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# Parabolic equations for curves on surfaces
# Part I. Curves with $p$-integrable curvature

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## Abstract

This is the first of a two-part paper in which we develop a theory of parabolic equations for curves on surfaces which can be applied to the so-called curve shortening of flow-by-mean-curvature problem, as well as to a number of models for phase transitions in two dimensions.

We introduce a class of equations for which the initial value problem is solvable for initial data with $p$-integrable curvature, and we also give estimates for the rate at which the $p$-norms of the curvature must blow up, if the curve becomes singular in finite time.

A detailed discussion of the way in which solutions can become singular and a method for "continuing the solution through a singularity" will be the subject of the second part.

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## Introduction

In this paper and its sequel we study the motion of curves on a surface whose normal velocity is a given function of its position and its curvature. A particular case is the *curve shortening problem*, or *flow by mean curvature* for curves on surfaces. Here, one studies curves whose normal velocity and geodesic curvature coincide. This case has been examined in great detail in the last few years by Gage, Gage and Hamilton, Abresch and Langer, Epstein and Weinstein and M. Grayson. Their papers are listed in the references. Intuitively, the problem is that of describing the motion of a rubber band on a very sticky surface, if you assume that the potential energy of the rubber band is proportional to its length, and that the friction between the rubber band and the surface is so large that it causes the band to move according to the gradient flow of the length function on the space of smooth curves on the surface.

Another special case of the problem we shall be looking at comes from the theory of phase transitions. M. Gurtin has formulated a model for the evolution of a two phase system in which both phases are perfect heat conductors. (See [GuA].) If the system is two dimensional, the free boundary between the two phases will be a plane curve. If this curve is assumed to be smooth, then its motion is determined by the law $v^{\perp} = \phi(\theta)k - \psi(\theta)$, where $\phi, \psi$ are given functions, $\theta$ is the angle the tangent to the curve makes with the $x$-axis, $k$ is its curvature and $v^{\perp}$ is the normal velocity of the curve.

Motivated by these and other examples (such as Gage's variation on the curve shortening problem, in which $v^{\perp} = k/R$, where $R$ is the Gaussian curvature of the surface and $R$ is assumed to be positive), we have tried to find the most general law of motion of the form

$$v^{\perp} = V(t, k) \tag{1}$$

for curves on some surface $M$ with a Riemannian metric $g$, for which the initial value problem is well posed for a large class of initial curves.

One cannot expect that the initial value problem for (1) will have a solution which exists for all time. It is known, for example, that solutions of the curve shortening problem in the plane always become singular in finite time. When the initial curve has no self-intersections, Gage and Hamilton, and Grayson have shown that the solution will shrink to a "round point" in finite time. In fact, the time it takes is $A/2\pi$, where $A$ is the area enclosed by the initial curve. If the initial curve does have self-intersections, then small loops may contract in finite time, causing the curvature to become infinite. In this case, one would expect that the family of curves converges to some singular limit curve which is piecewise smooth, with a finite number of cusp-like singularities. By drawing pictures, one can easily convince oneself that there should be a solution of (1) which has this singular limit curve as initial value, in some weak sense.

Our ultimate goal in this two-part paper is to find a large class of $V$'s for which these expectations can be proved, i.e. for which a detailed description of the limit curve is possible, and for which the class of allowable initial curves is so large that it contains the limit curve.

Using the theory of parabolic PDE's which has been developed over the last three decades, it is a fairly straightforward matter to prove short time existence of solutions to (1) for initial curves which are $C^2$, provided the function $V$ satisfies some parabolicity condition. Moreover, a very simple trick allows us to prove that solutions are actually as smooth as the manifold $M$, its metric $g$ and the function $V$: $S^1(M) \times \mathbf{R} \to \mathbf{R}$, even in the real analytic context. In fact if $M$, $g$ and $V$ are real analytic, then so is any solution of (1.1) (below), and we can show this without using any of the existing theorems on analyticity of solutions of parabolic equations!

In this first part we deal with the most general class of $V$'s for which we can solve the initial value problem for initial curves whose curvature belongs to some $L_p$ class. The sequel will be devoted to a smaller class of $V$'s, for which one can allow locally Lipschitz, and even locally graph-like, curves as initial data for (1). The methods which are used in Parts I and II are quite different. In Part I integral estimates and a blow-up argument are our main tools; in Part II estimates for the regularity of solutions of (1) are obtained by more geometrical arguments, e.g. by comparing general solutions with special solutions, and counting their intersections. In the next section we give a precise description of the results obtained here.

## 1. The initial value problem

We consider a fixed smooth (i.e., $C^\infty$) two-dimensional oriented Riemannian manifold $(M; g)$, and denote its unit tangent bundle by

$$S^1(M) = \{\xi \in T(M) | g(\xi, \xi) = 1\}.$$

It is a smooth submanifold of the tangent bundle of $M$, and therefore carries a natural Riemannian metric. Moreover, the tangent bundle to the unit tangent bundle splits into the Whitney sum of the bundle of horizontal vectors and the bundle of vertical vectors. We can identify the horizontal vectors with the pull-back of $T(M)$ under the bundle projection $\tau$: $S^1(M) \to M$, i.e., $\tau^*T(M)$; the bundle of vertical vectors is naturally isomorphic to a subbundle of $\tau^*T(M)$, namely

$$\mathrm{Vert} = \{(\mathbf{t}, v) \in \tau^*T(M) | v \perp \mathbf{t}\}.$$

The orthogonal splitting $TS^1(M) = \tau^* T(M) \oplus \mathrm{Vert}$ permits us to decompose the connection $\nabla$ on $S^1(M)$ into two components, one coming from differentiation in the horizontal direction, $\nabla^{\mathrm{h}}$, and its vertical counterpart $\nabla^{\mathrm{v}}$. Thus we have $\nabla = \nabla^{\mathrm{v}} \oplus \nabla^{\mathrm{h}}$.

A $C^1$ regular curve in our manifold $M$ is, by definition, an equivalence class of $C^1$ immersions of the circle $S^1$ into $M$; two such immersions which only differ by an orientation-preserving reparametrisation will be considered to be the same regular curve on the surface.

We let $\Omega(M)$ stand for the space of all $C^1$ regular curves in $M$. For a given $C^1$ curve we write $\mathbf{t}$ and $\mathbf{n}$ for its unit tangent and unit normal vectors, respectively; we shall always assume that $\{\mathbf{t}, \mathbf{n}\}$ is a positively oriented basis of $T_{\gamma(s)}(M)$.

The geodesic curvature of $\gamma \in \Omega(M)$, if it exists, will be denoted by $k_\gamma$, or just $k$.

Given a $C^1$ family of immersions $\gamma(t, \cdot): S^1 \to M$ one can decompose the time derivative $\gamma_t(t, s)$ as $\gamma_t(t, s) = \nu^{\|}\mathbf{t} + \nu^{\perp}\mathbf{n}$. The second component $v^{\perp}$ is independent of the chosen parametrisation of each $\gamma(t, \cdot)$; it is the *normal velocity* of the family of curves.

For any function $V$: $S^1(M) \times \mathbf{R} \to \mathbf{R}$, one can formulate the following *initial value problem*. Given a curve $\gamma_0 \in \Omega(M)$, find a family of curves $\gamma(t) \in \Omega(M)$ $(0 \le t < t_{\mathrm{Max}})$ which, for $t > 0$, have continuous curvature, whose normal velocity satisfies

$$\nu^{\perp} = V(t, k) \tag{1.1}$$

and whose initial value is $\gamma|_{t=0} = \gamma_0$.

Throughout the paper we shall assume that $V$ satisfies at least the following conditions:

($V_1$) $V$ is a locally Lipschitz continuous function,

($V_2$) $\lambda \le \partial V/\partial k \le \lambda^{-1}$ for almost all $(\mathbf{t}, k) \in S^1(M) \times \mathbf{R}$,

($V_3$) $|V(\mathbf{t}, 0)| \le \mu$ for almost all $t \in S^1(M)$

where $\lambda, \mu > 0$ are constants. In addition, we shall often assume that $V$ also has one of the following properties.

($V_4$) $|\nabla(V)| \le \hat{\mu}$ for almost all $(\mathbf{t}, k)$ with $|k| \le 1$,

($V_5$) $|\nabla_{\mathbf{t} \oplus k\mathbf{n}}(V)| \le \nu(1 + |k|^{1+\kappa})$ for almost all $(\mathbf{t}, k) \in S^1(M) \times \mathbf{R}$,

($V_5^*$) $|\nabla^{\mathrm{h}} V| + |k|\,|\nabla^{\mathrm{v}} V| \le \nu(1 + |k|^2)$ for almost all $(\mathbf{t}, k) \in S^1(M) \times \mathbf{R}$.

