \text{at } (x_k, t_k) \tag{2.21}$$

if $D\phi(x_k, t_k) \neq 0$, or

$$\phi_t \leq (\delta_{ij} - \eta_i^k \eta_j^k)\phi_{x_i x_j} \quad \text{at } (x_k, t_k) \tag{2.22}$$

for some $\eta^k \in \mathbb{R}^n$ with $|\eta^k| \leq 1$, if $D\phi(x_k, t_k) = 0$.

2. Assume first $D\phi(x_0, t_0) \neq 0$. Then $D\phi(x_k, t_k) \neq 0$ for all large enough k. Hence we may pass to limits in the equalities (2.21) to discover

$$\phi_t \leq (\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2) \quad \text{at } (x_0, t_0). \tag{2.23}$$

3. Next, suppose $D\phi(x_0, t_0) = 0$. We set

$$\xi^k \equiv \begin{cases} (D\phi/|D\phi|)(x_k, t_k) & \text{if } D\phi(x_k, t_k) \neq 0, \\ \eta^k & \text{if } D\phi(x_k, t_k) = 0. \end{cases} \tag{2.24}$$

Passing if necessary to a subsequence we may assume $\xi^k \to \eta$. Then $|\eta| \leq 1$. Utilizing now (2.22), we deduce as well

$$\phi_t \leq (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j} \quad \text{at } (x_0, t_0). \tag{2.25}$$

4. If $u - \phi$ has only a local maximum at (x_0, t_0) we apply the above argument to

$$\psi(x, t) \equiv \phi(x, t) + |x - x_0|^4 + (t - t_0)^4,$$

so that $u - \psi$ has a strict local maximum at (x_0, t_0). Hence u is a weak subsolution. Similar reasoning verifies that u is a weak supersolution as well.

Theorem 2.8. *Assume* u *is a weak solution of* (2.5) *and* $\Psi: \mathbb{R} \to \mathbb{R}$ *is continuous. Then* $v \equiv \Psi(u)$ *is a weak solution.*

Proof. 1. Assume first Ψ is smooth, with

$$\Psi' > 0 \quad \text{on } \mathbb{R}. \tag{2.26}$$

Let $\phi \in C^\infty(\mathbb{R}^{n+1})$ and suppose $v - \phi$ has a local maximum at (x_0, t_0). Adding as necessary a constant to ϕ, we may assume

$$\begin{cases} v(x_0, t_0) = \phi(x_0, t_0), \\ v(x, t) \leq \phi(x, t) \quad \text{for all } (x, t) \text{ near } (x_0, t_0). \end{cases} \tag{2.27}$$

In view of (2.26), $\Phi \equiv \Psi^{-1}$ is defined and smooth near $u(x_0, t_0)$, with

$$\Phi' > 0. \tag{2.28}$$

From (2.27) therefore we see

$$\begin{cases} u(x_0, t_0) = \psi(x_0, t_0), \\ u(x, t) \leq \psi(x, t) \quad \text{for all } (x, t) \text{ near } (x_0, t_0), \end{cases} \tag{2.29}$$

where

$$\psi \equiv \Phi(\phi). \tag{2.30}$$

2. Since u is a weak solution we conclude

$$\psi_t \leq (\delta_{ij} - \psi_{x_i}\psi_{x_j}/|D\psi|^2)\psi_{x_i x_j} \quad \text{at } (x_0, t_0) \tag{2.31}$$

if $D\psi(x_0, t_0) \neq 0$, and

$$\psi_t \leq (\delta_{ij} - \eta_i \eta_j)\psi_{x_i x_j} \quad \text{at } (x_0, t_0) \tag{2.32}$$

for some $|\eta| \leq 1$, if $D\psi(x_0, t_0) = 0$. Now $D\phi(x_0, t_0) = 0$ if and only if $D\psi(x_0, t_0) = 0$. Consequently (2.31) is obtained if $D\phi(x_0, t_0) \neq 0$; in which case we substitute (2.30) to compute

$$\Phi'\phi_t \leq \left(\delta_{ij} - \frac{(\Phi')^2\phi_{x_i}\phi_{x_j}}{(\Phi')^2|D\phi|^2}\right)(\Phi'\phi_{x_i x_j} + \Phi''\phi_{x_i}\phi_{x_j}) \quad \text{at } (x_0, t_0).$$

Since $\Phi' > 0$, we simplify and obtain

$$\phi_t \leq (\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2)\phi_{x_ix_j} \quad \text{at } (x_0, t_0). \tag{2.33}$$

Suppose on the other hand $D\phi(x_0, t_0) = 0$. Then (2.32) is valid for some $|\eta| \leq 1$. We substitute (2.30) and compute

$$\Phi'\phi_t \leq (\delta_{ij} - \eta_i\eta_j)(\Phi'\phi_{x_ix_j} + \Phi''\phi_{x_i}\phi_{x_j}) \quad \text{at } (x_0, t_0).$$

Since $D\phi = 0$, the term involving Φ'' is zero. Thus

$$\phi_t \leq (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j} \quad \text{at } (x_0, t_0). \tag{2.34}$$

We similarly have the opposite inequalities to (2.33), (2.34) should $v - \phi$ have a local minimum at (x_0, t_0).

3. Now assume instead of (2.20) that

$$\Psi' < 0 \quad \text{on } \mathbb{R}. \tag{2.35}$$

Then $\Phi' < 0$ on $\mathbb{R}$ as well. Thus (2.27) now implies

$$\begin{cases} u(x_0, t_0) = \psi(x_0, t_0), \\ u(x, t) \geq \psi(x, t) \quad \text{for all } (x, t) \text{ near } (x_0, t_0). \end{cases}$$

Since u is a weak solution either

$$\psi_t \geq (\delta_{ij} - \psi_{x_i}\psi_{x_j}/|D\psi|^2)\psi_{x_ix_j} \quad \text{at } (x_0, t_0)$$

if $D\psi(x_0, t_0) \neq 0$, or

$$\psi_t \geq (\delta_{ij} - \eta_i\eta_j)\psi_{x_ix_j} \quad \text{at } (x_0, t_0)$$

for some $|\eta| \leq 1$, if $D\psi(x_0, t_0) = 0$. Since now $\Phi' < 0$, we deduce as above either (2.33) or (2.34).

4. We have so far shown that $v = \Psi(u)$ is a weak solution provided Ψ is smooth, with $\Psi' \neq 0$. Approximating and using Theorem 2.7 we draw the same conclusion if $\Psi' \geq 0$ or $\Psi' \leq 0$ on $\mathbb{R}$.

5. Next assume Ψ is smooth and there exist finitely many points $-\infty = a_0 < a_1 < a_2 < \cdots < a_m < a_{m+1} = +\infty$ such that

(2.36) Ψ is monotone on the intervals (a_j, a_{j+1}) $(j = 0, \cdots, m)$

and

(2.37) Ψ is constant on the intervals $(a_j - \sigma, a_j + \sigma)$ $(j = 1, \cdots, m)$

for some $\sigma > 0$.

Suppose $v - \phi$ has a maximum at (x_0, t_0). Then

$$u(x_0, t_0) \in (a_j - \sigma/2, a_{j+1} + \sigma/2) \quad \text{for some } j \in \{0, \cdots, m\}.$$

As Ψ is monotone on $(a_j - \sigma, a_{j+1} + \sigma)$ and u is continuous, we can apply steps 1–4 in some neighborhood of (x_0, t_0) to deduce (2.33) or (2.34). The reverse inequalities are similarly obtained if $v - \phi$ has a minimum.

6. Finally suppose only that Ψ is continuous. We construct a sequence of smooth functions $\{\Psi^k\}_{k=1}^{\infty}$ each verifying the structural assumptions (2.36), (2.37) so that $\Psi^k \to \Psi$ uniformly on $[-\|u\|_{L^\infty}, \|u\|_{L^\infty}]$. Hence

$$v^k = \Psi^k(u) \to v \equiv \Psi(u)$$

bounded and uniformly. Then Theorem 2.7 asserts v to be a weak solution.

3. Uniqueness and comparison of weak solutions

3.1. Preliminaries. Our plan, as in Jensen [25] and Jensen-Lions-Souganidis [26], is to regularize using sup and inf convolutions, defined as follows. Assume $w: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is continuous and bounded. If $\epsilon > 0$, then we write

$$w^\epsilon(x, t) \equiv \sup_{\substack{y\in\mathbb{R}^n \\ s\in[0,\infty)}} \{w(y, s) - \epsilon^{-1}(|x-y|^2 + (t-s)^2)\}, \tag{3.1}$$

$$w_\epsilon(x, t) \equiv \inf_{\substack{y\in\mathbb{R}^n \\ s\in[0,\infty)}} \{w(y, s) + \epsilon^{-1}(|x-y|^2 + (t-s)^2)\}, \tag{3.2}$$

for $x \in \mathbb{R}^n$, $t \in [0, \infty)$. Note that since w is continuous and bounded, the "sup" and "inf" above can be replaced by "max" and "min".

Lemma 3.1 (Properties of sup and inf convolutions). *There exist constants A, B, C, depending only on $\|w\|_{L^\infty(R^n\times[0,\infty))}$, such that for $\epsilon > 0$ the following hold:*

(i) $w_\epsilon \leq w \leq w^\epsilon$ *on* $\mathbb{R}^n \times [0, \infty)$.

(ii) $\|w^\epsilon, w_\epsilon\|_{L^\infty(R^n\times[0,\infty))} \leq A$.

(iii) *If* $y \in \mathbb{R}^n$, $s \in [0, \infty)$, *and* $w^\epsilon(x, t) = w(y, x) - \epsilon^{-1}(|x-y|^2 + (t-s)^2)$, *then*

$$|x-y|, |t-s| \leq C\epsilon^{1/2} \equiv \sigma(\epsilon). \tag{3.3}$$

A similar assertion holds for w_ϵ.

(iv) $w^\epsilon, w_\epsilon \to w$ *as* $\epsilon \to 0^+$, *uniformly on compact subsets of* $\mathbb{R}^n \times [0, \infty)$.

(v) $\mathrm{Lip}(w^\epsilon), \mathrm{Lip}(w_\epsilon) \leq B/\epsilon$.

(vi) *The mapping*

$$(x, t) \mapsto w^\epsilon(x, t) + \epsilon^{-1}(|x|^2 + t^2)$$

is convex, and the mapping

$$(x, t) \mapsto w_\epsilon(x, t) - \epsilon^{-1}(|x|^2 + t^2)$$

is concave.

(vii) *Assume* w *is a weak solution of* (2.5) *in* $\mathbb{R}^n \times (0, \infty)$. *Then* w^ϵ *is a weak subsolution on* $\mathbb{R}^n \times (\sigma(\epsilon), \infty)$. *Similarly, if* w *is a weak supersolution of* (2.5), w_ϵ *is a weak supersolution.*

(viii) *The function* w^ϵ *is twice differentiable a.e. and satisfies*

$$w^\epsilon_t \le (\delta_{ij} - w^\epsilon_{x_i} w^\epsilon_{x_j}/|Dw^\epsilon|^2) w^\epsilon_{x_i x_j} \tag{3.4}$$

at each point of twice differentiability in $\mathbb{R}^n \times (\sigma(\epsilon), \infty)$, *where* $Dw^\epsilon \ne 0$. *Similarly,* w_ϵ *is twice differentiable a.e. and satisfies*

$$w_{\epsilon t} \ge (\delta_{ij} - w_{\epsilon x_i} w_{\epsilon x_j}/|Dw_\epsilon|^2) w_{\epsilon x_i x_j} \tag{3.5}$$

at each point of twice differentiability in $\mathbb{R}^n \times (\sigma(\epsilon), \infty)$, *where* $Dw_\epsilon \ne 0$.

Proof. 1. Assertions (i) and (ii) are clear from the definitions, for $A = \|w\|_{L^\infty(R^n \times [0, \infty))}$. Statement (iii) follows from (ii), and then (iv) is a consequence of the uniform continuity of w on compact sets. In light of estimate (3.3) we have (v) as well.

2. For each $y \in \mathbb{R}^n$, $s \in [0, \infty)$, the mapping

$$(x, t) \mapsto w(y, s) - \epsilon^{-1}(|x - y|^2 + (t - s)^2) + \epsilon^{-1}(|x|^2 + t^2)$$

is affine. Consequently

$$\begin{aligned}(x, t) \mapsto &\sup_{\substack{y \in \mathbb{R}^n \\ s \in [0, \infty)}} [w(y, s) - \epsilon^{-1}(|x - y|^2 + (t - s)^2) + \epsilon^{-1}(|x|^2 + t^2)] \\ &= w^\epsilon(x, t) + \epsilon^{-1}(|x|^2 + t^2)\end{aligned}$$

is convex, and (v) is proved.