Here, as above, $\hat{\mu}$ and $\nu$ are positive constants, and $\kappa$ is a constant in the range $1 \le \kappa < \infty$. By $\nabla(V)$ we mean the gradient of $V$ with respect to its first argument $\mathbf{t} \in S^1(M)$, and $\nabla^{\mathrm{v}} V$ and $\nabla^{\mathrm{h}} V$ denote the vertical and horizontal components of $\nabla(V)$.

One can verify that all examples which we mentioned in the introduction satisfy these conditions, just as does any $V$ of the form $V(\mathbf{t}, k) = f(\mathbf{t})k + g(\mathbf{t})$, where $f, g\colon S^1(M) \to \mathbf{R}$ are uniformly Lipschitz functions which satisfy $\lambda \leq f(\mathbf{t}) \leq \lambda^{-1}$ and $|g(\mathbf{t})| \leq \mu$.

One may show easily, using standard results on parabolic equations and assuming $V_1$, $V_2$ and $V_3$ and also some extra smoothness of $V$ (say $V \in C^2$), that the initial value problem will have a local solution (in time) for any initial curve which is $C^{2+\alpha}$, i.e., which has Hölder continuous curvature. This is done in Section 3. By means of an approximation argument this result could then be extended to arbitrary initial curves whose curvature is bounded.

Under assumption $V_5$ we can enlarge the class of allowable initial data. We get the following:

THEOREM A. *If $V$ satisfies $V_1 \cdots V_5$ with $\kappa > 1$, then the initial value problem has a short time solution for any initial curve $\gamma_0$ whose curvature belongs to $L_p$, i.e. for which*

$$\int_{\gamma_0} |k(s)|^p \, ds < \infty$$

*holds for some $p \in (\kappa, \infty)$. Moreover, if $[0, t_{\mathrm{Max}})$ is the maximal time interval on which the solution exists, then either $t_{\mathrm{Max}} = \infty$, or the $L_p$ norm of the curvature becomes unbounded as $t \to t_{\mathrm{Max}}$.*

The proof is given in Section 8, which uses the pointwise estimates for the curvature in terms of its $L_\kappa$ norm. These estimates are derived in Section 7, by means of a Nash-Moser-like iteration method.

If we replace $V_5$ by the stronger hypothesis $V_5^*$, then we get a stronger statement.

THEOREM B. *Let $V$ satisfy $V_1 \cdots V_5^*$; then the initial value problem is solvable for any initial curve which is locally the graph of a Lipschitz function. If the maximal solution exists for a finite time, say $t_{\mathrm{Max}}$, then*

$$\liminf_{t \to t_{\mathrm{Max}}} \left\{ \sup_{|s_1 - s_0| < \varepsilon} \left| \int_{s_0}^{s_1} k(s, t) \, ds \right| \right\} \geq \pi$$

*holds for any $\varepsilon > 0$.*

So in this case a solution can only blow up if it develops a kink of at least 180 degrees. The proof is spread out over Parts I and II. In this part we shall show that the theorem holds for initial curves whose curvature is $p$-integrable, for some $p > 1$ (this is just Theorem A), and that the description of blow-up holds with "lim inf" replaced by "lim sup" (Theorem 9.1). These restrictions will be removed in Part II.

The strongest results we get hold in the case where the evolution of the curve does not depend on its orientation. This is exactly the case if $V$ satisfies the following symmetry condition:

(S) $\quad V(\mathbf{t}, k) = -V(-\mathbf{t}, -k) \quad \text{for all} \quad \mathbf{t} \in S^1(M) \quad \text{and} \quad k \in \mathbf{R}.$

THEOREM C. *Let V satisfy* $V_1 \cdots V_5^*$ *and* S; *then the initial value problem has a solution for any initial curve which is* $C^1$ *locally graph-like.*

We shall call a continuous map $\gamma\colon S^1 \to M$ a parametrised $C^1$ locally graph-like curve, if $\gamma$ is locally a homeomorphism, and if for each $\xi \in S^1$, one can find $C^1$ coordinates $(x, y)$ on $M$ near $\gamma(\xi)$ such that the image under $\gamma$ of a small interval $(\xi - \delta, \xi + \delta)$ is the graph of a continuous function $y = f(x)$. A $C^1$ locally graph-like curve is an equivalence class of parametrised $C^1$ locally graph-like curves, where two such curves are equivalent if and only if they differ by a continuous reparametrisation.

In particular, locally Lipschitz curves are $C^1$ locally graph-like, but a $C^1$ locally graph-like curve can also have isolated cusps, and worse singularities.

## 2. The space of regular curves

Any regular curve admits a constant speed parametrisation $\gamma\colon S^1 \to M$, i.e., one for which the vector $\gamma'(s) \in T_{\gamma(s)}(M)$ has constant length. Since $S^1 = \mathbf{R}/\mathbf{Z}$ has length one, the length of the vector $\gamma'(s)$ is exactly the length of the curve $\gamma$. This constant speed parametrisation is unique, up to a rigid rotation of $S^1$. In other words, if we define

$$\hat{\Omega} = \{\gamma \in C^1(S^1, M) \colon |\gamma'(s)| \neq 0 \text{ is constant}\},$$

then we have an $S^1$ action on $\hat{\Omega}$ given by

$$(\theta \cdot \gamma)(s) = \gamma(s + \theta)$$

and we can define the space of regular curves in $M$ to be the quotient of $\hat{\Omega}$ by this action:

$$\Omega(M) = \hat{\Omega}/S^1.$$

Using the $C^1$ topology on $\hat{\Omega}$, we get a topology on $\Omega(M)$, which turns out to be metrisable and complete.

One can give $\Omega(M)$ the structure of a topological Banach manifold; i.e., every point in $\Omega(M)$ has a neighbourhood which is homeomorphic to an open subset of a Banach space ($C^1(S^1)$ to be precise). The construction of such neighbourhoods goes as follows.

Let $\gamma_0 \in \Omega(M)$ be a regular curve, with parametrisation $\gamma_0: S^1 \to M$. This parametrisation can be extended to an immersion $\sigma: [-1, 1] \times S^1 \to M$, where $\sigma|\{0\} \times S^1 = \gamma_0$. Clearly, any regular curve which is $C^1$ close to $\gamma_0$ can be parametrised as $\gamma_u(s) = \sigma(s, u(s))$ for some $C^1$ function $u$ with $|u(s)| < 1$ ($s \in S^1$). The correspondence $u \in C^1(S^1) \to \gamma_u \in \Omega(M)$ is the desired homeomorphism.

The homeomorphisms we have just defined show that $\Omega(M)$ is a topological Banach manifold. It turns out that the coordinate transformations that go with these homeomorphisms are, in general, not $C^1$, so that we cannot claim that we have given $\Omega(M)$ a differentiable structure.

We shall occasionally talk about the *lift* or *pull-back* of a curve under a local homeomorphism, in which case we shall have the following in mind.

If $\gamma: S^1 \to M$ is a continuous map, and $\sigma: S^1 \times [-1, 1] \to M$ is a local homeomorphism such that $\gamma$ can be lifted to a map $\Gamma: S^1 \to S^1 \times [-1, 1]$ (i.e., so that $\gamma = \sigma \circ \Gamma$), then we shall write $\sigma^*(\gamma)$ for the curve $\Gamma$. Given $\sigma$ and $\gamma$, the lift $\sigma^*(\gamma)$ need not be uniquely determined, unless we choose one specific value for $\Gamma(t_0) \in \sigma^{-1}(\gamma(t_0))$ for one $t_0 \in S^1$. However, once the lift $\Gamma$ is chosen, there is a unique lift $\Gamma_1 = \sigma^*(\gamma_1)$ which is close to $\Gamma$, for any curve $\gamma_1$ close to $\gamma$ (in the $C^0$ topology).

## 3. Short time existence for smooth initial data

We shall say that $\gamma: [0, t_0) \to \Omega(M)$ is a *classical solution*, or just a solution, for short, of (1.1), if

(i) $\gamma \in C([0, t_0); \Omega(M))$,

(ii) for each $t \in (0, t_0)$, $\gamma(t)$ has continuous curvature and normal velocity, and $\gamma(t)$ (of course) satisfies $\nu^\perp = V(\mathbf{t}, k)$.

A solution $\gamma: [0, t_0) \to \Omega(M)$ will be called *maximal*, if it cannot be extended to a classical solution on a strictly larger interval $[0, t_1) \supset [0, t_0)$.

THEOREM 3.1. *Assume* $V: S^1(M) \times \mathbf{R} \to \mathbf{R}$ *is a* $C^{1,1}$ *function which satisfies*

$$\frac{\partial V}{\partial k} > 0 \quad \textit{for all } (\mathbf{t}, k) \in S^1(M) \times \mathbf{R}.$$

*Let* $\gamma_0$ *be a regular curve with Hölder continuous curvature. Then there exists a unique maximal solution* $\gamma: [0, t_{\mathrm{Max}}) \to \Omega(M)$ *with initial value* $\gamma(0) = \gamma_0$.

*If V is a* $C^{m,1}$ *function, for some* $m \geq 1$, *then the solution* $\gamma(t)$ *is a* $C^{m+2,\alpha}$ *curve for any* $t > 0$, *and any* $0 < \alpha < 1$.

*If V, the manifold M and its metric g are real analytic, then so is the solution* $\gamma(t)$ *for* $t > 0$.

*Proof.* As in the previous section, we can extend $\gamma_0$: $S^1 \to M$ to an immersion $\hat{\sigma}$ of the annulus $[-1,1] \times S^1$ into $M$, and perturb it slightly, so that it becomes $C^\infty$ smooth. If we keep this perturbation small enough, then our curve $\gamma_0$ can be parametrised by a small $C^1$ function $u_0$: $S^1 \to (-1,1)$, i.e., by $x \to \sigma(x, u_0(x))$. Nearby curves in the $C^1$ topology will have a similar parametrisation. Since our curve has Hölder continuous curvature and $\sigma$ is smooth, the function $u_0$ will be $C^{2+\alpha}$, for some $0 < \alpha < 1$.

Any classical solution $\gamma$: $[0, t_{\mathrm{Max}}) \to \Omega(M)$ starts off close to its initial data, so that we may represent an initial section of this solution as the image under $\sigma$ of the graph of a function $u(t,x)$ of two variables, i.e., as $\gamma(t,x) = \sigma(x, u(t,x))$.

To compute the curvature of $\gamma(t,\cdot)$ in terms of $u$ and its derivatives, we consider the pull-back of the metric $g$ on $M$ under $\sigma$:

$$\sigma^*(g) = (ds)^2 = A(x,y)(dx)^2 + 2B(x,y)\,dx\,dy + C(x,y)(dy)^2.$$

Here $A$, $B$ and $C$ are $C^\infty$ functions on $S^1 \times [-1,1]$ which satisfy

$$D = AC - B^2 > 0.$$

If we define $l = A + 2Bu_x + Cu_x^2$, then the unit tangent $\mathbf{t}$ to the curve $\gamma(t,\cdot)$ is the image under $d\sigma$ of

$$T = l^{-1/2}(\partial_x + u_x \partial_y),$$

and the unit normal $\mathbf{n}$ is the image of

$$N = (lD)^{-1/2}\{-(B + Cu_x)\partial_x + (A + Bu_x)\partial_y\}$$

where the $A$, $B$, $C$ and $D$ should be evaluated at $(x, u(t,x))$. Using the Frenet formulae one then arrives at the following expression for the geodesic curvature of $\gamma(t,\cdot)$:

$$k = l^{-3/2} D^{1/2}\left(u_{xx} + P + Qu_x + Ru_x^2 + Su_x^3\right). \tag{3.1}$$

Again, $P$, $Q$, $R$ and $S$ are $C^\infty$ functions of $(x,y)$ evaluated at $y = u(t,x)$. They enter the expression for the geodesic curvature as the covariant derivatives of the vector fields $\partial_x$ and $\partial_y$, and can be expressed in terms of the Christoffel symbols of the metric in the $(x,y)$ coordinates.

The vertical velocity of the family of curves $\gamma(t,x)$ is given by the vector $u_t\partial_y$, so that its normal velocity is given by

$$v^\perp = \sigma^* g(u_t \partial_y, N) = l^{-1/2} D^{1/2} u_t.$$

We conclude from these computations that $\gamma$: $[0, t_0) \to \Omega(M)$ satisfies (1.1) if and only if the function $u$ satisfies

$$u_t = F(x, u, u_x, u_{xx}) \tag{3.2}$$

where $F$ is given by

$$F(x,u,u_x,u_{xx}) = l^{1/2}D^{-1/2}V(d\sigma\cdot T,k), \tag{3.3}$$

and $T, k$ depend on $(x,u,u_x,u_{xx})$ as above.

Clearly, $F$ is a $C^{1,1}$ function of its four arguments; it is well-defined for all $(x,u,u_x,u_{xx})$ with $|u|\le 1$, and it satisfies

$$\frac{\partial}{\partial q}F(x,u,p,q) = l^{-1}\frac{\partial V}{\partial k} > 0, \tag{3.4}$$

for any $(x,u,p,q)$ in the domain of $F$ (i.e. in $S^1\times[-1,1]\times\mathbf{R}^2$). Therefore (3.2) is a parabolic partial differential equation, as promised.

This construction allows us to appeal to the strong maximum principle for linear parabolic equations, and to conclude that the classical solution of (1.1) is indeed unique, if it exists.

We can also apply the existing theory for parabolic initial value problems to construct local solutions of (3.2). If all ingredients such as $V$ and the initial curve are smooth, then we can apply the results in Eidelman's treatise [Ei] (in particular Theorem 7.3 on page 311) to conclude the existence of a local solution which is smooth.

We shall now outline an approach due to DaPrato and Grisvard ([DPG]), and extended in [A1]. As we tried to point out in [A1], one of the advantages of this approach is the ease with which one can prove smooth and even analytic dependence of the solution on parameters. This in turn leads to an almost trivial proof of the smoothing effect of the parabolic equation.

The procedure is as follows: Introduce the Banach spaces

$$E_0 = h^\alpha(S^1);\qquad E_1 = h^{2+\alpha}(S^1)$$

where $h^\beta(S^1)$ denotes the little Hölder space of exponent $\beta$, i.e., the closure of $C^\infty(S^1)$ in the usual Hölder space equipped with the usual Hölder norm. Then when we assume that $V$, and therefore also $F$, are $C^{m,1}$ functions, the nonlinear differential operator

$$\mathbf{F}\colon u\in O_1 \to F(x,u,u_x,u_{xx})\in E_0$$

is a $C^{m-1,1}$ mapping of the open subset

$$O_1 = \{u\in E_1 | -1 < u < 1\}$$

of $E_1$ to the Banach space $E_0$. Its Fréchet derivative at a $u_0\in O_1$ is given by the linear operator

$$d\mathbf{F}(u_0)\cdot v = F_q v_{xx} + F_p v_x + F_u v.$$

Since the operator $d\mathbf{F}(u_0)$ with domain $h^{2+\beta}$ generates an analytic semigroup in $h^\beta$ for any $\beta \in (0, \alpha]$, we can apply Theorem 4.1 of [DPG] to conclude the existence of a possibly short-lived solution $u$: $[0, t_0] \to E_1$ of (3.2).

Using the uniqueness of the solution, given the initial value $\gamma_0$, one easily shows that there is a unique maximal solution; indeed, if one orders the solutions by inclusion (i.e., $\gamma_1 \le \gamma_2$ if $\gamma_1$ is obtained from $\gamma_2$ by restriction to a smaller time interval), then this ordering is linear, and the maximal solution is nothing but the union of all possible solutions.