3. Assume $\phi \in C^\infty(\mathbb{R}^{n+1})$ and $w^\epsilon - \phi$ has a local maximum at a point (x_0, t_0), with $t_0 > \sigma(\epsilon)$. We then employ (3.3) to choose $(y_0, s_0) \in \mathbb{R}^n \times (0, \infty)$ so that

$$w^\epsilon(x_0, t_0) = w(y_0, s_0) - \epsilon^{-1}(|x_0 - y_0|^2 + (t_0 - s_0)^2).$$

Set

$$\psi(x, t) \equiv \phi(x + x_0 - y_0, t + t_0 - s_0). \tag{3.6}$$

Since $w^\epsilon - \phi$ has a local maximum at (x_0, t_0) we compute

$$\begin{aligned} w(y_0, s_0) - \epsilon^{-1}(|x_0 - y_0|^2 + (t_0 - s_0)^2) - \phi(x_0, t_0) \\ = w^\epsilon(x_0, t_0) - \phi(x_0, t_0) \ge w^\epsilon(x, t) - \phi(x, t) \\ \ge w(y, s) - \epsilon^{-1}(|x - y|^2 + (t - s)^2) - \phi(x, t)\end{aligned}$$

for all (x, t) near (x_0, t_0) and all $(y, s) \in \mathbb{R}^n \times [0, \infty)$. Fix (y, s) close to (y_0, s_0) and set $x = y + x_0 - y_0$, $t = s + t_0 - s_0$ as above, to discover

$$w(y_0, s_0) - \phi(x_0, t_0) \ge w(y, s) - \phi(y + x_0 - y_0, s + t_0 - s_0).$$

Using (3.6) we rewrite this as

$$w(y_0, s_0) - \psi(y_0, s_0) \ge w(y, s) - \psi(y, s)$$

for all (y, s) near (y_0, s_0). Hence $w - \phi$ has a local maximum at (y_0, s_0) and thus

$$\psi_t \le (\delta_{ij} - \psi_{x_i}\psi_{x_i}/|D\psi|^2)\psi_{x_i x_j} \quad \text{at } (y_0, s_0)$$

if $D\psi(y_0, s_0) \ne 0$, and

$$\psi_t \le (\delta_{ij} - \eta_i \eta_j)\psi_{x_j x_j} \quad \text{at } (y_0, s_0)$$

for some $|\eta| \le 1$, if $D\psi(y_0, s_0) = 0$. Since

$$\begin{gathered} D\psi(y_0, s_0) = D\phi(x_0, t_0), \ \psi_t(y_0, s_0) = \phi_t(x_0, t_0), \\ D^2\phi(y_0, s_0) = D^2\phi(x_0, t_0), \end{gathered}$$

we immediately obtain

$$\phi_t \le (\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2)\phi_{x_i x_j} \quad \text{at } (x_0, t_0)$$

or

$$\phi_t \le (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j} \quad \text{at } (x_0, t_0)$$

according as $D\phi(x_0, t_0) = 0$ or not, and (vii) is proved.

4. Owing to (vi), $w^\epsilon(x, t) + \epsilon^{-1}(|x|^2 + t^2)$ is convex in (x, t) and so is twice differentiable a.e. according to a theorem of Alexandroff (see, e.g., Krylov [30, Appendix 2]). Thus w^ϵ is twice differentiable a.e. In view of (vii) and Theorem 2.6, (3.4) holds at points of twice differentiability, where $Dw^\epsilon \ne 0$. Hence (viii) is proved.

3.2. Comparison principle, uniqueness. We establish now a comparison assertion for weak solutions of our mean curvature evolution PDE. Many of the key technical devices in the proof are taken from Jensen [25] and Ishii [24].

Theorem 3.2. *Assume that u is a weak subsolution and v is a weak supersolution of* (2.5). *Suppose further*

$$u \le v \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

Finally assume

$$\left\{ \begin{array}{l} u \text{ and } v \text{ are constant, with } u \le v, \\ \text{on } \mathbb{R}^n \times [0, \infty) \cap \{|x| + t \ge R\} \end{array} \right. \tag{3.8}$$

for some constant $R \ge 0$. Then

$$u \le v \quad \text{on } \mathbb{R}^n \times [0, \infty). \tag{3.9}$$

In particular, a weak solution of (2.5), (2.6) *is unique.*

Proof. 1. Should (3.9) fail, then

$$\max_{(x,t)\in R^n\times[0,\infty)} (u - v) \equiv a > 0;$$

and so for $\alpha > 0$ small enough,

$$\max_{(x,t)\in R^n\times[0,\infty)} (u - v - \alpha t) \ge a/2 > 0. \tag{3.10}$$

According to (3.8) we have

$$u^\epsilon = u, \quad v_\epsilon = v \quad \text{on } \{|x| + t \ge 2R\} \tag{3.11}$$

for all small $\epsilon > 0$. Note further $u^\epsilon \to u$ and $v_\epsilon \to v$ uniformly. Consequently if we fix $\epsilon > 0$ small enough,

$$\max_{(x,t)\in R^n\times[0,\infty)} (u^\epsilon - v_\epsilon - \alpha t) \ge a/4 > 0. \tag{3.12}$$

2. Given $\delta > 0$ define for $x, y \in \mathbb{R}^n$ and $t, t+s \in [0, \infty)$

$$\Phi(x, y, t, s) \equiv u^\epsilon(x+y, t+s) - v_\epsilon(x, t) - \alpha t - \delta^{-1}(|y|^4 + s^4). \tag{3.13}$$

Owing to (3.12) we see

$$\max_{(x,t),(x+y,t+s)\in R^n\times[0,\infty)} \Phi \ge a/4 > 0. \tag{3.14}$$

Choose now $(x_1, t_1), (x_1 + y_1, t_1 + s_1) \in \mathbb{R}^n \times [0, \infty)$ so that

$$\Phi(x_1, y_1, t_1, s_1) = \max_{(x,t),(x+y,t+s)\in R^n\times[0,\infty)} \Phi. \tag{3.15}$$

Note in view of (3.11), (3.13) and Lemma 3.1(ii) that such points exist. Since $\Phi(x_1, y_1, t_1, s_1) > 0$, (3.13) implies

$$|y_1|, |s_1| \le C\delta^{1/4}. \tag{3.16}$$

3. We *claim* next that if $\epsilon, \delta > 0$ are fixed small enough, we have

$$t_1, t_1 + s > \sigma(\epsilon), \tag{3.17}$$

with $\sigma(\epsilon)$ defined in (3.3). Indeed if $t_1 \le \sigma(\epsilon)$, then

$$\begin{aligned} a/4 &\le \Phi(x_1, y_1, t_1, s_1) \\ &\le u^\epsilon(x_1 + y_1, t_1 + s_1) - v_\epsilon(x_1, t_1) \\ &= u(x_1 + y_1, t_1 + s_1) - v(x_1, t_1) + o(1) && \text{as } \epsilon \to 0 \\ &= u(x_1 + y_1, s_1) - v(x_1, 0) + o(1) && \text{as } \epsilon \to 0 \\ &= u(x_1, 0) - v(x_1, 0) + o(1) && \text{as } \epsilon, \delta \to 0 \\ &\le o(1) && \text{as } \epsilon, \delta \to 0, \end{aligned}$$

where we employed Lemma 3.1(ii), (3.16), (3.7), and the continuity of u, v. This is a contradiction for $\epsilon, \delta > 0$ small enough; whence $t_1 > \sigma(\epsilon)$. Owing to (3.16) we may as necessary adjust δ smaller to ensure (3.17). *Hereafter in the proof, $\alpha, \epsilon, \delta > 0$ are fixed.*

According to Lemma 3.1(vii),

$$u^\epsilon \text{ is a weak subsolution of (2.5) near } (x_1 + y_1, t_1 + s_1) \tag{3.18}$$

and

$$v_\epsilon \text{ is a weak supersolution of (2.5) near } (x_1, t_1). \tag{3.19}$$

4. We now demonstrate

$$y_1 \ne 0. \tag{3.20}$$

Assume for contradiction that in fact $y_1 = 0$. Then (3.13), (3.15) imply

$$\begin{aligned} &u^\epsilon(x_1, t_1 + s_1) - v_\epsilon(x, t_1) - \alpha t_1 - \delta^{-1} s_1^4 \\ &\qquad \ge u^\epsilon(x + y, t + s) - v_\epsilon(x, t) - \alpha t - \delta^{-1}(|y|^4 + s^4) \end{aligned}$$

for all $(x, t), (x + y, t + s) \in \mathbb{R}^n \times [0, \infty)$. Put $x = x_1$ and $t = t_1$ as above, and simplify to obtain the inequality

$$u^\epsilon(x_1 + y, t_1 + s) \le u^\epsilon(x_1, t_1 + s_1) + \delta^{-1}|y|^4 + \delta^{-1}(s^4 - s_1^4)$$

for $(x_1 + y, t_1 + s) \in \mathbb{R}^n \times [0, \infty)$. Set $r = s - s_1$ and rewrite to find

$$\begin{aligned} u^\epsilon(x_1 + y, t_1 + s_1 + r) \le u^\epsilon(x_1, t_1 + s_1) &+ 4s_1^3 r/\delta + 6s_1^2 r^2/\delta \\ &+ O(|r|^3 + |y|^4) \quad \text{as } (y, r) \to (0, 0). \end{aligned}$$

Since u^ϵ is a weak subsolution near $(x_1+y_1, t_1+s_1) = (x_1, t_1+s_1)$, we may invoke (2.13), (2.15) with $x_0 = x_1$, $t_0 = t_1+s_1$, $p=0$, $q = 4s_1^3/\delta$, $r_{n+1,n+1} = 12s_1^2/\delta$, $r_{ij}=0$ otherwise. This gives

$$4s_1^3/\delta \le 0. \tag{3.22}$$

Now go back and insert $y = x_1 - x$ and $s = t_1+s_1-t$ into (3.21). This yields after simplifications:

$$\begin{aligned} v_\epsilon(x,t) \ge v_\epsilon(x_1,t_1) &+ (4s_1^3/\delta - \alpha)(t-t_1) - 6s_1^2(t-t_1)^2/\delta \\ &+ O(|x-x_1|^4 + |t-t_1|^3) \quad \text{as } (x,t)\to(x_1,t_1). \end{aligned}$$

Now v_ϵ is a weak supersolution near (x_1, t_1). Thus (2.16), (2.18) with $x_0 = x_1$, $t_0 = t_1$, $p=0$, $q = 4s_1^3/\delta - \alpha$, $r_{n+1,n+1} = -12s_1^2/\delta$, and $r_{ij}=0$ otherwise, imply

$$4s_1^3/\delta - \alpha \ge 0. \tag{3.23}$$

But now we have a contradiction with (3.22), since $\alpha > 0$. This establishes (3.20).

5. Note next that in general if $f\colon \mathbb{R}^m \to \mathbb{R}$ is convex, then so is the mapping $(w,z) \mapsto f(w+z)$ on $\mathbb{R}^{2m}$. Consequently Lemma 3.1(vi) asserts

$$(x,y,t,s) \mapsto u^\epsilon(x+y, t+s) + \epsilon^{-1}(|x-y|^2 + (t+s)^2)$$

is convex. As

$$(x,t) \mapsto -v_\epsilon(x,t) + \epsilon^{-1}(|x|^2 + t^2)$$

is convex as well, we see that

$$(x,y,t,s) \mapsto \Phi(x,y,t,s) + C(|x|^2 + |y|^2 + t^2 + s^2)$$

is convex near (x_1, y_1, t_1, s_1), for some sufficiently large constant $C = C(\epsilon, \delta)$. Since Φ additionally attains its maximum at (x_1, y_1, t_1, s_1) we may invoke Jensen [25]: there exist points $\{(x^k, y^k, t^k, s^k)\}_{k=1}^\infty$ such that

$$(x^k, y^k, t^k, s^k) \to (x_1, y_1, t_1, s_1), \tag{3.24}$$

$$\begin{aligned} &\Phi,\ u^\epsilon \text{ and } v_\epsilon \text{ are each twice differentiable} \\ &\quad \text{at } (x^k, y^k, t^k, s^k)\ (k=1,2,\dots), \end{aligned} \tag{3.25}$$

$$D_{x,y,t,s}\Phi(x^k, y^k, t^k, s^k) \to 0, \tag{3.26}$$

$$D^2_{x,y,t,s}\Phi(x^k, y^k, t^k, s^k) \le o(1) I_{2n+2} \quad \text{as } k\to\infty. \tag{3.27}$$

6. Using (3.13), (3.25), we see

$$\begin{aligned} D_x\Phi(x^k, y^k, t^k, s^k) &= Du^\epsilon(x^k+y^k, t^k+s^k) - Dv_\epsilon(x^k, t^k) \\ &\equiv p^k - \overline{p}^k, \end{aligned} \tag{3.28}$$

$$\begin{aligned} D_y\Phi(x^k, y^k, t^k, s^k) &= Du^\epsilon(x^k+y^k, t^k+s^k) - 4|y^k|^2 y^k/\delta \\ &= p^k - 4|y^k|^2 y^k/\delta. \end{aligned}$$

Since $y^k \to y_1$, we deduce from (3.26) that

$$p^k,\ \overline{p}^k \to 4|y_1|^2 y_1/\delta \equiv p \quad \text{in } \mathbb{R}^n. \tag{3.30}$$

Assertion (3.20) tells us $p \ne 0$ and so $p^k, \overline{p}^k \ne 0$ for large enough k.

Again employing (3.13), (3.26) we note

$$\begin{aligned} \Phi_t(x^k, y^k, t^k, s^k) &= u^\epsilon_t(x^k+y^k, t^k+s^k) - v_{\epsilon t}(x^k, t^k) - \alpha \\ &\equiv q^k - \overline{q}^k - \alpha. \end{aligned} \tag{3.31}$$

As u^ϵ and v_ϵ are Lipschitz, we may assume, upon passing to a subsequence and reindexing if necessary, that

$$q^k \to q,\ \overline{q}^k \to \overline{q} \quad \text{in } \mathbb{R}. \tag{3.32}$$

Then (3.26) and (3.31) ensure

$$q - \overline{q} = \alpha > 0. \tag{3.33}$$

7. Next, (3.13) and (3.25) imply

$$\begin{aligned} D_x^2\Phi(x^k, y^k, t^k, s^k) &= D^2u^\epsilon(x^k+y^k, t^k+s^k) - D^2v_\epsilon(x^k, t^k) \\ &\equiv R^k - \overline{R}^k. \end{aligned} \tag{3.34}$$

Now (3.27) forces

$$R^k - \overline{R}^k \le \epsilon_k I_n, \tag{3.35}$$

where $\epsilon_k \to 0$. Furthermore, Lemma 3.1(vi) shows $R^k \ge -CI_n$ and $\overline{R}^k \le CI_n$, for $C = C(\epsilon)$. Thus

$$-CI_n \le R^k \le \overline{R}^k + \epsilon_k I_n \le CI_n.$$

We may consequently suppose, passing as necessary to subsequences, that

$$R^k \to R,\ \overline{R}^k \to \overline{R} \quad \text{in } S^{n\times n}, \tag{3.36}$$

with

$$R \le \overline{R}. \tag{3.37}$$

8. Now recall (3.25) holds and $p^k \equiv Du^\epsilon(x^k + y^k, t^k + s^k)$, $\overline{p}^k \equiv Dv_\epsilon(x^k, t^k)$ are nonzero for large k. Since u^ϵ is a weak subsolution near $(x_1 + y_1, t_1 + s_1)$ and v_ϵ is a weak supersolution near (x_1, t_1), we thus have

$$q^k \le (\delta_{ij} - p_i^k p_j^k/|p^k|^2) r_{ij}^k \quad \text{and} \quad \overline{q}^k \ge (\delta_{ij} - \overline{p}_i^k \overline{p}_j^k/|\overline{p}^k|^2)\overline{r}_{ij}^k$$

for all large k. We send k to infinity, recalling (3.30), (3.32), and (3.36) to obtain

$$q \le (\delta_{ij} - p_i p_j/|p|^2) r_{ij} \quad \text{and} \quad \overline{q} \ge (\delta_{ij} - p_i p_j/|p|^2)\overline{r}_{ij},$$

and, by subtracting,

$$q - \overline{q} \le (\delta_{ij} - p_i p_j/|p|^2)(r_{ij} - \overline{r}_{ij}).$$

Now the matrix $((\delta_{ij} - p_i p_j/|p|^2))$ is nonnegative and $R - \overline{R}$ is nonpositive, by (3.37). Consequently $q - \overline{q} \le 0$, a contradiction to (3.33).