In [A1] we remarked that the construction in [DPG], combined with the implicit function theorem on Banach spaces, shows that the solution $u \in C([0, t_0]; E_1)$ depends $C^{m-1,1}$ on any parameters which occur in the nonlinearity $\mathbf{F}$. In particular, this implies that equation (3.2) generates a $C^{m-1,1}$ local semiflow

Another trivial consequence of this construction is the smoothing effect of the equation. Indeed, if we define

$$u_{a,b}(t, x) = u(at, x + bt),$$

then for $a$ close to 1 and any $b \in \mathbf{R}$, $u_{a,b}$ satisfies

$$u_t = \mathbf{F}_{a,b}(x, t, u, u_x, u_{xx}),$$

where $\mathbf{F}_{a,b} = a \cdot F(x - bt, u, u_x, u_{xx})$. The corresponding nonlinear differential operator $u \to \mathbf{F}_{a,b}(u)$ is $C^{m-1,1}$ both in $u$ and in the parameters $a$ and $b$. Hence the solution $u_{a,b} \in C([0, t_0]; E_1)$ depends $C^{m-1,1}$ on $a$ and $b$. In particular, for any $t \in (0, t_0]$, the partial derivatives

$$\left.\frac{\partial^{j+k} u_{a,b}}{\partial a^j \partial b^k}\right|_{a=1, b=0} = t^{j+k} \frac{\partial^{j+k} u}{\partial t^j \partial x^k}$$

belong to $L_\infty([0, t_0]; C^\alpha(S^1))$ for $j + k \le m$, and we have the estimate

$$\left\| \frac{\partial^{j+k} u}{\partial t^j \partial x^k} \right\|_{C^{2+\alpha}} \le c_{j,k} t^{-j-k}.$$

Also, if $m = \omega$, i.e. if $\mathbf{F}$ is real analytic, then so is the solution. Thus, if the manifold $M$, the metric $g$ and the velocity function $V$ are real analytic, then all classical solutions are real analytic. □

## 4. Bounds for the length and the total curvature

In this section we shall assume that $V$ satisfies $V_1, \dots, V_4$. We shall show that if $\gamma$: $[0, t_0] \to \Omega(M)$ is a family of curves which evolves according to (1.1), then the length and total curvature of $\gamma(t)$ remain bounded on finite time intervals. More precisely, we prove the following:

THEOREM 4.1. *The length $L(t)$ and total curvature $K(t)$ of a classical solution $\gamma$: $[0, t_0) \to \Omega(M)$ satisfy*

$$L(t) \leq L(0)e^{(\mu^2/4\lambda)t},$$

$$K(t) \leq c_1(K(0) + L(0))e^{c_2 t},$$

*where the constants $c_1, c_2$ depend only on $\lambda$, $\mu$, $\hat{\mu}$, and $R^*$.*

One should view these estimates as a generalisation of those of Abresch and Langer [AL, Theorem B]. Theorem 4.1 will follow from some identities, which we shall now derive.

*Proof.* We may assume that $V$ and the solution $\gamma$ are as smooth as we like: given the estimates for smooth solutions, an approximation argument will prove that they also hold for the general classical solution.

Let $\gamma$: $[0, t_0) \to \Omega(M)$ be a solution of the initial value problem. Fix a $t_1 \in (0, t_0)$, choose a constant speed parametrisation of $\gamma(t_1)$ and let $x, y$ be Fermi coordinates near the regular curve $\gamma(t_1)$. In other words, extend the immersion $\gamma$: $S^1 \to M$ to an immersion $\sigma$: $S^1 \times [-\varepsilon, +\varepsilon] \to M$ for which the pull-back of the metric on $M$ has the form

$$\sigma^* g = A(x, y)(dx)^2 + (dy)^2$$

with $A(x, 0) \equiv$ constant.

The scalar curvature of $M$ is given by Gauss's formula

$$R = -\frac{1}{\sqrt{A}} \frac{\partial^2 \sqrt{A}}{\partial y^2},$$

and the geodesic curvature of the graph $y = u(x, t)$ is

$$k = \frac{A^{1/2}}{(A + p^2)^{3/2}} \left\{ q - \frac{A_y}{A} p^2 - \frac{A_x}{2A} p - \frac{1}{2} A_y \right\} \tag{4.1}$$

where $p = u_x$, $q = u_{xx}$, and $A_x$ and $A_y$ denote the partial derivatives of $A$ evaluated at $y = u(x)$.

In these coordinates the curve $\gamma(t)$ (with $t$ close to $t_1$) is given as the graph of a function $y = u(t, x)$, where $u(t_1, x) \equiv 0$. Thus, at $t = t_1$ the normal velocity of the curve is $v^\perp = u_t$; if we differentiate (4.1) with respect to $t$, and put $t = t_1$, then we can eliminate $u_t$ from the resulting equation for $k_t$. If we also use the fact that, at $t = t_1$, $u$ vanishes, so that $\partial/\partial s = A^{-1/2}\partial/\partial x$, then we find that

$$\frac{\partial k}{\partial t} = \frac{\partial^2 v^\perp}{\partial s^2} + (k^2 + R)v^\perp . \tag{4.2}$$

The time derivative $k_t$ can be described without referring to the coordinates $x, y$ in the following manner. Given the family of curves $\gamma$: $[0, t_0) \to \Omega(M)$, choose a parametrisation $\gamma$: $S^1 \times [0, t_0) \to M$ for which the time derivative $\gamma_t$ is always orthogonal to the curve; in this way any quantity, such as the geodesic curvature $k$, which is defined on the curve, may be considered as a function of $(x, t) \in S^1 \times [0, t_0)$. Then $k_t$ is its derivative with respect to time, with $x$ constant.

The time derivative can also be described as the covariant derivative in the direction of the vector field $v^\perp \mathbf{n}$: $k_t = \nabla_{v^\perp n}(k)$.

If we write

$$ds = \sqrt{A + p^2}\, dx$$

for the arclength element along the graph $y = u(t, x)$, then its time derivative is

$$\frac{\partial}{\partial t} ds = -kv^\perp\, ds.$$

This implies that the length of the curve grows according to

$$L'(t) = -\int_{\gamma(t)} kv^\perp\, ds.$$

The hypotheses $V_2$ and $V_3$ imply that $kV(\mathrm{t}, k) \geq \mu^2/4\lambda$, so that we have $L'(t) \leq \mu^2/4\lambda L(t)$, from which the first estimate in Theorem 4.1 follows immediately.

To get the estimate for the total curvature we consider a more general quantity, which will also be of use later on. Let $\psi \in C^2(\mathbf{R})$ be a nonnegative convex function, and consider

$$\Psi(t) = \int_{\gamma(t)} \psi(k)\, ds. \tag{4.3}$$

Then one has

$$\begin{aligned} \Psi'(t) &= \int_{\gamma(t)} \{\psi'(k)\big(v_{ss}^\perp + (R + k^2)v^\perp\big) - \psi(k)kv^\perp\}\, ds \\ &= \int_{\gamma(t)} \{-\psi''(k)k_s v_s^\perp + \big(R\psi'(k) + k^2\psi'(k) - k\psi(k)\big)v^\perp\}\, ds \\ &= I_1 + I_2. \end{aligned} \tag{4.4}$$

If the family of curves evolves according to (1.1), then one has

$$v_s^\perp = \nabla_{\mathbf{t}\oplus k\mathbf{n}}(V) + \frac{\partial V}{\partial k}k_s.$$

Therefore, since $v^\perp = V(t, k)$, one has for any $\varepsilon > 0$

$$\nu_s^\perp k_s \geq \frac{\partial V}{\partial k}k_s^2 - \frac{1}{4\varepsilon}\left(\nabla_{\mathbf{t}\oplus k\mathbf{n}}(V)\right)^2 - \varepsilon k_s^2.$$

If we choose $\varepsilon = \lambda/2$, and recall hypothesis $V_2$, then this leads us to

$$\nu_s^\perp k_s \geq \frac{\lambda}{2}k_s^2 - \frac{1}{2\lambda}\left(\nabla_{\mathbf{t}\oplus k\mathbf{n}}(V)\right)^2. \tag{4.5}$$

Now let $\psi$ be a function for which $\psi(k) = |k|$ holds when $|k| \geq 1$, and $0 \leq \psi''(k) \leq 1$ when $|k| < 1$ (e.g. $\psi(k) = (1 + k^2)/2$). Then, from (4.5) and hypothesis $V_4$, we get $\psi''(k)\nu_s^\perp k_s \leq \hat{\mu}^2/2\lambda$, and therefore

$$I_1 \leq \frac{\hat{\mu}^2}{2\lambda}L(t).$$

To estimate the second term, we observe that $0 \leq \psi(k) - k\psi'(k) \leq 1/2$, and that $\psi(k) - k\psi'(k)$ vanishes for $|k| \geq 1$. In addition to this we also have $|\psi'(k)| \leq 1$ for all $k$, and finally, it follows from $V_2$ and $V_3$ that $|\nu^\perp| \leq \mu + \lambda^{-1}|k|$. Together, these properties of $\psi$ and $V$ imply the following estimate for $I_2$:

$$\begin{aligned} I_2 &\leq \int_{\gamma(t)} (\mu + \lambda^{-1}|k|)R^*\,ds + \int_{|k|\leq 1} \tfrac{1}{2}|k\nu^\perp|\,ds \\ &\leq \lambda^{-1}R^*K(t) + \left(\left(R^* + \tfrac{1}{2}\right)\mu + (2\lambda)^{-1}\right)L(t). \end{aligned}$$

Adding $I_1$ and $I_2$ together one finds that $\Psi'(t) \leq c(K(t) + L(t))$, and using $K(t) \leq \Psi(t) \leq K(t) + L(t)$ one can derive the exponential bound on the total absolute curvature in Theorem 4.1. □

## 5. The average speed

Using the maximum principle, we shall obtain an upper bound for the distance a family of curves, which evolves according to (1.1), can cover in a given time.

We assume in this section that $V$ satisfies $V_1$, $V_2$ and $V_3$.

THEOREM 5.1. *There is a constant $t_* > 0$, which only depends on $\lambda$, $\mu$ and $R^*$, such that for any family of curves $\gamma$: $[0, t_0) \to \Omega(M)$ which satisfies* (1.1),

$$\gamma(t) \subset \mathbf{N}_{2\sqrt{t/\lambda}}(\gamma_0)$$

*for* $0 \leq t < \min(t_0, t_*)$.

(We abuse notation slightly by identifying the curves $\gamma(t)$ with the corresponding subsets of $M$; $\mathbf{N}_\varepsilon(A)$ denotes the closed $\varepsilon$ neighbourhood of the set $A \subset M$.)

COROLLARY. *There are constants $a, b > 0$ which only depend on $\lambda$, $\mu$ and $R^*$ such that*

$$\gamma(t) \subset \mathbf{N}_{a+bt}(\gamma_0)$$

*holds for all $t > 0$.*

*Proof.* We consider an arbitrary point $p \in M$, which does not lie on the initial curve $\gamma_0$, and define $d(t)$ to be the distance from $p$ to the curve $\gamma(t)$. Since the family of curves has a $C^1$ parametrisation $\gamma$: $(0, t_0) \times S^1 \to M$, this distance is a Lipschitz continuous function of time.

The exponential map $\exp_p$: $T_pM \to M$ is an immersion on a disk of radius $\rho = \pi/\sqrt{R^*}$. If at some moment in time $t$ the distance function becomes less than $\rho$, then we can choose a point $q$ on $\gamma(t)$ which minimizes $\operatorname{dist}(p, q)$; this point must lie in the image of the $\rho$-disk in $T_pM$ under the exponential map.

Let $\Gamma$ be the pre-image of $\gamma(t)$ in the $\rho$-disk in $T_pM$ under $\exp_p$. Choosing polar coordinates $(r, \phi)$ in the tangent space $T_pM$, we can represent $\Gamma$ near $\exp_p^{-1}(q)$ as a graph $r = u(\phi)$. If $q$ has polar coordinates $r = d(t)$, $\phi = 0$, then $u(0) = d(t)$ and $u'(0) = 0$. Moreover, $u$ has a local minimum at $\phi = 0$, so that $u''(0) \geq 0$. Hence the geodesic curvature of $\gamma(t)$ at $q$ is at least the curvature of the geodesic circle with radius $d(t)$ and centre $p$ (this follows from (4.1)). See Figure 5.1.

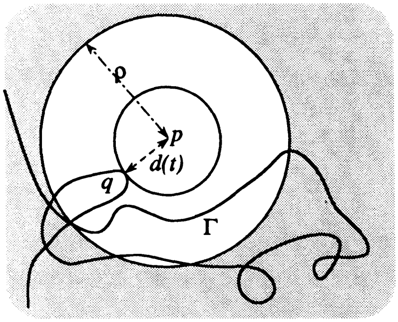


FIGURE 5.1

If the metric in polar coordinates is given by

$$(ds)^2 = (dr)^2 + A(r, \phi)(d\phi)^2,$$

then curvature of the circle $r =$ constant, which we denote by $k(r, \phi)$, and the scalar curvature $R$ of $M$ satisfy

$$k = -\frac{1}{\sqrt{A}}\frac{\partial\sqrt{A}}{\partial r}; \qquad R = -\frac{1}{\sqrt{A}}\frac{\partial^2\sqrt{A}}{\partial r^2},$$

so that $k$ also satisfies the following Ricatti equation

$$\frac{\partial k}{\partial r} = k^2 + R.$$

Our assumption that the scalar curvature $R$ is bounded by $|R| \le R^*$ then implies that

$$k(r,\phi) \ge -\sqrt{R^*}\coth(r\sqrt{R^*})$$

(in fact, we only need a lower bound for the scalar curvature for this). Using the calculus inequality $\coth(x) < 1 + x^{-1}$ we therefore find

$$k(r,\phi) \ge -r^{-1} - \sqrt{R^*}.$$

Recall that $V(t,k) \ge -\mu + \min(0,k)/\lambda$ (hypotheses $V_2$ and $V_3$), so that the velocity of the curve $\gamma(t)$ at $q$ satisfies

$$\nu^\perp \ge -\mu - \left(r^{-1} + \sqrt{R^*}\right)/\lambda,$$

with $r = d(t)$. Therefore the distance function $d(t)$ satisfies

$$d'(t) \ge -\mu - \lambda^{-1}\sqrt{R^*} - (\lambda d(t))^{-1} \ge -2(\lambda d(t))^{-1}, \tag{5.1}$$

whenever

$$d(t) \le d_* =_{\text{def}} \left(\lambda\mu + \sqrt{R^*}\right)^{-1}$$

and $d(t)$ is less than the immersivity radius of $(M, g)$, which exceeds $\rho$. Since $\rho = \pi/\sqrt{R^*}$ this automatically holds if $d(t) \le d_*$.

Now define

$$t_* = \frac{\lambda}{4}\left(\lambda\mu + \sqrt{R^*}\right)^{-2}.$$

If $p$ lies on $\gamma(t)$ for some $0 < t < t_*$, then $d(t) = 0$, and, by integrating (5.1), one finds that

$$0 = d(t)^2 \ge d(0)^2 - 4t/\lambda,$$

which means that $p$ lies in a $2\sqrt{t/\lambda}$ neighbourhood of $\gamma_0$. Since $p$ was any point on $\gamma(t)$, the theorem follows.

This proof gives us an explicit estimate for $t_*$ and allows us to estimate the maximal large scale speed a family of curves obeying (1.1) can have (i.e. the coefficient $b$ of the corollary to Theorem 5.1). Indeed, in a time interval $t_*$ the curve cannot travel further than $d_* = 2\sqrt{t_*/\lambda}$, so that

$$b \leq \frac{d_*}{t_*} \leq 4\left(\mu + \frac{\sqrt{R^*}}{\lambda}\right). \qquad \square$$

We conclude this section with two examples, to illustrate the theorem and its corollary.

In the first example we let $(M, g)$ be the hyperbolic plane, with constant negative curvature $R \equiv -1$, and we let $V(\mathbf{t}, k) \equiv k$; in other words, we consider the standard curve shortening problem. If we choose a circle with radius $r(0)$ as our initial curve, then it follows from symmetry considerations that corresponding solutions of the initial value problem will also consist of (concentric) circles.

The metric in geodesic polar coordinates is

$$(ds)^2 = (dr)^2 + \sinh^2(r)(d\phi)^2,$$

so that the geodesic curvature of a circle with radius $r$ is $-\coth(r)$. Therefore the radius of the shrinking family of circles satisfies $r'(t) + \coth(r(t)) = 0$, and thus

$$r(t) = \operatorname{arcosh}(e^{T-t}) \quad (0 \leq t < T),$$

where $T = \log\cosh r(0)$.

The point of this example is that initially the curve will shrink with speed close to one, and that we can make the interval on which this happens as long as we like by choosing $T$ large enough. The upper bound for the large scale speed which our theorem gives is $b = 4$.

In our other example we choose $M = S^1 \times (0, \infty)$, with coordinates $(\phi, r)$ again (identify $S^1$ and $\mathbf{R}/2\pi\mathbf{Z}$), and we let the metric be given by

$$(ds)^2 = (dr)^2 + e^{-2r^\alpha}(d\phi)^2$$

where $\alpha$ is a positive constant. As above, we consider the curve shortening problem, $V \equiv k$, and determine the evolution of a circle $r = r(t)$. The geodesic curvature of such a circle is given by $k = \alpha r^{\alpha-1}$, so that a circle with radius $r(t)$ will evolve according to $r'(t) = \alpha r^{\alpha-1}$. After integrating this equation one finds that

$$\begin{aligned} r(t) &= (\alpha(2-\alpha)t)^{1/(2-\alpha)} && (0 < \alpha < 2) \\ &= e^t && (\alpha = 2) \\ &= (\alpha(2-\alpha)(T-t))^{1/(2-\alpha)} && (\alpha > 2). \end{aligned}$$

Thus, if $\alpha \leq 1$ the (large scale) speed of the circle remains bounded, but if

$\alpha > 1$ the large scale speed becomes unbounded, and even blows up in finite time if $\alpha > 2$.