3.3. Contraction property.

Theorem 3.3. *Assume that u and v are weak solutions of (2.5), such that*

$$u \text{ and } v \text{ are constant on } \mathbb{R}^n \times [0, \infty) \cap \{|x| + t \ge R\} \tag{3.38}$$

for some constant $R > 0$. Then

$$\max_{0 \le t < \infty} \|u(\cdot, t) - v(\cdot, t)\|_{L^\infty(R^n)} = \|u(\cdot, 0)0 - v(\cdot, 0)\|_{L^\infty(R^n)}. \tag{3.39}$$

Proof. Should (3.39) fail, we may assume

$$\max_{(x,t)\in R^n \times [0,\infty)} (u - v) \equiv a > \|u(\cdot, 0) - v(\cdot, 0)\|_{L^\infty(R^n)} \equiv b.$$

Then as in the proof of Theorem 3.2 as above, there exist $\alpha, \epsilon, \delta > 0$ such that $\max_{(x,t),(x+y,t+s)\in R^n\times[0,\infty)} \Phi > b$, where Φ is defined by (3.13). We find a point (x_1, y_1, t_1, s_1) satisfying (3.15) and check (3.17) is valid provided $\epsilon, \delta > 0$ are small enough. The rest of the proof follows from that for Theorem 3.2.

4. Existence of weak solutions

4.1. Approximation; geometric interpretation. We turn our attention now to constructing a weak solution of the initial value problem (2.5), (2.6). We will assume that

$$g \text{ is constant on } \{\mathbb{R}^n\} \cap \{|x| \ge S\} \tag{4.1}$$

for some constant $S > 0$ and additionally, for the moment at least, g is smooth.

Our intention is to approximate (2.5), (2.6) by the PDE

$$u_t^\epsilon = \left(\delta_{ij} - \frac{u_{x_i}^\epsilon u_{x_j}^\epsilon}{|Du^\epsilon|^2 + \epsilon^2}\right) u_{x_i x_j}^\epsilon \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{4.2}$$

$$u^\epsilon = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{4.3}$$

for $0 < \epsilon < 1$. (The superscript ϵ here and hereafter is only a label and does not mean the sup-convolution (3.1).)

We interpret (4.2), (4.3) geometrically as follows. Assuming for the moment $u^\epsilon = u^\epsilon(x, t)$ to be a smooth solution of (4.2), (4.3), write $y = (x, x_{n+1}) \in \mathbb{R}^{n+1}$ and define

$$v^\epsilon(y, t) \equiv u^\epsilon(x, t) - \epsilon x_{n+1}. \tag{4.4}$$

Then $|D_y v^\epsilon|^2 = |Du^\epsilon|^2 + \epsilon^2$, and thus our PDE (4.2) becomes

$$v_t^\epsilon = (\delta_{ij} - v_{y_i}^\epsilon v_{y_j}^\epsilon/|Dv^\epsilon|^2) v_{y_i y_j}^\epsilon \quad \text{in } \mathbb{R}^{n+1} \times [0, \infty), \tag{4.5}$$

$$v^\epsilon = g^\epsilon \quad \text{on } \mathbb{R}^{n+1} \times \{t = 0\}, \tag{4.6}$$

for $g^\epsilon(y) \equiv g(x) - \epsilon x_{n+1}$. As noted in §2, the PDE (4.5) says that each level set of v^ϵ evolves according to its mean curvature. This is, in particular, the case for the zero level sets

$$\Gamma_t^\epsilon \equiv \{y \in \mathbb{R}^{n+1} | v^\epsilon(y, t) = 0\}.$$

But according to (4.4) each Γ_t^ϵ is a graph:

$$\Gamma_t^\epsilon = \{y = (x, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} = \epsilon^{-1} u^\epsilon(x, t)\},$$

and Ecker and Huisken [10] have shown the evolution of an entire graph by mean curvature remains a smooth entire graph for all time.

Geometrically, if as in §1 we are given Γ_0 as the boundary of a smooth, bounded, simply connected open set U in $\mathbb{R}^n$, we select a smooth function g with $g = 0$ on Γ_0, $g < 0$ in U, $g > 0$ in $\mathbb{R}^n - \overline{U}$. Then $\Gamma_0^\epsilon \subset \mathbb{R}^{n+1}$ is the graph $\{x_{n+1} = \epsilon^{-1} g(x)\}$ as drawn in Figure 3 (next page).

For small ϵ, Γ_0^ϵ roughly approximates the cylinder $\Gamma_0 \times \mathbb{R}$. We may thus hope that for moderate $t > 0$ and small $\epsilon > 0$, the smooth graph Γ_t^ϵ will be close to the cylinder $\Gamma_t \times \mathbb{R}$, Γ_t denoting the evolution of Γ_0 via its mean curvature in $\mathbb{R}^n$ (see Figure 4).

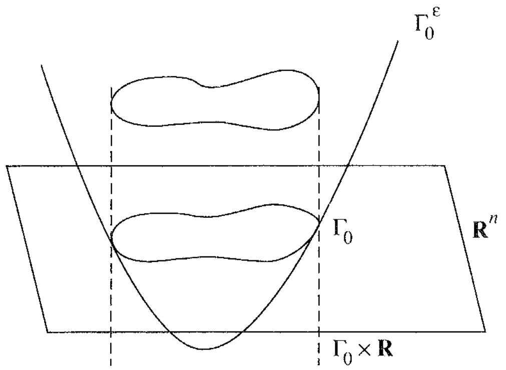

FIGURE 3

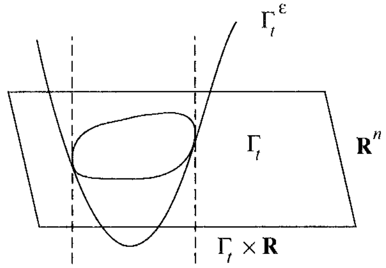

FIGURE 4

The idea then is that the complicated, possibly singular behavior of $\{\Gamma_t\}_{t\geq 0}$ in $\mathbb{R}^n$ will be approximated by the smooth evolution $\{\Gamma_t^\epsilon\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$; in the sense that for a given $t > 0$, $\Gamma_t^\epsilon \approx \Gamma_t \times \mathbb{R}$ if $\epsilon > 0$ is very small. The illustrations provided make this expectation appear plausible, although there are a number of subtleties.

4.2. Solution of the approximate equations. We now investigate the approximations (4.2), (4.3) analytically.

Theorem 4.1. (i) *For each* $0 < \epsilon < 1$ *there exists a unique smooth, bounded solution* u^ϵ *of* (4.2), (4.3).

(ii) *Additionally,*

$$\sup_{0<\epsilon<1} \|u^\epsilon, Du^\epsilon, u_t^\epsilon\|_{L^\infty(R^n\times[0,\infty))} \leq C\|g\|_{C^{1,1}(R^n)}. \tag{4.7}$$

Proof. 1. For each $0 < \sigma < 1$, consider the PDE

$$u_t^{\epsilon,\sigma} = a_{ij}^{\epsilon,\sigma}(Du^{\epsilon,\sigma})u_{x_ix_j}^{\epsilon\sigma} \quad \text{in } \mathbb{R}^n \times [0,\infty), \tag{4.8}$$

$$u^{\epsilon,\sigma} = g \quad \text{on } \mathbb{R}^n \times \{t=0\}, \tag{4.9}$$

for

$$a_{ij}^{\epsilon,\sigma}(p) \equiv (1+\sigma)\delta_{ij} - \frac{p_ip_j}{|p|^2+\epsilon^2} \qquad (p \in \mathbb{R}^n,\ 1 \leq i, j \leq n).$$

The smooth bounded coefficients $\{a_{ij}\}$ satisfy also the uniform ellipticity condition

$$\sigma|\xi|^2 \leq a_{ij}^{\epsilon,\sigma}(p)\xi_i\xi_j \qquad (\xi \in \mathbb{R}^n)$$

for each $p \in \mathbb{R}^n$, and consequently classical PDE theory gives the existence of a unique smooth bounded solution $u^{\epsilon,\sigma}$ (see, e.g., Ladyzhenskaja, Solonnikov, and Ural'tseva [31]). By the maximum principle,

$$\|u^{\epsilon,\sigma}\|_{L^\infty(R^n\times[0,\infty))} = \|g\|_{L^\infty(R^n)}. \tag{4.10}$$

2. Now differentiate (4.8) with respect to x_l:

$$u_{x_lt}^{\epsilon,\sigma} = a_{ij}^{\epsilon,\sigma}(Du^{\epsilon,\sigma})u_{x_lx_ix_j}^{\epsilon,\sigma} + a_{ij,p_k}^{\epsilon,\sigma}(Du^{\epsilon,\sigma})u_{x_lx_k}^{\epsilon,\sigma}u_{x_ix_j}^{\epsilon,\sigma}.$$

The maximum principle then implies

$$\|Du^{\epsilon,\sigma}\|_{L^\infty(R^n\times[0,\infty))} = \|Dg\|_{L^\infty(R^n)}. \tag{4.11}$$

Similarly

$$\|u_t^{\epsilon,\sigma}\|_{L^\infty(R^n\times[0,\infty))} = \|u_t^{\epsilon,\sigma}(\cdot,0)\|_{L^\infty(R^n)} \leq C\|D^2g\|_{L^\infty(R^n)}. \tag{4.12}$$

3. Since

$$\left(1 - \frac{L^2}{L^2+\epsilon^2}\right)|\xi|^2 \leq a_{ij}^{\epsilon,\sigma}(p)\xi_i\xi_j \qquad (\xi \in \mathbb{R}^n)$$

provided $|p| \leq L$, we deduce from (4.10)–(4.12) and classical estimates that we have bounds, uniform in $0 < \sigma < 1$, on the derivatives of all orders of $\{u^{\epsilon,\sigma}\}_{0<\sigma<1}$. Consequently, uniqueness of the limit implies for each multi-index α,

$$D^\alpha u^{\epsilon,\sigma} \to D^\alpha u^\epsilon \quad \text{locally uniformly as } \sigma \to 0,$$

for a smooth function u^ϵ solving (4.2), (4.3). Estimate (4.7) follows from (4.10)–(4.12).

4.3. Passage to limits.

Theorem 4.2. *Assume* $g: \mathbb{R}^n \to \mathbb{R}$ *is continuous and satisfies* (4.1). *Then there exists a weak solution* u *of* (2.5), (2.6), *such that*

$$u \text{ is constant on } \mathbb{R}^n \times [0,\infty) \cap \{|x| + t \geq R\} \tag{4.13}$$

for some $R > 0$, *depending only on the constant* S *from* (4.1).

Proof. 1. Suppose temporarily g is smooth. Employing estimate (4.7) we can extract a subsequence $\{u^{\epsilon_k}\}_{k=1}^\infty \subset \{u^\epsilon\}_{0<\epsilon\leq 1}$ so that $\epsilon_k \to 0$ and

$u^{\epsilon_k} \to u$ locally uniformly in $\mathbb{R}^n \times [0, \infty)$, for some bounded, Lipschitz function u.

2. We assert now that u is a weak solution of (2.5), (2.6). For this, let $\phi \in C^\infty(\mathbb{R}^{n+1})$ and suppose $u - \phi$ has a *strict* local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. As $u^{\epsilon_k} \to u$ uniformly near (x_0, t_0), $u^{\epsilon_k} - \phi$ has a local maximum at a point (x_k, t_k), with

$$(x_k, t_k) \to (x_0, t_0) \quad \text{as } k \to \infty. \tag{4.14}$$

Since u^{ϵ_k} and ϕ are smooth, we have

$$Du^{\epsilon_k} = D\phi, \quad u_t^{\epsilon_k} = \phi_t, \quad D^2 u^{\epsilon_k} \le D^2\phi \quad \text{at } (x_k, t_k).$$

Thus (4.2) implies

$$\phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2 + \epsilon_k^2}\right)\phi_{x_i x_j} \le 0 \quad \text{at } (x_k, t_k). \tag{4.15}$$

Suppose first $D\phi(x_0, t_0) \ne 0$. Then $D\phi(x_k, t_k) \ne 0$ for large k. We consequently may pass to limits in (4.15), recalling (4.14) to deduce

$$\phi_t \le (\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2)\phi_{x_i x_j} \quad \text{at } (x_0, t_0). \tag{4.16}$$

Next, assume instead $D\phi(x_0, t_0) = 0$. Set

$$\eta^k \equiv \frac{D\phi(x_k, t_k)}{(|D\phi(x_k, t_k)|^2 + \epsilon_k^2)^{1/2}},$$

so that (4.15) becomes

$$\phi_t \le (\delta_{ij} - \eta_i^k \eta_j^k)\phi_{x_i x_j} \quad \text{at } (x_k, t_k). \tag{4.17}$$

Since $|\eta^k| \le 1$, we may assume, upon passing to a subsequence and reindexing if necessary, that $\eta^k \to \eta$ in $\mathbb{R}^n$ for some $|\eta| \le 1$. Sending k to infinity in (4.17), we discover

$$\phi_t \le (\delta_{ij} - \eta_i\eta_j)\phi_{x_i x_j} \quad \text{at } (x_0, t_0). \tag{4.18}$$

If $u-\phi$ has a local maximum, but not necessarily a strict local maximum at (x_0, t_0), we repeat the argument above with $\phi(x, t)$ replaced by

$$\tilde{\phi}(x, t) = \phi(x, t) + |x - x_0|^4 + (t - t_0)^4,$$

again to obtain (4.16) or (4.18).

Consequently, u is a weak subsolution. That u is a weak supersolution follows analogously.

3. Finally we verify u satisfies (4.13). Upon rescaling as necessary, we may as well assume

$$|g| \le 1 \quad \text{on } \mathbb{R}^n, \qquad g = 0 \quad \text{on } \mathbb{R}^n \cap \{|x| \ge 1\}. \tag{4.19}$$

Consider now the auxiliary function (cf. Brakke [5, p. 25])

$$v(x, t) \equiv \Psi(|x|^2/2 + (n-1)t) \qquad (x \in \mathbb{R}^n, \ t > 0), \tag{4.20}$$

for

$$\Psi(s) \equiv \begin{cases} 0 & (s \ge 2), \\ (s-2)^3 & (0 \le s \le 2). \end{cases}$$

Then for $\Psi \in C^2([0, \infty))$,

$$\Psi'(s) = \begin{cases} 0 & (s \ge 2), \\ 3(s-2)^2 & (0 \le s \le 2), \end{cases} \qquad \Psi''(s) = \begin{cases} 0 & (s \ge 2), \\ 6(s-2) & (0 \le s \le 2). \end{cases}$$

In particular,

$$|\Psi''(s)| \le C(\Psi'(s))^{1/2} \qquad (s \ge 0). \tag{4.21}$$

Now

$$\begin{aligned} v_t &- \left(\delta_{ij} - \frac{v_{x_i} v_{x_i}}{|Dv|^2 + \epsilon^2}\right) v_{x_i x_j} \\ &= (n-1)\Psi' - \left(\delta_{ij} - \frac{(\Psi')^2 x_i x_j}{(\Psi')^2|x|^2 + \epsilon^2}\right)(\Psi'\delta_{ij} + \Psi'' x_i x_j) \\ &= \Psi'\left[(n-1) - \left(\delta_{ij} - \frac{(\Psi')^2 x_i x_j}{(\Psi'^2|x|^2 + \epsilon^2}\right)\delta_{ij}\right] \\ &\quad - \Psi''\left[\left(\delta_{ij} - \frac{(\Psi')^2 x_i x_j}{(\Psi')^2|x|^2 + \epsilon^2}\right) x_i x_j\right] \\ &\equiv A + B. \end{aligned} \tag{4.22}$$

We further compute

$$A = -\Psi' \frac{\epsilon^2}{(\Psi')^2|x|^2 + \epsilon^2} \le 0, \tag{4.23}$$

since $\Psi' \ge 0$. Moreover,

$$|B| = |\Psi''| \frac{\epsilon^2 |x|^2}{(\Psi')^2|x|^2 + \epsilon^2}.$$

Now if $|\Psi'| \le \epsilon$, then

$$(4.24)\qquad \begin{aligned}|B| &\le |\Psi''|\,|x|^2 \le C|\Psi''| \quad (\text{since } \Psi''=0 \text{ if } |x|\ge 2)\\ &\le C(\Psi')^{1/2} \quad (\text{by } (4.21))\\ &\le C\epsilon^{1/2}.\end{aligned}$$

On the other hand if $|\Psi'| \ge \epsilon$, we have

$$(4.25)\qquad |B| \le |\Psi''|\frac{\epsilon^2}{(\Psi')^2} \le \frac{C\epsilon^2}{|\Psi'|^{3/2}} \le C\epsilon^{1/2}.$$

Combining (4.22)–(4.25) yields

$$v_t - \left(\delta_{ij} - \frac{v_{x_i}v_{x_j}}{|Dx|^2+\epsilon^2}\right) v_{x_ix_j} \le C\epsilon^{1/2}$$

and so

$$(4.26)\qquad w_t^\epsilon \le \left(\delta_{ij} - \frac{w_{x_i}^\epsilon w_{x_j}^\epsilon}{|Dw^\epsilon|^2+\epsilon^2}\right) w_{x_ix_j}^\epsilon \quad \text{in } \mathbb{R}^n\times(0,\infty)$$

for

$$(4.27)\qquad w^\epsilon(x,t) \equiv v(x,t) - Ct\epsilon^{1/2}.$$

Now

$$w^\epsilon(x,0) = \Psi(|x|^2/2) = 0 \quad \text{if } |x|\ge 2$$

and

$$w^\epsilon(x,0) = \Psi(|x|^2/2) \le -1 \quad \text{if } |x|\le 1.$$

Consequently, we see from (4.19) that

$$(4.28)\qquad w^\epsilon \le g \quad \text{on } \mathbb{R}^n\times\{t=0\}.$$

Applying the maximum principle to (4.2), (4.3), (4.26), and (4.27), we deduce $w^\epsilon \le u^\epsilon$ in $\mathbb{R}^n\times[0,\infty)$ for each $0<\epsilon<1$. Sending $\epsilon=\epsilon_k$ to zero, we then have

$$\Psi(|x|^2/2+(n-1)t) = v(x,t) \le u(x,t)$$

for all $x\in\mathbb{R}^n$, $t\ge 0$. Thus, $u\ge 0$ if $|x|^2/2+(n+1)t\ge 2$. Similarly,

$$(4.29)\qquad \tilde w_t^\epsilon \ge \left(\delta_{ij} - \frac{\tilde w_{x_i}^\epsilon \tilde w_{x_j}^\epsilon}{|D\tilde w^\epsilon|^2+\epsilon^2}\right) \tilde w_{x_ix_j}^\epsilon \quad \text{in } \mathbb{R}^n\times(0,\infty),$$

$$(4.30)\qquad \tilde w^\epsilon \ge g \quad \text{on } \mathbb{R}^n\times(0,\infty),$$

for $\tilde w^\epsilon \equiv -w^\epsilon$. As above we consequently deduce

$$u\le 0 \quad \text{if } |x|^2/2+(n+1)t\ge 2.$$

Assertion (4.13) is proved.

4. According to the uniqueness assertion Theorem 3.2, in fact the full limit $\lim_{\epsilon\to 0} u^\epsilon = u$ exists. Note also from Theorem 3.3 that

$$(4.31)\qquad \|u-\tilde u\|_{L^\infty(R^n\times[0,\infty))} = \|g-\tilde g\|_{L^\infty(R^n)}$$

if $\tilde u$ is the solution built as above for a smooth initial function $\tilde g$ satisfying (4.1).

Suppose at last g satisfies (4.1), but is only continuous. We select smooth $\{g^k\}_{k=1}^\infty$, satisfying (4.1) (for the same S) so that $g^k\to g$ uniformly on $\mathbb{R}^n$. Denote by u^k the solution of (2.5), (2.6) constructed above with initial function g^k. Utilizing (4.31) we see that the limit $\lim_{k\to\infty} u^k = u$ exists uniformly on $\mathbb{R}^n\times[0,\infty)$. According to Theorem 2.7 u is a weak solution of (2.5), (2.6).

5. Definition of the generalized evolution by mean curvature

We now make precise the definition of the motion $\{\Gamma_t\}_{t>0}$ for a given initial hypersurface Γ_0. In fact, let us assume now only that

$$(5.1)\qquad \Gamma_0 \text{ is a compact subset of } \mathbb{R}^n.$$

Choose then any continuous function $g:\mathbb{R}^n\to\mathbb{R}$ satisfying

$$(5.2)\qquad \Gamma_0 = \{x\in\mathbb{R}^n \mid g(x)=0\}$$

and

$$(5.3)\qquad g \text{ constant on } \mathbb{R}^n\cap\{|x|\ge S\}$$

for some $S>0$. Utilizing Theorems 3.2 and 4.1, we see that there is a unique weak solution of the mean curvature evolution equation

$$(5.4)\qquad u_t = (\delta_{ij} - u_{x_i}u_{x_j}/|Du|^2)u_{x_ix_j} \quad \text{in } \mathbb{R}^n\times(0,\infty),$$

$$(5.5)\qquad u=g \quad \text{on } \mathbb{R}^n\times\{t=0\},$$

with

$$(5.6)\qquad u \text{ constant on } \mathbb{R}^n\times[0,\infty)\cap\{|x|+t\ge R\}$$

for some $R>0$.

Define then the compact set

$$\Gamma_t \equiv \{x \in \mathbb{R}^n | u(x, t) = 0\} \tag{5.7}$$

for each $t > 0$. We call $\{\Gamma_t\}_{t>0}$ the *generalized evolution by mean curvature* of the original compact set Γ_0.

We must first verify that $\{\Gamma_t\}_{t>0}$ is well defined.

Theorem 5.1. *Assume $\hat{g}: \mathbb{R}^n \to \mathbb{R}$ is continuous, with*

$$\Gamma_0 = \{x \in \mathbb{R}^n | \hat{g}(x) = 0\} \tag{5.8}$$

and

$$\hat{g} \text{ constant on } \mathbb{R}^n \cap \{|x| \geq S\}. \tag{5.9}$$

Suppose $\hat{u}$ is the unique weak solution of (5.4)–(5.6), *with $\hat{g}$ replacing g. Then*

$$\Gamma_t = \{x \in \mathbb{R}^n | \hat{u}(x, t) = 0\} \tag{5.10}$$

for each $t > 0$. Consequently our definition (5.7) *does not depend upon the particular choice of initial function g satisfying* (5.2), (5.3).

A related assertion for the level sets of solutions to homogeneous Hamilton-Jacobi PDE may be found in Evans-Souganidis [13, §7].

Proof. 1. First, we may as well assume $g \geq 0$ on $\mathbb{R}^n$ and thus $u \geq 0$ in $\mathbb{R}^n \times (0, \infty)$. Indeed, if g is negative somewhere, we can consider the PDE (5.4)–(5.6) with $|g|$ replacing g, the unique solution of which, owing to Theorems 2.8 and 3.2, is $|u|$. Our definition (5.7) is unchanged if we replace u by $|u|$. Similarly we may suppose $\hat{g}, \hat{u} \geq 0$. Set

$$\widehat{\Gamma}_t = \{x \in \mathbb{R}^n | \hat{u}(x, t) = 0\} \qquad (t \geq 0).$$

2. For $k = 1, 2, \ldots$ write $E_0 = \varnothing$ and $E_k = \{x \in \mathbb{R}^n | g(x) \geq 1/k\}$, so that

$$E_1 \subset \cdots \subset E_k \subset E_{k+1} \subset \cdots, \qquad \mathbb{R}^n - \Gamma_0 = \bigcup_{k=1}^{\infty} E_k. \tag{5.11}$$

Define

$$a_k \equiv \max_{R^n - E_{k-1}} \hat{g} > 0 \qquad (k = 1, 2, \ldots). \tag{5.12}$$

Then $a_1 \geq a_2 \geq \cdots$ and $\lim_{k\to\infty} a_k = 0$, according to (5.8) and (5.11). Next define the continuous function $\Psi: [0, \infty) \to [0, \infty)$ satisfying

$$\Psi(0) = 0,$$
$$\Psi(1/k) = a_k \qquad (k = 1, 2, \ldots),$$
$$\Psi \text{ linear on } [1/(k+1), 1/k] \qquad (k = 1, 2, \ldots),$$
$$\Psi \text{ constant on } [1, \infty).$$

3. Write $\tilde{g} = \Psi(g)$ and $\tilde{u} = \Psi(u)$. Then $\tilde{u}$ solves (5.4)–(5.6), with $\tilde{g}$ replacing g. Now $\tilde{g} = \hat{g} = 0$ on Γ_0. Furthermore, if $x \in E_k - E_{k-1}$, then

$$\tilde{g}(x) = \Psi(g(x)) \geq \Psi(1/k) = a_k \geq \hat{g}(x) \quad \text{by (5.12)}.$$

Thus $\tilde{g} \geq \hat{g}$ on $\mathbb{R}^n$. Consequently, Theorem 3.2 asserts

$$\tilde{u} = \Psi(u) \geq \hat{u} \geq 0 \quad \text{on } \mathbb{R}^n \times [0, \infty).$$

Thus if $x \in \Gamma_t$, then $\hat{u}(x, t) = 0$ and so $x \in \widehat{\Gamma}_t$. Hence $\Gamma_t \subseteq \widehat{\Gamma}_t$. The opposite inclusion is similarly proved, and therefore $\Gamma_t = \widehat{\Gamma}_t$ for each $t > 0$. q.e.d.

In light of this theorem, we can regard the mappings $\Gamma_0 \mapsto \Gamma_t$ $(t \geq 0)$ as comprising a time-dependent evolution on the collection $\mathscr{K}$ of compact subsets of $\mathbb{R}^n$. Let us write

$$\mathscr{M}(t)\Gamma_0 \equiv \Gamma_t \qquad (t \geq 0) \tag{5.13}$$

explicitly to display the dependence of Γ_t on t and Γ_0. Then $\mathscr{M}(t): \mathscr{K} \to \mathscr{K}$ for each $t \geq 0$, and $\mathscr{M}(0)$ is the identity operator. We will call $\{\mathscr{M}(t)\}_{t\geq 0}$ the *mean-curvature semigroup on $\mathscr{K}$*.