On the other hand the scalar curvature of $(M, g)$ is given by

$$R = \frac{\partial k}{\partial r} - k^2 = \alpha(\alpha - 1)r^{\alpha-2} - \alpha^2 r^{2\alpha-2},$$

which is bounded from below if and only if $\alpha \leq 1$.

## 6. The limit curve at blow-up time

We assume $V$ satisfies $V_1, \ldots, V_4$, and consider a classical solution, $\gamma\colon [0, t_0) \to \Omega(M)$, of (1.1).

THEOREM 6.1. *As $t \to t_0$, the curves $\gamma(t)$ converge, in the Hausdorff metric, to a curve $\gamma^*$, which has finite total absolute curvature.*

The limit curve $\gamma^*$ need not be smooth, and, in particular, does not have to belong to $\Omega(M)$.

*Proof.* We begin by choosing a parametrisation $\gamma\colon S^1 \times [0, t_0) \to M$ of our family of curves, which, for each $t$, is a constant speed parametrisation of $\gamma(t)$.

By Theorem 5.1 all the $\gamma(t)$ lie in some bounded, and hence compact, region of $M$. Moreover, their lengths are uniformly bounded, so that the $\gamma(t, \cdot)$ are equicontinuous maps from $S^1$ to $M$. The Ascoli-Arzela theorem allows us to extract a uniformly convergent subsequence $\gamma(t_n, \cdot)$, whose limit $\gamma^*$ is a Lipschitz continuous map from $S^1$ to $M$. The bound for the total absolute curvature, which we derived in Section 4, implies that $\gamma^*$ also has finite total absolute curvature. In other words, except at a finite number of points, $\gamma^*$ is locally the graph of a Lipschitz continuous function, whose derivative is of bounded variation.

Clearly, the sequence $\gamma(t_n)$ converges in the Hausdorff metric on compact subsets of $M$ to $\gamma^*$. We complete the proof by showing that all the $\gamma(t)$ converge to $\gamma^*$ when $t \to t_0$.

Let $\varepsilon > 0$ be given, and choose a $t_n$ for which $\gamma(t_n) \subset \mathbf{N}_{\varepsilon/2}(\gamma^*)$, and $t_n < t_0 - \lambda\varepsilon^2/16$ holds. If $\varepsilon$ is small enough then we also have $t_0 - t_n < t_*$, so that we can apply Theorem 5.1. We find that for any $t_n < t < t_0$,

$$\gamma(t) \subset \mathbf{N}_{\varepsilon/2}(\gamma(t_n)) \subset \mathbf{N}_\varepsilon(\gamma^*)$$

and also (choosing $t = t_k$, and letting $k \to \infty$),

$$\gamma^* \subset \mathbf{N}_{\varepsilon/2}(\gamma(t)) \subset \mathbf{N}_\varepsilon(\gamma(t)).$$

Since $\varepsilon > 0$ is arbitrary we may conclude that the $\gamma(t)$ do indeed converge in the Hausdorff metric. $\square$

Let $\gamma$: $[0, t_{\mathrm{Max}}) \to \Omega(M)$ be a maximal classical solution, and choose a parametrisation $\Gamma$: $S^1 \times [0, t_{\mathrm{Max}}) \to M$ of $\gamma$ whose time derivative $\Gamma_t$ is always orthogonal to the curve $\gamma(t)$. For this particular parametrisation the following holds.

THEOREM 6.2. *As $t \to t_{\mathrm{Max}}$, $\Gamma(t, \cdot\,)$ converges uniformly to a continuous map $\Gamma^* \in C^0(S^1; M)$.*

*Proof.* If we denote the coordinate on $S^1$ by $u$, then arclength on $\gamma(t)$ is given by

$$ds = J(u, t)\, du,$$

where $J(u, t)$ is the length of $\Gamma_u(u, t)$. In Section 4 we observed that $J_t = -kv^{\perp} J$. This implies that $e^{-\mu^2 t/4\lambda} J$ is a nonincreasing function of $t$, so that its limit for $t \to t_{\mathrm{Max}}$ must exist. Dividing by the exponential we see that $J(u, t)$ converges pointwise to some function $J^*(u)$ as $t \to t_{\mathrm{Max}}$.

From our bound on the total absolute curvature we get

$$\int_0^{t_{\mathrm{Max}}} \int_{S^1} |k(u, t)| J(u, t)\, du\, dt < \infty.$$

Fubini's theorem implies that for almost every $u \in S^1$,

$$\int_0^{t_{\mathrm{Max}}} |k(u, t)| J(u, t)\, dt < \infty. \tag{6.1}$$

Since $|\Gamma_u| = J$ is bounded from above, the $\Gamma(\cdot, t)$ are uniformly Lipschitz, and it suffices to prove pointwise convergence of the $\Gamma(\cdot, t)$. We may also assume that the length of $\gamma(t)$ is bounded away from zero, for otherwise the arguments of the previous theorem show that $\Gamma(\cdot, t)$ converges uniformly to a constant.

Let $u_0 \in S^1$ be given, and assume (6.1) holds for this $u_0$. If $J^*(u_0) > 0$, then $J(u_0, t)$ is bounded from below on $[0, t_{\mathrm{Max}})$. Now (6.1), together with the inequality $|\Gamma_t| = |v^{\perp}| \le \mu + |k/\lambda|$, imply that

$$\int_0^{t_{\mathrm{Max}}} |\Gamma_t(u_0, t)|\, dt < \infty.$$

Hence $\Gamma(u_0, t)$ converges as $t \to t_{\mathrm{Max}}$.

If (6.1) does not hold for $u_0$, or if $J^*(u_0) = 0$, then, for any given $\varepsilon > 0$ we can find a $u_1 < u_0$ such that

$$\int_{u_1}^{u_0} J^*(u)\, du < \varepsilon,$$

and for which both (6.1) and $J^*(u_1) > 0$ hold (here we use the assumption that the length of the curve $\gamma(t)$ does not vanish as $t \to t_{\mathrm{Max}}$).

By the dominated convergence theorem there will be a $\delta > 0$ such that

$$\operatorname{dist}(\Gamma(u_1, t), \Gamma(u_0, t)) \leq \int_{u_1}^{u_0} J(u, t)\, du < \varepsilon$$

and

$$\operatorname{dist}(\Gamma(u_1, t), \Gamma(u_1, s)) < \varepsilon$$

hold for $t_{\mathrm{Max}} - \delta < t, s < t_{\mathrm{Max}}$.

The triangle inequality then implies

$$\operatorname{dist}(\Gamma(u_0, t), \Gamma(u_0, s)) < 3\varepsilon$$

for $t_{\mathrm{Max}} - \delta < t, s < t_{\mathrm{Max}}$, so that $\Gamma(u_0, t)$ also converges. □

## 7. Integral bounds for the curvature

The results in Section 3 imply that, if $\gamma$: $[0, t_{\mathrm{Max}}) \to M$ is a maximal solution of (1.1), whose lifespan $t_{\mathrm{Max}}$ is finite, the Hölder norm of the curvature $k(t, \cdot\,)$ of $\gamma(t)$ must blow up as $t \to t_{\mathrm{Max}}$. Indeed, if the $h^\alpha$ norm of $k$ remained bounded, for some $\alpha > 0$, then the family of curves $\gamma(t)$ $(0 \leq t < t_{\mathrm{Max}})$ would be precompact in the $h^{2+\beta}$ topology for any $\beta < \alpha$; the limit $\gamma^*$ of the $\gamma(t)$ which exists, according to Theorem 6.1, would be an $h^{2+\beta}$ curve, and we could continue the solution beyond $t_{\mathrm{Max}}$.

The next theorem improves upon this observation.

THEOREM 7.1. *Let V be as in Theorem 3.1. If* $\gamma$: $[0, t_{\mathrm{Max}}) \to M$ *is a maximal solution of* (1.1), *and if its lifespan is finite, i.e.,* $t_{\mathrm{Max}} < \infty$, *then the maximal curvature of* $\gamma(t)$ *becomes unbounded as* $t \to t_{\mathrm{Max}}$.