To justify this terminology, let us verify the semigroup property.

Theorem 5.2. *We have*

$$\mathscr{M}(t+s) = \mathscr{M}(t)\mathscr{M}(s) \qquad (t, s \geq 0). \tag{5.14}$$

Proof. If $t, s > 0$ and $\Gamma_0 \in \mathscr{K}$, choose any continuous function g satisfying (5.2), (5.3). Let u be the corresponding unique weak solution of (5.4)–(5.6). Then

$$\mathscr{M}(t+s)\Gamma_0 = \Gamma_{t+s} = \{x \in \mathbb{R}^n | u(x, t+s) = 0\}, \tag{5.15}$$

$$\mathscr{M}(s)\Gamma_0 = \Gamma_s = \{x \in \mathbb{R}^n | u(x, s) = 0\}. \tag{5.16}$$

To compute $\mathscr{M}(t)\Gamma_s$ we select any continuous function $\hat{g}$ so that

$$\Gamma_s = \{x \in \mathbb{R}^n | \hat{g}(x) = 0\} \tag{5.17}$$

and $\hat{g}$ is constant outside some large ball. We then find the unique weak solution $\hat{u}$ of (5.4)–(5.6) (with $\hat{g}$ replacing g) and set

$$\mathscr{M}(t)\Gamma_s = \widehat{\Gamma}_t = \{x \in \mathbb{R}^n | \hat{u}(x, t) = 0\}. \tag{5.18}$$

According to Theorem 5.1, this construction is independent of the particular choice of $\hat{g}$ satisfying (5.17). In particular, we may as well take $\hat{g}(x) = u(x, s)$ $(x \in \mathbb{R}^n)$. Owing then to the uniqueness of a weak solution to (5.4)–(5.6) we have

$$\hat{u}(x, t) = u(x, t+s) \qquad (x \in \mathbb{R}^n, \ t > 0).$$

Consequently (5.15) and (5.18) imply

$$\mathscr{M}(t+s)\Gamma_0 = \mathscr{M}(t)\mathscr{M}(s)\Gamma_0,$$

as required. This establishes (5.14). q.e.d.

Note that we make no assertions concerning continuity of the mapping $(t, \Gamma_0) \mapsto \mathscr{M}(t)\Gamma_0$.

6. Consistency with classical motion by mean curvature

We must now check that our generalized evolution by mean curvature agrees with the classical motion, if and so long as the latter exists. Let us therefore suppose for this section that Γ_0 is a smooth hypersurface, the connected boundary of a bounded open set $U \subset \mathbb{R}^n$. According to Hamilton [22], Gage-Hamilton [17], and Evans-Spruck [14], there exists a time $t_* > 0$ and a family $\{\Sigma_t\}_{0\le t<t_*}$ of smooth hypersurfaces evolving from $\Sigma_0 = \Gamma_0$ according to classical motion by mean curvature. In particular for each $0 \le t < t_*$, Σ_t is diffeomorphic to Γ_0, and is the boundary of an open set U_t diffeomorphic to $U_0 \equiv U$.

Theorem 6.1. *We have* $\Sigma_t = \Gamma_t$ $(0 \le t < t_*)$, *where* $\{\Gamma_t\}_{t\ge 0}$ *is the generalized evolution by mean curvature defined in* §5.

Proof. 1. Fix $0 < t_0 < t_*$, and define then for $0 \le t \le t_0$ the (signed) distance function

$$d(x, t) \equiv \begin{cases} -\operatorname{dist}(x, \Sigma_t) & \text{if } x \in U_t, \\ \operatorname{dist}(x, \Sigma_t) & \text{if } x \in \mathbb{R}^n \setminus \overline{U}_t. \end{cases}$$

As $\Sigma \equiv \bigcup_{0\le t\le t_0} \Sigma_t \times \{t\}$ is smooth, d is smooth in the regions

$$Q^+ \equiv \{(x, t) | 0 \le d(x, t) \le \delta_0, \ 0 \le t \le t_0\}$$

and

$$Q^- \equiv \{(x, t) | -\delta_0 \le d(x, t) \le 0, \ 0 \le t \le t_0\}$$

for $\delta_0 > 0$ sufficiently small.

2. Now if $\delta_0 > 0$ is small enough, for each point $(x, t) \in Q^+$ there exists a unique point $y \in \Sigma_t$ verifying $d(x, t) = |x - y|$. Consider now near (y, t) the smooth unit vector field $\nu \equiv Dd$ pointing from Σ into Q^+. Then

$$d_t(x, t) = (\operatorname{div} \nu)(y, t) \tag{6.1}$$

since $\{\Sigma_t\}_{0\le t\le t_*}$ is a classical evolution by mean curvature. Additionally, the eigenvalues of $D^2 d(x, t)$ are (see, e.g., Gilbarg-Trudinger [18, p. 355])

$$\left\{ \frac{-\kappa_1}{1-\kappa_1 d}, \cdots, \frac{-\kappa_{n-1}}{1-\kappa_{n-1} d}, 0 \right\}, \tag{6.2}$$

$\kappa_1, \cdots, \kappa_{n-1}$ denoting the principal curvatures of Σ_t at the point y, calculated with respect to the unit normal field ν. Thus,

$$\Delta d(x, t) = -\sum_{i=1}^{n-1} \frac{\kappa_i}{1-\kappa_i d}. \tag{6.3}$$

However, $(\operatorname{div} \nu)(y, t) = -(\kappa_1 + \cdots + \kappa_{n-1})$, and so (6.1) and (6.3) imply

$$d_t - \Delta d = \left(\sum_{i=1}^{n-1} \frac{\kappa_i^2}{1-\kappa_i d} \right) d \quad \text{at } (x, t). \tag{6.4}$$

Since the quantity $\sum_{i=1}^{n-1} \kappa_i^2/(1-\kappa_i d)$ is uniformly bounded and $d \ge 0$ in Q^+, we deduce from (6.4) that

$$\underline{d} \equiv \alpha e^{-\lambda t} d \tag{6.5}$$

satisfies

$$\underline{d}_t - \Delta \underline{d} \le 0 \quad \text{in } Q^+ \tag{6.6}$$

if $\lambda > 0$ is fixed large enough and $\alpha > 0$ (to be selected later). Furthermore, $|Dd|^2 = |\nu|^2 = 1$ and so $d_{x_i} d_{x_i x_j} = 0$ in Q^+ $(1 \le j \le n)$. The function $\underline{d}$ satisfies the same identity, whence (6.6) implies for each $\epsilon \ge 0$ that

$$\underline{d}_t - \left(\delta_{ij} - \frac{\underline{d}_{x_i} \underline{d}_{x_j}}{|D\underline{d}|^2 + \epsilon^2} \right) \underline{d}_{x_i x_j} \le 0 \quad \text{in } Q^+. \tag{6.7}$$

We see therefore that $\underline{d}$ is a smooth subsolution of the approximate mean curvature evolution PDE (4.2) in Q^+.

3. Choose any Lipschitz function $g: \mathbb{R}^n \to \mathbb{R}^+$ so that $g(x) = \operatorname{dist}(x, \Sigma_0)$ near Σ_0, $\{g = 0\} = \Sigma_0$, and $g(x)$ is a positive constant for large $|x|$. For $0 < \epsilon < 1$ the approximating PDE (4.2), (4.3) then has a continuous solution u^ϵ, which is smooth in $\mathbb{R}^n \times (0, \infty)$. Additionally we have $u^\epsilon \to u$ locally uniformly, where

$$\Gamma_t = \{x \in \mathbb{R}^n | u(x, t) = 0\}, \qquad t \ge 0. \tag{6.8}$$

Now $u = g = \delta_0 > 0$ on $\{(x, 0) | \operatorname{dist}(x, \Sigma_0) = \operatorname{dist}(x, \Gamma_0) = \delta_0\}$; and, as u is continuous, we thus have

$$u \ge \delta_0/2 > 0 \quad \text{on } \{(x, t) | d(x, t) = \delta_0\} \tag{6.9}$$

for $0 \le t \le t_0$, provided $t_0 > 0$ is small enough. Hence (6.9) implies

$$u^\epsilon \ge \delta_0/4 \quad \text{on } \{(x, t) | d(x, t) = \delta_0\}$$

for $0 \le t \le t_0$, $0 < \epsilon \le \epsilon_0$, if $\epsilon_0 > 0$ is sufficiently small. Consequently there exists $0 < \alpha < 1$ so that

$$u^\epsilon \ge \underline{d} \quad \text{on } \{(x, t) | d(x, t) = \delta_0\} \tag{6.10}$$

for $0 \le t \le t_0$, $0 < \epsilon < \epsilon_0$, $\underline{d}$ defined by (6.5). Since $0 < \alpha < 1$, we have

$$u^\epsilon \ge \underline{d} \quad \text{on } \{(x, 0) | 0 \le d(x, 0) \le \delta_0\}. \tag{6.11}$$

Furthermore, $g \ge 0$ implies $u^\epsilon \ge 0$ and so

$$u^\epsilon \ge \underline{d} \quad \text{on } \{(x, t) | d(x, t) = 0\}. \tag{6.12}$$

4. Combining (6.10)–(6.12) we see that $u^\epsilon \ge \underline{d}$ on the parabolic boundary of Q^+. Since $\underline{d}$ solves (6.7) and u^ϵ solves (4.2), the maximum principle implies $u^\epsilon \ge \underline{d}$ in Q^+. Let $\epsilon \to 0$ to conclude

$$u > 0 \text{ in the interior of } Q^+. \tag{6.13}$$

A similar argument using instead $\underline{d} = -\alpha e^{-\lambda t} d$ shows

$$u > 0 \text{ in the interior of } Q^-. \tag{6.14}$$

Since $u > 0$ in $(R^n \setminus \{x | \operatorname{dist}(x, \Sigma_0) \le \delta_0\}) \times [0, t_0]$, we deduce from (6.13), (6.14), and (6.8) that

$$\Gamma_t \subseteq \Sigma_t = \{x | d(x, t) = 0\} \qquad (0 \le t \le t_0). \tag{6.15}$$

5. Now define a new function $\hat{g}: \mathbb{R}^n \to \mathbb{R}$ so that $\hat{g}(x) = d(x, 0)$ (the *signed* distance function to Σ_0) near $\Sigma_0 = \Gamma_0$, $\{\hat{g} = 0\} = \Sigma_0$, and $\hat{g}(x)$ is a positive constant for large $|x|$. Let $\hat{u}$ denote the unique weak solution of (2.5), (2.6), (4.13) for this new initial function $\hat{g}$. According to Theorem 5.1

$$\Gamma_t = \{x \in \mathbb{R}^n | \hat{u}(x, t) = 0\} \qquad (t \ge 0). \tag{6.16}$$

Since $\hat{g} < 0$ in U_0 we know by continuity that $\hat{u} < 0$ somewhere in U_t, provided $0 \le t \le t_0$ and t_0 is small. Similarly $\hat{u} > 0$ somewhere in $\mathbb{R}^n - \overline{U}_t$ for each $0 \le t \le t_0$. Fix any point $x_0 \in \Sigma_t$ and draw a smooth curve C in $\mathbb{R}^n$, intersecting Σ_t precisely at x_0 and connecting a point $x_1 \in U_t$, where $\hat{u}(x_1, t) < 0$, to a point $x_2 \in \mathbb{R}^n - \overline{U}_t$, where $\hat{u}(x_2, t) > 0$. As $\hat{u}$ is continuous, we must have $\hat{u}(x, t) = 0$ for some point x on the curve C. However (6.15) and (6.16) say that the set $\{x | \hat{u}(x, t) = 0\}$ lies in Σ_t. Thus $\hat{u}(x_0, t) = 0$. Since x_0 denotes any point on Σ_t we deduce from (6.15), (6.16) that

$$\Gamma_t = \Sigma_t \quad \text{if } 0 \le t \le t_0. \tag{6.17}$$

We have consequently demonstrated that the classical motion $\{\Sigma_t\}_{0 \le t < t_*}$ and the generalized motion $\{\Gamma_t\}_{t \ge 0}$ agree at least on some short time interval $[0, t_0]$.

6. Write

$$s \equiv \sup_{0 \le t < t_*} \{t | \Gamma_\tau = \Sigma_\tau \text{ for all } 0 \le \tau \le t\}$$

and suppose $s < t_*$. Then $\Gamma_t = \Sigma_t$ for all $0 \le t < s$, and so, applying the continuity of the solution u to (2.5) and (2.6) for g as above, we have $\Gamma_s \supseteq \Sigma_s$. On the other hand if $x \in \mathbb{R}^n - \Sigma_s$, there exists $r > 0$ so that $B(x, r) \subset \mathbb{R}^n - \Sigma_t$ for all $s - \epsilon \le t \le s$, $\epsilon > 0$ small enough. Using this we easily deduce $x \notin \Gamma_s$. Hence $\Gamma_s = \Sigma_s$. But then applying steps 1–5 we deduce $\Gamma_t = \Sigma_t$ for all $s \le t \le s + s_0 < t_*$, if $s_0 > 0$ is small enough. This contradicts the definition of s, and so in fact $s = t_*$. q.e.d.

Observe carefully that our argument in step 5 above improving (6.15) to (6.17) depends critically upon the possibility of finding an initial function $\hat{g}$ which changes sign above. Compare this with the geometric situation in Theorem 8.1 below.

7. Geometric properties of generalized evolution by mean curvature

We devote this section to establishing some elementary properties of the generalized evolution by mean curvature

$$\Gamma_0 \mapsto \mathscr{M}(t)\Gamma_0 \equiv \Gamma_t \qquad (t \ge 0) \tag{7.1}$$

for Γ_0 a compact subset of $\mathbb{R}^n$.

7.1. Localization and extinction. First of all, it is known that if Γ_0 is the sphere $\partial B(0, R)$, then

$$\Gamma_t = \begin{cases} \partial B(0, R(t)) & \text{if } 0 \le t < t^*, \\ \{0\} & \text{if } t = t^*, \\ \varnothing & \text{if } t > t^*, \end{cases} \tag{7.2}$$

where

$$R(t) \equiv (R^2 - 2(n-1)t)^{1/2} \quad \text{for } 0 \le t \le t^* \equiv R^2/2(n-1). \tag{7.3}$$

This assertion follows in our approach by noting $u(x, t) = \Psi(|x|^2 + 2(n-1)t)$ is a weak solution of (5.4), where $\Psi: \mathbb{R} \to \mathbb{R}$ is smooth with

$$\begin{cases} \Psi' \ge 0, \quad \Psi < 0 \quad \text{on } [0, R), \\ \Psi > 0 \quad \text{on } (R, 3R), \qquad \Psi \equiv 1 \quad \text{on } [3R, \infty). \end{cases} \tag{7.4}$$

By making comparisons with the shrinking sphere (7.2) we derive now some elementary properties of the general motion (7.1) (cf. Brakke [5, pp. 29–30]).

Theorem 7.1. (a) *If* $\Gamma_0 \subset B(0, R)$, *then*

$$\Gamma_t = \emptyset \quad \text{for } t > R^2/2(n-1). \tag{7.5}$$

(b) *We have*

$$\Gamma_t \subseteq \operatorname{conv}(\Gamma_0) \qquad (t \geq 0), \tag{7.6}$$

where $\operatorname{conv}(\Gamma_0)$ *denotes the convex hull of* Γ_0.

Proof. 1. Assume first $\Gamma_0 \subset B(0, R-\epsilon)$ for some $\epsilon > 0$. Let $g: \mathbb{R}^n \to \mathbb{R}$ be continuous, with $\Gamma_0 = \{g = 0\}$, $g = 1$ on $\mathbb{R}^n \cap \{|x| \geq 2R\}$. Set $\hat{g}(x) = \Psi(|x|^2)$, with Ψ satisfying (7.4) selected so that $\hat{g} \leq g$ on $\mathbb{R}^n$. Then

$$\hat{u} \leq u \quad \text{on } \mathbb{R}^n \times [0, \infty),$$

for $\hat{u}(x, t) = \Psi(|x|^2 + 2(n-1)t)$ and u the weak solution of (5.4)–(5.6). Thus $u > 0$, and so $\Gamma_t = \emptyset$, if $t > \frac{1}{2}R^2/(n-1)$.

In the general case, replace R by $R + \epsilon$ in this argument and send $\epsilon \to 0$.

2. Suppose $\Gamma_0 \subset \mathbb{R}^n_+ = \{x_n > 0\}$. Choose $R \gg 1$ so large that $\Gamma_0 \subset B(Re_n, R)$, for $e_n = (0, 0, \cdots, 0, 1)$. By the argument in step 1, we deduce $\Gamma_t \subset B(Re_n, R(t))$ for $0 \leq t \leq \frac{1}{2}R^2/(n-1)$, $R(t)$ defined as above. In particular, $\Gamma_t \subseteq \mathbb{R}^n_+$ for all $t \geq 0$. Replacing $\mathbb{R}^n_+$ in this argument by an open half-space containing Γ_0, we obtain (7.6).

7.2. Comparison of different sets moving by mean curvature.

Theorem 7.2. *Let* Γ_0 *and* $\hat{\Gamma}_0$ *be compact subsets of* $\mathbb{R}^n$, *and denote by* $\{\Gamma_t\}_{t\geq 0}$ *and* $\{\hat{\Gamma}_t\}_{t\geq 0}$ *the corresponding generalized motions by mean curvature. Suppose also*

$$\Gamma_0 \subseteq \hat{\Gamma}_0. \tag{7.7}$$

Then

$$\Gamma_t \subseteq \hat{\Gamma}_t \quad \text{for each } t > 0. \tag{7.8}$$

We see therefore that if a compact set Γ_0 lies within another $\hat{\Gamma}_0$ at time zero, then the subsequent evolution Γ_t of Γ_0 lies within the subsequent evolution $\hat{\Gamma}_t$ of $\hat{\Gamma}_0$, for each $t > 0$. We will see in §8 that this assertion provides us with a useful tool for studying specific examples.

Proof. Choose continuous functions $g, \hat{g}: \mathbb{R}^n \to [0, \infty)$ so that $\Gamma_0 = \{g = 0\}$ and $\hat{\Gamma}_0 = \{\hat{g} = 0\}$, and g and $\hat{g}$ are constant on $\mathbb{R}^n \cap \{|x| \geq S\}$

for some $S > 0$. Replacing g by $g + \hat{g}$ if necessary, we may assume

$$\hat{g} \leq g \quad \text{on } \mathbb{R}^n. \tag{7.9}$$

Now let $\hat{u}, u$ denote the corresponding weak solutions of (5.4)–(5.6). Then (7.9) implies $0 \leq \hat{u} \leq u$ on $\mathbb{R}^n \times (0, \infty)$. Thus, $x \in \Gamma_t$ implies $x \in \hat{\Gamma}_t$, and so (7.8) is valid.

Theorem 7.3. *Assume* Γ_0 *and* $\hat{\Gamma}_0$ *are nonempty compact sets, and* $\{\Gamma_t\}_{t\geq 0}$ *and* $\{\hat{\Gamma}_t\}_{t\geq 0}$ *are the subsequent generalized motions by mean curvature. Then*

$$\operatorname{dist}(\Gamma_0, \hat{\Gamma}_0) \leq \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) \qquad (t \geq 0). \tag{7.10}$$

By definition, $\operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) = +\infty$ *if* $\Gamma_t = \emptyset$, $\hat{\Gamma}_t = \emptyset$, *or both.*

Proof. 1. We may assume $\operatorname{dist}(\Gamma_0, \hat{\Gamma}_0) > 0$. Choose $g: \mathbb{R}^n \to \mathbb{R}$ so that

$$\begin{cases} \Gamma_0 = \{g = 0\}, \qquad \hat{\Gamma}_0 = \{g = 1\}, \\ g = 2 \quad \text{on } \mathbb{R}^n \cap \{|x| \geq S\} \text{ for some } S, \\ \operatorname{Lip}(g) = \operatorname{dist}(\Gamma_0, \hat{\Gamma}_0)^{-1}. \end{cases} \tag{7.11}$$

Then

$$\Gamma_t = \{u = 0\}, \qquad \hat{\Gamma}_t = \{u = 1\}, \tag{7.12}$$

with u denoting the corresponding weak solution of (5.4)–(5.6).

2. From the contraction property Theorem 3.3, we see that

$$\operatorname{Lip}(u(\cdot, t)) \leq \operatorname{Lip}(g) \qquad (t \geq 0). \tag{7.13}$$

If $\Gamma_t \neq \emptyset$ and $\hat{\Gamma}_t \neq \emptyset$, choose points $x \in \Gamma_t$, $\hat{x} \in \hat{\Gamma}_t$ so that

$$|x - \hat{x}| = \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t).$$

Then using (7.11)–(7.13) we compute

$$1 = u(\hat{x}, t) - u(x, t) \leq \operatorname{Lip}(u)|x - \hat{x}| \leq \operatorname{dist}(\Gamma_0, \hat{\Gamma}_0)^{-1} \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t).$$

This proves (7.10). q.e.d.

Inequality (7.10) implies in particular that two hypersurfaces evolving under generalized motion by mean curvature do not ever move closer to each other than they were initially. In particular, $\Gamma_t \cap \hat{\Gamma}_t = \emptyset$ for all $t > 0$ provided $\Gamma_0 \cap \hat{\Gamma}_0 = \emptyset$. Notice that this property is essential for our approach of representing the evolving surfaces as the level sets of a continuous function.

7.3. Positive mean curvature. Now let us assume that Γ_0 is a smooth connected hypersurface, the boundary of a bounded open set $U \subset \mathbb{R}^n$. We will suppose additionally that

$$\operatorname{div}(\nu) < 0 \quad \text{on } \Gamma_0, \tag{7.14}$$

ν denoting the inner unit normal vector field to Γ_0 (extended smoothly to some neighborhood of Γ_0). Inequality (7.14) says that Γ_0 has positive mean curvature with respect to the inner unit normal field. Consequently, if Γ_0 evolves according to mean curvature, we see from (2.2) that initially at least the motion is directed into U.

We show now that in fact Γ_t lies in U for all $t \geq 0$, and that Γ_t continues to have positive mean curvature, this last statement interpreted in an appropriate weak sense.

Expanding upon a suggestion of L. Caffarelli, our idea is to solve the mean curvature equation (5.4)–(5.6) by separating variables. Indeed we will show

$$u(x, t) \equiv v(x) - t \qquad (x \in U,\ t > 0), \tag{7.15}$$

where v is the (unique) weak solution of the stationary problem

$$-(\delta_{ij} - v_{x_i}v_{x_j}/|Dv|^2)v_{x_ix_j} = 1 \quad \text{in } U, \tag{7.16}$$

$$v = 0 \quad \text{on } \partial U = \Gamma_0. \tag{7.17}$$

We will further prove that

$$\Gamma_t = \{x \in U | v(x) = t\} \qquad (t \geq 0), \tag{7.18}$$

so that $\Gamma_t \subset U$ $(t \geq 0)$ and $\Gamma_t = \varnothing$ for $t > t^* \equiv \|v\|_{L^\infty(U)}$. Note also that in any open region where v is smooth and $|Dv| \neq 0$, we can rewrite (7.16) as

$$-\operatorname{div}(\nu) = 1/|Dv| > 0 \quad \text{for } \nu \equiv Dv/|Dv|.$$

As ν is the inward pointing unit normal field along $\Gamma_t \equiv \{v = t\}$, we informally interpret our PDE (7.16) as implying "Γ_t has positive mean curvature" for $0 \leq t < t^*$.

To carry out the foregoing program rigorously, let us first define $v \in C(\overline{U})$ to be a weak solution to (7.16) provided that if $u - \phi$ has a local maximum (minimum) at a point $x_0 \in U$ for each $\phi \in C^\infty(\mathbb{R}^n)$, then

$$-(\delta_{ij} - \phi_{x_i}\phi_{x_j}/|D\phi|^2)\phi_{x_ix_j} \leq (\geq) 1 \quad \text{at } x_0 \text{ if } D\phi(x_0) \neq 0 \tag{7.19}$$

and

$$\begin{cases} -(\delta_{ij} - \eta_i\eta_j)\epsilon_{x_ix_j} \leq (\geq) 1 & \text{at } x_0 \text{ for some } \eta \in \mathbb{R}^n \\ \text{with } |\eta| \leq 1, & \text{if } D\phi(x_0) = 0. \end{cases} \tag{7.20}$$

Theorem 7.4. *There exists a unique weak solution v of (7.16), (7.17). Furthermore, there are constants A, $a > 0$ so that*

$$\begin{gathered} a \operatorname{dist}(x, \Gamma_0) \leq v(x) \leq A \operatorname{dist}(x, \Gamma_0) \qquad (x \in \overline{U}), \\ |Dv(x)| \leq A. \end{gathered} \tag{7.21}$$

Proof. 1. Similarly to §4, we approximate (7.16), (7.17) by the uniformly elliptic PDE

$$-\left(\delta_{ij} - \frac{v^\epsilon_{x_i}v^\epsilon_{x_j}}{|Dv^\epsilon|^2 + \epsilon^2}\right)v^\epsilon_{x_ix_j} = 1 \quad \text{in } U, \tag{7.22}$$

$$v^\epsilon = 0 \quad \text{on } \partial U = \Gamma_0 \tag{7.23}$$

for $0 < \epsilon \leq 1$. We will construct upper and lower barriers for (7.22), (7.33) of the form

$$w(x) = \lambda g(d(x)) \qquad (\lambda \in \mathbb{R},\ d(x) = \operatorname{dist}(x, \Gamma_0))$$

in a neighborhood $V \equiv \{0 < d(x) < 2\delta_0\}$ of Γ_0 in which d is smooth. Owing to the mean curvature condition (7.14), d satisfies

$$0 < b \leq -\Delta d \leq B, \quad d_{x_i}d_{x_j}d_{x_ix_j} \equiv 0 \tag{7.24}$$

in this region. We then use (7.24) to compute

$$\begin{aligned} Mw &\equiv \left(\delta_{ij} - \frac{w_{x_i}w_{x_j}}{|Dw|^2 + \epsilon^2}\right)w_{x_ix_j} \\ &= \lambda(g'\Delta d + g'') - \frac{\lambda^3}{\lambda^2 g'^2 + \epsilon^2}g'^2 g'' \\ &= \lambda g'\Delta d + \frac{\epsilon^2\lambda g''}{\lambda^2 g'^2 + \epsilon^2}. \end{aligned} \tag{7.25}$$

Choosing $g(t) = \delta_0^2 - (t - \delta_0)^2$ we find from (7.24), (7.25) that $Mw \geq -c\lambda\delta_0 - 2\lambda > -1$ for λ sufficiently small. Since $w = 0$ on ∂V, $w < v^\epsilon$ in V by the maximum principle. In particular, $v^\epsilon(x) \geq ad(x)$ $(x \in U)$, where the constant a is independent of ϵ. To obtain the corresponding upper bound, we choose

$$g(t) = \log(2\delta_0/(2\delta_0 - t)).$$

Then $g(t)$ is convex on $[0, 2\delta_0)$, and satisfies

$$g(0) = 0, \quad g' \geq 1/(2\delta_0), \quad g'' = g'^2, \quad g'(2\delta_0) = +\infty. \tag{7.26}$$

Again using (7.24)–(7.26), we find

$$Mw \leq -c\lambda + \epsilon^2/\lambda < -1$$

for λ sufficiently large. Since $\partial w/\partial\nu = +\infty$ on $\{d = 2\delta_0\}$, where ν denotes the exterior normal to V, we find that $v^\epsilon < w$ in V by a simple variant of the maximum principle. This gives the estimate

$$v^\epsilon(x) \leq Ad(x) \qquad (x \in V). \tag{7.27}$$

To complete our preliminary estimates, we observe that (7.27) implies $|Dv^\epsilon| \leq A$ on Γ_0. By differentiating (7.22) with respect to x_l, we see that any derivative $v^\epsilon_{x_l}$ achieves its maximum and minimum on Γ_0. Thus $|Dv^\epsilon|$ is uniformly bounded in U and in particular $v^\epsilon \leq Ad$ in U.