*Proof.* We argue by contradiction: Let $\gamma(t)$ be a maximal solution with a finite lifespan, whose curvature satisfies $|k| \leq c$ for some constant $c$. Since $\gamma(t)$ satisfies (1.1), a bound on the curvature implies a bound on the normal velocity $v^\perp$, so that $\gamma(t) \to \gamma^*$ for some limit curve $\gamma^*$. This limit curve will also have bounded curvature, and we may assume that the $\gamma(t)$ converge in the $C^1$ topology to $\gamma^*$ (by compactness, and uniqueness of the possible limit). As in Section 3 we may represent $\gamma^*$, and the $\gamma(t)$ for $t$ close to $t_{\mathrm{Max}}$, as the image under some smooth immersion $\sigma$: $S^1 \times [-1, +1] \to M$ of the graph of a function $y = u(t, x)$. This function is a solution of the parabolic equation (3.2), and the boundedness of the curvature of the $\gamma(t)$ implies that $q = u_{xx}$ is uniformly bounded. Differentiating (3.2) twice with respect to $x$ leads to

$$q_t = \frac{\partial}{\partial x}\left(F_q \frac{\partial q}{\partial x} + b\right)$$

in which $b = F_x + F_u u_x + F_p u_{xx}$. Since $u$, $u_x$ and $u_{xx}$ are all bounded, this equation is uniformly parabolic and the term $b$ is bounded, so that we can apply the results in [LSV], in particular Theorem III.10.1; the conclusion is that $q$ is $\alpha$-Hölder continuous for some $\alpha > 0$. But this means that the curvature of the $\gamma(t)$ also remains Hölder continuous, and therefore contradicts our assumption that $\gamma(t)$ was a maximal solution. □

From here on we shall assume that $V$ satisfies $\mathrm{V}_1, \ldots, \mathrm{V}_5$.

For any $p \geq 1$, we define the $p$ norm of the curvature

$$X_p(t) = \left\{ \int_{\gamma(t)} |k|^p \, ds \right\}^{1/p}$$

The following theorem is an analogue of Theorem (4.1), giving an estimate for the rate at which $X_p$ grows.

THEOREM 7.2. *If* $1 \leq \kappa < p < \infty$, *then*

$$X_p(t) \leq (A - t)^{(1/2p - 1/2\kappa)}$$

*holds for some constant* $A$ *which depends only on* $\lambda$, $\mu$, $\nu$, $\kappa$, $R^*$, $L(0)$, $p$ *and* $X_p(0)$.

*If* $\kappa > 1$, *then for any* $A > 0$ *there is a* $T_A > 0$ *such that* $X_\kappa(0) < A$ *implies that* $X_\kappa(t) < 2A$ *for all* $t \in [0, T_A]$; *the constant* $T_A$ *depends only on* $\lambda$, $\mu$, $\nu$, $\kappa$, $R^*$, $L(0)$ *and* $A$.

(Recall that $L(t)$ is the length of the curve at time $t$.)

In addition to this growth estimate, we also have the following pointwise estimate for the curvature, in terms of the $p$ norm.

THEOREM 7.3. *If* $1 \leq \kappa < p < \infty$, *and* $\gamma\colon [0, t_0) \to \Omega(M)$ *is a classical solution of* (1.1), *then one has the following pointwise inequality*:

$$|k| \leq cA^{p/(p-\kappa)} t^{-1/2p}$$

*where* $A = \max(1, \sup_{0<t<t_0} X_p(t))$ *and* $c$ *is a constant which depends on* $\lambda$, $\mu$, $\nu$, $\kappa$, $R^*$, $L(0)$, $t_0$ *and* $p$. *If* $p = \kappa \geq 1$, *then there is an* $\varepsilon > 0$, *such that*

$$|k| \leq ct^{-1/2\kappa}$$

*holds for* $t \in (0, t_0)$, *provided* $X_\kappa(t) \leq \varepsilon$ *holds in the same time interval. Again, the constants* $c$ *and* $\varepsilon$ *only depend on* $\lambda$, $\mu$, $\nu$, $\kappa$, $R^*$, $L(0)$, $t_0$ *and* $p$.

*Proof of Theorem* 7.2. Throughout this proof and the next we shall use the letter $c$ to denote any constant which depends on $\lambda$, $\mu$, $\nu$, $\kappa$, $R^*$, $L(0)$, $t_0$, but *not* on $p$; its precise value may change from line to line.

For the time being, we shall assume that $p \geq 2$, since the case $1 < p < 2$ turns out to be a little bit more involved. If $p \geq 2$, then $\psi(k) = |k|^p$ is a $C^2$ convex function, so that we can use (4.4) to compute the rate with which $X_p(t)$ changes:

$$\begin{aligned}\frac{d}{dt}X_p(t)^p &= -p(p-1)\int_{\gamma(t)} |k|^{p-2}k_s\nu_s^\perp\, ds \\ &\quad + \int_{\gamma(t)}\big(p(R+k^2)|k|^{p-2} - |k|^p\big)k\nu^\perp\, ds \\ &= I_1 + I_2.\end{aligned}$$

Our hypothesis $V_5$ implies that

$$\nabla_{\mathbf{t}\oplus k\mathbf{n}}(V)^2 \leq 2\nu^2\big(1 + |k|^{2+2\kappa}\big),$$

so that, by (4.5), we can estimate the first term, $I_1$, as follows:

$$\begin{aligned}I_1 &\leq p(p-1)\int_{\gamma(t)}\left\{-\frac{\lambda}{2}(k_s)^2|k|^{p-2} + \frac{\nu^2}{\lambda}\big(|k|^{p-2} + |k|^{p+2\kappa}\big)\right\} ds \\ &\leq -\lambda\int_{\gamma(t)}(k^{p/2})_s^2\, ds + cp^2\int_{\gamma(t)}\{|k|^{p-2} + |k|^{p+2\kappa})\}\, ds.\end{aligned}$$

To estimate the other term we use the bound $|R| \leq R^*$, and the inequality

$$-\frac{\mu^2}{4\lambda} \leq kv^\perp \leq \mu|k| + \lambda^{-1}|k|^2,$$

which follows from the hypotheses on V. One finds that

$$\begin{aligned}I_2 &\leq \int_{\gamma(t)}\left\{p\big(R^*|k|^{p-1} + |k|^{p+1}\big) + \frac{\mu^2}{4\lambda}|k|^p\right\} ds \\ &\leq cp\int_{\gamma(t)}\{|k|^{p-1} + |k|^{p+2}\}\, ds.\end{aligned}$$

We add these two estimates together, and remember that $|k|^p$ is a convex function of $p$, so that $|k|^q \leq 1 + |k|^p$ holds whenever $0 \leq q \leq p$. The result is

$$\frac{d}{dt}X_p(t)^p \leq -\lambda\big\|(k^{p/2})_s\big\|_2^2 + cp^2\big(L(t) + X_{p+2\kappa}^{p+2\kappa}\big),$$

where we write $\|u\|_p$ for the $L_p$ norm of a function $u$ on $\gamma(t)$. Using our a priori estimate for $L(t)$ from Section 4, we are led to

$$\frac{d}{dt}X_p(t)^p \leq -\lambda\big\|(k^{p/2})_s\big\|_2^2 + cp^2\big(1 + X_{p+2\kappa}^{p+2\kappa}\big)$$

(this the only place where the dependence of $c$ on $t_0$ and $L(0)$ is introduced). Using Hölder's inequality and the interpolation inequality

$$\|u\|_\infty \leq c\|u\|_2^{1/2}\|u_s\|_2^{1/2},$$

one finds that

$$X_{p+2\kappa}^{p+2\kappa} \leq \|k^{p/2}\|_2^2\|k^{p/2}\|_\infty^{4\kappa/p}$$
$$\leq c\|k^{p/2}\|_2^{2+2\kappa/p}\|(k^{p/2})_s\|_2^{2\kappa/p}.$$

From the inequality $x^\theta y^{1-\theta} \leq \varepsilon\theta x + \varepsilon^{-\theta/(1-\theta)}(1-\theta)y$ with $\theta = \kappa/p$ we get

$$X_{p+2\kappa}^{p+2\kappa} \leq \varepsilon\frac{c\kappa}{p}\|(k^{p/2})_s\|_2^2 + c\varepsilon^{-\kappa/(p-\kappa)}\|k^{p/2}\|_2^{2(p+\kappa)/(p-\kappa)}.$$

If we choose $\varepsilon$ small enough ($\varepsilon = \lambda/(2c^2p\kappa)$), and combine the inequalities we have found so far, then we obtain

$$\frac{d}{dt}X_p(t)^p \leq -\frac{\lambda}{2}\|(k^{p/2})_s\|_2^2 + cp^2\left(1 + X_p^{p(p+\kappa)/(p-\kappa)}\right)$$

and therefore

$$\frac{d}{dt}X_p(t) \leq -\frac{\lambda}{2p}\frac{\|(k^{p/2})_s\|_2^2}{X_p(t)^{p-1}} + cp\left(X_p(t)^{1-p} + X_p(t)^{1+\chi p}\right) \tag{7.1}$$

where $\chi = 2\kappa/(p-\kappa)$.