2. As a consequence of step 1, we derived the uniform bounds

$$\sup_{0<\epsilon\leq 1} \|v^\epsilon\|_{C^{0,1}(U)} < \infty.$$

Hence we may extract a subsequence $\{v^{\epsilon_k}\}_{k=1}^\infty \subset \{v^\epsilon\}_{0<\epsilon\leq 1}$ so that $\epsilon_k \to 0$ and $v^{\epsilon_k} \to v$ uniformly on $\overline{U}$. As in the proof of Theorem 4.2, we verify that v is a weak solution of (7.16).

3. The uniqueness of this weak solution v will follow from the characterization of $\{\Gamma_t\}_{t\geq 0}$ below.

Theorem 7.5. *Let $\{\Gamma_t\}_{t\geq 0}$ denote the generalized evolution by mean curvature starting with Γ_0. Then $\Gamma_t = \{x \in U | v(x) = t\}$ for each $t \geq 0$.*

Proof. 1. Define $u(x, t) \equiv v(x) - t$ for $x \in U$, $t > 0$. It is then straightforward to verify that u is a weak solution of the mean curvature evolution equation

$$u_t = (\delta_{ij} - u_{x_i}u_{x_j}/|Du|^2)u_{x_ix_j} \quad \text{in } U \times (0, \infty). \tag{7.28}$$

Set

$$\widehat{\Gamma}_t \equiv \{x \in U | v(x) = t\} = \{x \in U | u(x, t) = 0\} \qquad (t > 0). \tag{7.29}$$

2. Now let

$$\hat{u}(x, t) \equiv |u(x, t)| = |v(x) - t| \qquad (x \in U,\ t > 0).$$

In view of Theorem 2.8, $\hat{u}$ is a weak solution of

$$\begin{cases} \hat{u}_t = (\delta_{ij} - \hat{u}_{x_i}\hat{u}_{x_j}/|D\hat{u}|^2)\hat{u}_{x_ix_j} & \text{in } U \times (0, \infty), \\ \hat{u} = t & \text{on } \partial U \times [0, \infty), \\ \hat{\mu} = v & \text{on } \overline{U} \times \{t = 0\}. \end{cases} \tag{7.30}$$

3. Choose any smooth function $g: \mathbb{R}^n \to \mathbb{R}$ so that

$$\begin{cases} \Gamma_0 = \{g = 0\}, \quad g \geq 0, \quad Dg \neq 0 \quad \text{on } \Gamma_0, \\ g \text{ is constant on } \mathbb{R}^n \cap \{|x| \geq S\} \text{ for some } S > 0. \end{cases} \tag{7.31}$$

Let $w \geq 0$ be the unique weak solution of

$$\begin{cases} w_t = (\delta_{ij} - w_{x_i}w_{x_j}/|Dw|^2)w_{x_ix_j} & \text{in } \mathbb{R}^n \times (0, \infty), \\ w = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{7.32}$$

so that

$$\Gamma_t = \{x \in \mathbb{R}^n | w(x, t) = 0\} \qquad (t \geq 0). \tag{7.33}$$

According to our construction in §4, w is Lipschitz in t, and thus

$$|w(x, t)| \leq Ct \qquad (x \in \Gamma_0,\ t > 0)$$

for some constant C.

4. Employing now (7.21), we see that $\underline{w} \equiv \alpha w$ satisfies

$$\underline{w} \leq \begin{cases} v & \text{on } \overline{u} \times \{t = 0\}, \\ t & \text{on } \partial U \times (0, \infty) \end{cases}$$

if $\alpha > 0$ is sufficiently small.

Now the proof of our Comparison Theorem 3.2 can be modified to show from (7.30), (7.32) that $0 \leq \underline{w} \leq \hat{u}$ in $U \times [0, \infty)$. Thus $x \in \widehat{\Gamma}_t$ implies $x \in \Gamma_t$, and so $\widehat{\Gamma}_t \subseteq \Gamma_t$ for $t \geq 0$. Similarly, let us set

$$\overline{w} \equiv \beta w \tag{7.34}$$

for some large constant β. Now (7.14) yields that if t_0 is sufficiently small, then

$$\Gamma_t \subset U \qquad (0 \leq t \leq t_0). \tag{7.35}$$

Since $\partial U = \Gamma_0$, we may employ (7.35) and the semigroup property (5.14) to conclude $\Gamma_t \subset U$ $(t > 0)$. In particular, $w > 0$ on $\partial U \times (0, \infty)$. Consequently, for any $T > 0$ we may choose β so large that $\overline{w}$ defined by (7.34) satisfies $\hat{u} \leq \overline{w}$ on $U \times [0, T]$. Hence as above we find

$$\Gamma_t = \widehat{\Gamma}_t \qquad (0 \leq t \leq T).$$

7.4. Convexity. We next recover certain assertions of Huisken [23], by suitably adapting various methods of Korevaar [29] and Kennington [27] for studying the convexity of solutions to nonlinear elliptic PDE. Kennington had previously proposed this method in [28] (see also the concluding remarks in Trudinger [36]).

Theorem 7.6. *Assume Γ_0 is the boundary of a smooth convex bounded open set U. Then there exists a time $t^* > 0$ such that Γ_t is the boundary of a convex, nonempty open set for $0 \le t < t^*$ and Γ_t is empty for $t > t^*$.*

Proof. 1. Because of §7.3 it suffices to consider the stationary PDE

$$(7.36)\qquad \begin{cases} -(\delta_{ij} - v_{x_i}v_{x_j}/|Dv|^2)v_{x_ix_j} = 1 & \text{in } U, \\ v = 0 & \text{on } \Gamma_0 = \partial U. \end{cases}$$

We will show that $\{x \in U | v(x) > t\}$ is convex for $0 \le t < t^*$, $t^* = \|v\|_{L^\infty}$. In fact, we will show that $\sqrt{v}$ is concave.

Formally, if $w = \sqrt{v}$ and v satisfies (7.36), then w solves

$$(7.37)\qquad -(\delta_{ij} - w_{x_i}w_{x_j}/|Dw|^2)w_{x_ix_j} = 1/2w \quad \text{in } U.$$

This suggests we consider approximations $w^\epsilon = \sqrt{v^\epsilon}$ satisfying

$$(7.38)\qquad \begin{aligned} Mw^\epsilon \equiv \left(\delta_{ij} - \frac{w^\epsilon_{x_i}w^\epsilon_{x_j}}{|Dw^\epsilon|^2 + \epsilon^2}\right) w^\epsilon_{x_ix_j} &= -\frac{1}{2w^\epsilon} \quad \text{in } U, \\ w^\epsilon &= 0 \qquad \text{on } \Gamma_0, \end{aligned}$$

$$(7.39)\qquad -\left(\delta_{ij} - \frac{v^\epsilon_{x_i}v^\epsilon_{x_j}}{|Dv^\epsilon|^2 + 4\epsilon^2 v^\epsilon}\right) v^\epsilon_{x_ix_j} = \frac{-2\epsilon^2|Dv^\epsilon|^2}{|Dv^\epsilon|^2 + 4\epsilon^2 v^\epsilon} + 1 \quad \text{in } U.$$

Because the convexity arguments are very sensitive to the form of the equation, we are forced into making a nice approximation w^ϵ to (7.37) and then making due with nastier approximations v^ϵ to (7.36).

2. We first demonstrate the existence of a solution $w^\epsilon \in C^2(U) \cap C^{1/2}(\overline{U})$ to (7.38). Consider therefore the PDE

$$(7.40)\qquad \begin{aligned} Ms^{\epsilon,\delta} \equiv \left(\delta_{ij} - \frac{w^{\epsilon,\delta}_{x_i}w^{\epsilon,\delta}_{x_j}}{|Dw^{\epsilon,\delta}|^2 + \epsilon^2}\right) w^{\epsilon,\delta}_{x_ix_j} &= -\frac{1}{2(w^{\epsilon,\delta} + \delta)} \quad \text{in } U, \\ w^{\epsilon,\delta} &= 0 \quad \text{on } \Gamma_0, \end{aligned}$$

which has a unique smooth solution $w^{\epsilon,\delta} \ge 0$.

Choose a large ball $B(p, R)$ containing U with $\operatorname{dist}(p, U) \ge R/2$, and let $r \equiv |x - p|$. Set $w \equiv (2R - r)$. Then

$$Mw + \frac{1}{2(w+\delta)} = -\frac{(n-1)}{r} + \frac{1}{2(2R - r)} \le -\frac{(n-1)}{R} + \frac{1}{3R} < 0.$$

Since $w > 0$ on ∂U, $w > w^{\epsilon,\delta}$ in U by the maximum principle. Hence

$$(7.41)\qquad 0 \le w^{\epsilon,\delta} < 2R \quad \text{in } U,$$

with R independent of ϵ and δ.

Next, let $w \equiv \lambda\sqrt{d}$ in $V = \{0 < d(x) < \delta_0\}$. Using formula (7.25) of §7.3 (with $g(t) = \sqrt{t}$) we find

$$Mw + \frac{1}{2(w+\delta)} \le -\frac{c\lambda}{\sqrt{d}} + \frac{1}{2(\lambda\sqrt{d} + \delta)} < 0$$

for λ sufficiently large. If in addition, we choose λ so that $\lambda\sqrt{\delta_0} \ge 2R$, then $w \ge w^{\epsilon,\delta}$ on ∂V, and thus $w \ge w^{\epsilon,\delta}$ on V by the maximum principle. In particular

$$(7.42)\qquad 0 \le w^{\epsilon,\delta} \le A\sqrt{d} \quad \text{in } U,$$

with A independent of ϵ and δ.

Estimate (7.42) implies that

$$(7.43)\qquad |w^{\epsilon,\delta}(x) - w^{\epsilon,\delta}(y)| \le C|x - y|^{1/2} \quad \text{if } x \in U,\ y \in \Gamma_0,$$

with C independent of ϵ and δ. We show that (7.43) holds for all $x, y \in U$ by the following well-known argument. Given $x, y \in U$ we set $\tau \equiv y - x$, $U_\tau \equiv \{z \in \mathbb{R}^n | z - \tau \in U\}$, and $w^{\epsilon,\delta}_\tau(z) \equiv w^{\epsilon,\delta}(z - \tau)$. Note that U_τ is open and nonempty since $y \in U_\tau$. On $U \cap U_\tau$, both $w^{\epsilon,\delta}$ and $w^{\epsilon,\delta}_\tau$ satisfy (7.40) and hence the difference $w = w^{\epsilon,\delta} - w^{\epsilon,\delta}_\tau$ satisfies a linear elliptic equation of the form $Lw + c(x)w = 0$ with $c(x) \ge 0$. Hence by the maximum principle,

$$|w(y)| \le \max_{z \in \partial(U \cap U_\tau)} |w(z)| \qquad (y \in U \cap U_\tau).$$

Since for $z \in \partial(U \cap U_\tau)$ either $z \in \partial U$ or $z - \tau \in \partial U$, we have by (7.43) that

$$(7.44)\qquad |w^{\epsilon,\delta}(y) - w^{\epsilon,\delta}(x)| = |w^{\epsilon,\delta}(y) - w^{\epsilon,\delta}_\tau(y)| \le C|x - y|^{1/2}.$$

Finally, in order to pass to the limit for a sequence $\delta_k \searrow 0$, we need to establish some interior estimates for $w_k = w^{\epsilon,\delta_k}$. Let $W \subset\subset U$. Then we claim

$$(7.45)\qquad \|w_k\|_{C^{2+\alpha}(W)} \le M(\epsilon, \operatorname{dist}(W, \Gamma_0))$$

with M independent of δ_k. By Schauder theory, (7.45) follows from an interior gradient estimate

$$\|Dw_k\|_{L^\infty(W)} \le C(\epsilon, \operatorname{dist}(W, \Gamma_0)),$$

which in turn follows from Gilbarg-Trudinger [18, Theorem 15.5]. Therefore, we have established the existence of a (unique) solution w^ϵ of (7.38), and in addition the estimates

$$0 \le w^\epsilon \le A\sqrt{d}, \qquad 0 \le w^\epsilon \le 2R, \tag{7.46}$$
$$|w^\epsilon(x) - w^\epsilon(y)| \le C|x-y|^{1/2},$$

with A, C, R independent of ϵ.

3. Before we proceed to the proof of the concavity of w^ϵ, we shall need to establish the lower bound

$$w^\epsilon \ge ad \tag{7.47}$$

with a independent of ϵ.

Consider $w \equiv \lambda g(d)$ in $V = \{0 < d(x) < 2\delta_0\}$ with $g(t) = (\delta_0^2 - (t-\delta_0)^2)^{1/2}$. Then from formulas (7.24) and (7.25) we find

$$\begin{aligned} Mw &\ge -\frac{\lambda\delta_0}{g}\left(c + \frac{\epsilon^2\delta_0}{\lambda^2(d-\delta_0)^2 + \epsilon^2(\delta_0^2 - (d-\delta_0)^2)}\right) \\ &\ge -\frac{\lambda\delta_0}{g}\left(c + \frac{1}{\delta_0}\right) \quad \text{for } \lambda \ge \epsilon, \end{aligned}$$

and so

$$Mw + \frac{1}{2w} \ge -\frac{\lambda}{g}(\delta_0 c + 1) + \frac{1}{2\lambda g} \ge 0$$

for $\epsilon^2 \le \lambda^2 = 2(\delta_0 c + 1)$. With this choice, we see $w^\epsilon \ge w$ in $\{0 < d(x) < 2\delta_0\}$, and as in §7.3 the estimate (7.47) follows easily.

4. We can now show that w^ϵ is concave. For $x, y \in \overline{U}$ set $z = \lambda x + (1-\lambda)y$, $\lambda \in (0,1)$ being fixed. The concavity function of w^ϵ is defined by

$$\mathscr{C}^\lambda(x, y) \equiv w^\epsilon(z) - \lambda w^\epsilon(x) - (1-\lambda)w^\epsilon(y) \qquad (x, y \in U).$$

The fundamental concavity maximum principle for $\mathscr{C}$ was established by Korevaar [29] for a large class of elliptic equations. The case at hand fails to satisfy Korevaar's condition. However, Kennington's improved concavity maximum principle [27, Theorem 3.1] does apply and so the infimum of $\mathscr{C}^\lambda$ is not attained on $U \times U$.

To complete the proof we must essentially show that w^ϵ is concave near Γ_0. Since $w^\epsilon = \sqrt{v^\epsilon}$ satisfies (7.38), (7.39), it is straightforward to see that $v^\epsilon \in C^{2+\alpha}(\overline{U})$ and $Dv^\epsilon \cdot \nu \ge a > 0$ for ν the interior normal to Γ_0. Using the strict convexity of U it is easy to check that

$$w_{ll}^\epsilon = \frac{1}{2\sqrt{v^\epsilon}} v_{ll}^\epsilon - \frac{(v_{ll}^\epsilon)^2}{4(v^\epsilon)^{3/2}}$$

is strictly negative near Γ_0. It follows easily that $\mathscr{C}^\lambda \ge 0$ on $U \times U$ (for complete details, see Korevaar [29, Lemma 2.4] or Caffarelli-Spruck [6, Theorem 3.1]). This completes the proof that w^ϵ is concave.

5. Since w^ϵ is concave, it follows that $|Dw^\epsilon| \ne 0$ on each level set of w^ϵ below the maximum of w^ϵ. Hence all these level sets are smooth convex hypersurfaces.

We claim that these level sets have uniformly bounded principal curvatures. To see this, it suffices because of the convexity of these level sets to know that the mean curvature $\mathscr{H}$ with respect to the inward normal is uniformly bounded. But

$$\begin{aligned} \mathscr{H}|Dw^\epsilon| &= -\left(\delta_{ij} - \frac{w_{x_i}^\epsilon w_{x_j}^\epsilon}{|Dw^\epsilon|^2}\right) w_{x_i x_j}^\epsilon \\ &= \frac{1}{2w^\epsilon} + w_{x_i}^\epsilon w_{x_j}^\epsilon w_{x_i x_j}^\epsilon \left(\frac{1}{|Dw^\epsilon|^2} - \frac{1}{|Dw^\epsilon|^2 + \epsilon^2}\right). \end{aligned}$$

Since w^ϵ is concave we conclude that $0 \le \mathscr{H} \le 1/2w^\epsilon|Dw^\epsilon|$, and therefore $\mathscr{H}$ is uniformly bounded on each of the level sets below the maximum of w^ϵ.

6. We complete the proof of Theorem 7.6 by showing that $v^\epsilon \to v$ uniformly on $\overline{U}$, where v is the unique solution of (7.36) constructed in Theorem 7.4.

Since w^ϵ satisfies (7.38), v^ϵ satisfies

$$|v^\epsilon(x) - v^\epsilon(y)| \le 4RC|x-y|^{1/2}, \qquad x, y \in U.$$

Hence, we may choose a sequence $\epsilon_k \to 0$ with $v^{\epsilon_k} \to v$ uniformly on $\overline{U}$. We assert that v is a weak solution of (7.36). As before, it suffices to consider $\phi \in C^\infty(\mathbb{R}^n)$ with $v - \phi$ having a *strict* local maximum at a point $x_0 \in U$. As $v^{\epsilon_k} \to v$ uniformly near x_0, $v^{\epsilon_k} - \phi$ has a local maximum at a point x_k, with $x_k \to x_0$ as $k \to \infty$.

Since v^{ϵ_k} and ϕ are smooth, we have

$$Dv^{\epsilon_k} = D\phi, \quad D^2 v^{\epsilon_k} \le D^2\phi \quad \text{at } x_k.$$

Thus (7.39) implies

$$-\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2 + 4\epsilon_k^2 v^{\epsilon_k}}\right)\phi_{x_i x_j} \le -2\epsilon_k^2 \frac{|D\phi|^2}{|D\phi|^2 + 4\epsilon_k^2 v^{\epsilon_k}} + 1 \tag{7.48}$$

at x_k. Suppose first $D\phi(x_0) \ne 0$. Then $D\phi(x_k) \ne 0$ for large k. Consequently we may pass to the limit in (7.48) (since $0 \le v^{\epsilon_k} \le 4R^2$) to deduce

$$-\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} \le 1 \quad \text{at } x_0 .$$

Next, assume instead $D\phi(x_0) = 0$ and set

$$\eta^k \equiv \frac{D\phi(x_k)}{(|D\phi(x_k)|^2 + 4\epsilon_k^2 v^{\epsilon_k})^{1/2}},$$

so that (7.48) becomes

$$(7.49) \qquad -(\delta_{ij} - \eta_i^k \eta_j^k)\phi_{x_i x_j} \le -2\epsilon_k^2 \frac{|D\phi|^2}{|D\phi|^2 + 4\epsilon_k^2 v_k^\epsilon} + 1 \quad \text{at } x_k .$$

Since $|\eta^k| \le 1$, we may pass to a subsequence and reindex if necessary to ensure $\eta_k \to \eta$ in $\mathbb{R}^n$ for some $|\eta| \le 1$. Sending k to infinity in (7.49) we discover

$$(7.50) \qquad -(\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j} \le 1 \quad \text{at } x_0 .$$

Consequently v is a weak subsolution. Similarly, we find that v is a weak supersolution, and the proof of Theorem 7.6 is complete.

Remark 7.7. We have shown that if Γ_0 is smooth, then Γ_t is a $C^{1,1}$ convex hypersurface. In a subsequent paper, we will demonstrate that for arbitrary convex Γ_0, the surfaces $\{\Gamma_t\}_{t\ge 0}$ are actually smooth. Once this smoothness is demonstrated, it follows from the work of Huisken [23] that the $\{\Gamma_t\}_{t\ge 0}$ are strictly convex and shrink to a point.

8. Examples, pathologies, and conjectures

In this concluding section, we note various odd behavior allowed by our generalized mean curvature flow

$$\Gamma_0 \mapsto \mathscr{M}(t)\Gamma_0 = \Gamma_t \qquad (t \ge 0),$$

and set forth some related conjectures.

8.1. Instantaneous extinction. Suppose Σ_0 is the smooth, connected boundary of a bounded open subset $U \subset \mathbb{R}^n$, and let Γ_0 be a compact subset of Σ_0. If $\Gamma_0 = \Sigma_0$, then we know from Theorem 6.1 that, at least for small times $t > 0$, Γ_t is the classical evolution via mean curvature.

What happens if Γ_0 is a proper subset of Σ_0?

Theorem 8.1. *Assume that Γ_0 is compact, $\Gamma_0 \subseteq \Sigma_0$, $\Gamma_0 \ne \Sigma_0$. Then*

$$(8.1) \qquad \Gamma_t = \varnothing \quad \textit{for each } t > 0 .$$

If we take Γ_0 to be, say, Σ_0 with a small disk D removed, we may informally regard (8.1) as asserting Γ_0 "pops" instantly. In this heuristic interpretation, we may think of Γ_0 as somehow having so much mean curvature concentrated along its boundary within Σ_0 that the hole then widens infinitely fast (see Figure 5).

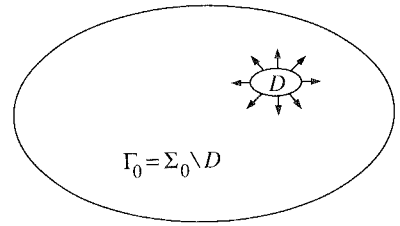

FIGURE 5

The proof of Theorem 8.1 will be given after the next assertion, of independent interest. Assume now that $\widehat{\Sigma}_0$ is the smooth connected boundary of a bounded open set $\widehat{U} \subset \mathbb{R}^n$ and that

$$(8.2) \qquad \widehat{\Sigma}_0 \subset \overline{U},$$

with Σ_0 and U as above. Thus the surface $\widehat{\Sigma}_0$ lies within the closed region $\overline{U}$ enveloped by Σ_0. Suppose further that

$$(8.3) \qquad \widehat{\Sigma}_0 \ne \Sigma_0 .$$

Then choose a time $t_0 > 0$ so small that the classical evolutions $\{\Sigma_t\}$ and $\{\widehat{\Sigma}_t\}$ starting at Σ_0 and $\widehat{\Sigma}_0$, respectively, exist at least for times $0 \le t \le t_0$.

Theorem 8.2. *We have*

$$(8.4) \qquad \Sigma_t \cap \widehat{\Sigma}_t = \varnothing \quad \textit{for } 0 < t \le t_0 .$$

We are thus asserting that even if Σ_0 and $\widehat{\Sigma}_0$ coincide except for a very small region (see Figure 6), then for any positive $t > 0$ the subsequent evolutions will have completely broken apart (as in Figure 7). The point

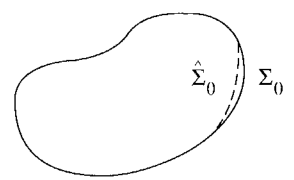

FIGURE 6

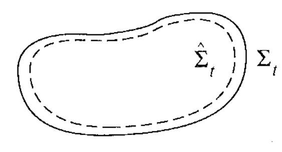

FIGURE 7

is that the PDE describing evolution by mean curvature is "uniformly parabolic along the surface" and thus admits infinite propagation speed for disturbances.

We will give the proof of Theorem 8.2 (as well as a new proof of the short time existence of classical mean curvature flow) in a separate paper [14]. Another proof follows by covering Σ_0 and $\widehat{\Sigma}_0$ by overlapping balls small enough so that the restrictions of Σ_t and $\widehat{\Sigma}_t$ to each ball can be written as graphs. Since the equation for the height function is uniformly parabolic for small t_0, and since $\widehat{\Sigma}_0 \neq \Sigma_0$, in at least one of the balls the surfaces Σ_t and $\widehat{\Sigma}_t$ must instantly separate. Thus in each ball the surfaces must also separate.

Proof of Theorem 8.1. Given Γ_0 and Σ_0 as in Theorem 8.1 we may choose a smooth, nearby surface $\widehat{\Sigma}_0$ to Σ_0 satisfying (8.2), (8.3), and $\Gamma_0 \subset \widehat{\Sigma}_0$. Then owing to Theorem 7.2 we have $\Gamma_t \subseteq \Sigma_t \cap \widehat{\Sigma}_t$ for small $t > 0$. Assertion (8.1) now follows from (8.4).

8.2. Development of an interior. The foregoing demonstrates that a "large" initial set Γ_0 can instantly vanish under the generalized mean curvature flow. An opposite and perhaps more surprising phenomenon is that the set Γ_t for $t > 0$ may develop an interior, even if Γ_0 had none.

The simplest example occurs if we take Γ_0 to be the union of the coordinate axes in the plane $\mathbb{R}^2$ (Figure 8). (Ignore for the moment that Γ_0 is not compact and so our theory in §5 is not really applicable.) To discover, heuristically at least, the subsequent evolution of Γ_0, consider instead the simpler figure as drawn in Figure 9. As for instance in Brakke [5, Figure 3] we expect this corner to evolve to the shape depicted in Figure 10 for times $t > 0$. Since Γ_0 is composed of four rotated copies of this corner, we expect from Theorem 7.2 that Γ_t will look like the shape in Figure 11. This assertion is at variance with Brakke [5, Figure 5]. Our Γ_t presumably contains the set shown in Figure 12, which he draws as one of the (nonunique!) evolutions for Γ_0. *We conjecture that our* Γ_t *contains all of the evolutions of* Γ_0 *allowed for by Brakke.*

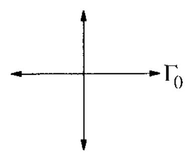

FIGURE 8

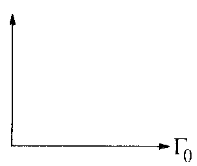

FIGURE 9

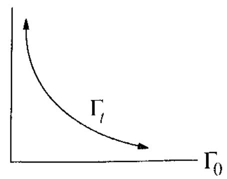

FIGURE 10

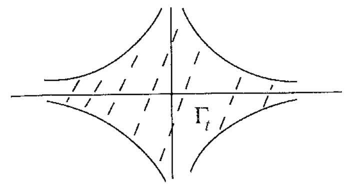

FIGURE 11

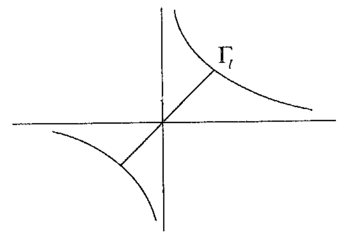

FIGURE 12

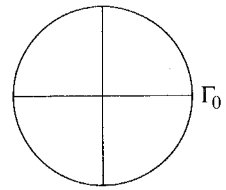

FIGURE 13

The discussion above can be modified to apply to various compact figures Γ_0, to which our theory does apply. We leave it to the reader to provide at least a heuristic proof that the set $\Gamma_0 \subset \mathbb{R}^2$ as drawn in Figure 13 will develop an interior.

Observe by the way that our approach regards a "figure eight" in R^2 as being embedded with a singularity at the crossing point. We in particular do not interpret this shape as an immersed circle, and consequently model its evolution completely differently than [1], [3], [11], etc.

We conjecture that if $\Gamma_0 = \Sigma_0$ *is, as above, the boundary of a smooth open set, then* Γ_t *will never have an interior.*

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