The first part of Theorem 7.2 now follows easily by integrating the inequality one obtains after discarding the first (negative) term in (7.1); i.e.,

$$\frac{dX_p}{dt} \leq cp\left(X_p^{1+p\chi} + X_p^{1-p}\right). \tag{7.2}$$

Indeed, whenever $X_p(t) \geq 1$, one has $X_p' \leq cpX_p^{1+p\chi}$, so that integration gives the following more precise form of the inequality which was claimed:

$$X_p(t) \leq \left(X_p(0)^{-p\chi} - cp^2\chi t\right)^{-1/p\chi}$$

(which only holds when $p \geq 2$).

Still assuming that $p \geq 2$, we consider the borderline case, $p = \kappa$. The constant $c$ in the inequality (7.2) is independent of $p$, so that we can take the limit $p \to \kappa$. In this limit, $\chi$ becomes infinite, and so, if $X_p < 1$, we see

$$\frac{dX_\kappa}{dt} \leq c.$$

Therefore, if $X_\kappa(0) < 1/2$, then $X_\kappa(t) \leq 1/2 + ct$, at least as long as this upper bound is less than 1, i.e., for $t \leq 1/2c$. So the second part of Theorem 7.2 is true when $A = 1/2$.

To get the same result for arbitrary $A$, we rescale the metric, i.e., we replace the metric $g$ on $M$ by $\sigma^{-2}g$. The new arclength and curvature of the curves $\gamma(t)$ become $\sigma^{1}\,ds$ and $\sigma k$, respectively. Therefore the new $\kappa$-norm of the curvature is $\sigma^{1-1/\kappa}X_\kappa$. This rescaling will change the parameters in the equation (such as $\lambda, \mu, \nu$) and also the quantities $R^*$ and $L(0)$; thus $X_\kappa$ will satisfy (7.2), but with a different constant $c$. By choosing $\sigma$ small enough, we can make the rescaled value of $X_\kappa$ less than $\frac{1}{2}$, and the foregoing argument shows that the rescaled $X_\kappa$ will stay less than 1 for a while (i.e. for $T_A = 1/2c(\sigma)$). This completes the proof in the case $p \geq 2$.

When $1 < p < 2$, the function $|k|^p$ is no longer $C^2$ in $k$; therefore we let $\psi(k)$ be a smooth convex function which coincides with $|k|^p$ for $|k| > 1$. Then, defining $\Psi(t)$ as in (4.3), and working out (4.4), one gets

$$\Psi'(t) \leq c\left(1 + \Psi(t)^{1+p\chi}\right).$$

As above, this shows that $\Psi(t) \leq c(T_A - t)^{1/2p - 1/2\kappa}$, for some constants $c$ and $T_A$. Since $\Psi$ dominates $X_p$, this proves the first part of the theorem. To prove the second part when $1 < p < 2$ and $p = \kappa$, one again lets $p$ tend to $\kappa$, and applies the rescaling argument. □

*Proof of Theorem* 7.3. The starting point of the proof is inequality 7.1. If we combine the interpolation inequality $\|u\|_\infty \leq 4\|u\|_2^{1/2}\|u_x\|_2^{1/2}$, which holds for all periodic functions with a square integrable derivative, with $\|u\|_2^2 \leq \|u\|_1\|u\|_\infty$, then

$$\left\|(k^{p/2})_s\right\|_2^2 \geq \frac{1}{256}\frac{\|k^{p/2}\|_2^6}{\|k^{p/2}\|_1^4} = \frac{1}{256}\frac{X_p^{3p}}{X_{p/2}^{2p}}.$$

Together with inequality (7.1) this shows that, for $p \geq 2$,

$$\text{(7.3)} \qquad \frac{dX_p}{dt} \leq -\frac{\lambda}{2p}\frac{X_p^{2p+1}}{X_{p/2}^{2p}} + cp\left(X_p^{1-p} + X_p^{1+p\chi}\right).$$

We observe that, if $p > 2\kappa$, one has $\chi = 2\kappa/(p-\kappa) < 2$, so that the negative term on the right-hand side of (7.3) dominates the positive term for large values of $X_p$. This will allow us to find an upper bound for $X_p(t)$ of the form $At^{-\alpha}$ if we are given a similar bound for $X_{p/2}(t)$. By induction we shall get a sequence of estimates of the same form $X_{2^jp}(t) \leq A_jt^{-\alpha_j}$ where we know the $\alpha_j$, and have a good estimate for the $A_j$. By taking the limit as $j \to \infty$ we get the desired pointwise estimate for the curvature, since $X_p(t) \to \|k\|_\infty$ as $p \to \infty$.

So assume that we know that $X_{p/2}(t) \leq At^{-\alpha}$, and assume also that $p > \kappa$. Then $X_p(t)$ will certainly satisfy $X_p(t) \leq Bt^{-\beta}$ if the function $Y_p(t) = Bt^{-\beta}$

satisfies the reverse inequality to (7.3), for $0 < t \leq 1$,

$$\frac{dY_p}{dt} \geq -\frac{\lambda}{2p}\frac{Y_p^{2p+1}}{X_{p/2}^{2p}} + cp\left(Y_p^{1-p} + Y_p^{1+p\chi}\right).$$

This, in turn, is implied by

$$-\beta Bt^{-\beta-1} \geq -\frac{\lambda}{2p}B^{2p+1}A^{-2p}t^{1+2p(\alpha-\beta)} + cpB^{1+p\chi}t^{-\beta(1+p\chi)},$$

where $B \geq 1$ is assumed, and we have only considered $t \in [0, 1]$, so that the term $Y_p^{1-p}$ could be neglected. Division by $Bt^{-1-\beta}$ leads to

$$\text{(7.4)} \qquad -\beta \geq -\frac{\lambda}{2p}\left(\frac{B}{A}\right)^{2p} t^{1+2p(\alpha-\beta)} + cpB^{p\chi}t^{1-p\chi\beta}.$$

This inequality prompts us to choose $\beta = \alpha + 1/2p$. If we assume, for the moment, that $p\beta\chi < 1$, then (7.4) is implied by

$$-\beta \geq -\frac{\lambda}{2p}\left(\frac{B}{A}\right)^{2p} + cpB^{p\chi}.$$

As we are going to choose $B > 1$, this inequality only becomes stronger if we replace $-\beta$ by $-\beta B^{p\chi}$; therefore (7.4) will certainly hold if

$$\left(\frac{B}{A}\right)^{2p} \geq \frac{2p}{\lambda}(\beta + cp)B^{p\chi},$$

e.g., if

$$\text{(7.5)} \qquad B = \left(\frac{2p}{\lambda}\left(\frac{1}{4\kappa} + cp\right)\right)^{1/(2-\chi)p} A^{2/(2-\chi)}$$

(where we have assumed that $\beta \leq 1/4\kappa$). Now we can make an induction argument. Let $p_0 > \kappa$ be given, and assume, as we do in Theorem 7.3, that $X_{p_0}(t) \leq A_0$ for $0 < t < 1$ (without loss in generality we assume that $t_0 = 1$). Then define

$$p_j = 2^j p_0; \qquad \alpha_0 = 0, \qquad \alpha_{j+1} = \alpha_j + \frac{1}{2p_j} = \frac{1}{2p_0}(1 - 2^{-j}).$$

Clearly, all the $\alpha_j$ are less than $1/2p_0$, and if one defines $\chi_j = 2\kappa/(p_j - \kappa)$, then

$$1 - p_j\chi_j\alpha_j = \frac{p_0 - \kappa}{p_0 - 2^{-j}\kappa} > 0$$

for all $j = 1, 2, 3, \ldots$ . So, if we have an estimate $X_{p_{j-1}}(t) \le A_{j-1} t^{-\alpha_{j-1}}$, for some $j \ge 1$, then the preceding arguments with $p = p_j$, $\alpha = \alpha_{j-1}$, $\beta = \alpha_j$, $A = A_{j-1}$ and $B = A_j$ show that we have a similar estimate for $X_{p_j}$, in which $A_{j-1}$ and $A_j$ are related by (7.5). For large $j$ the first factor in (7.5) is $1 + O(j2^{-j})$; taking logarithms, and using

$$\frac{2}{2 - \chi_j} = \frac{p_0 - 2^{-j}\kappa}{p_0 - 2^{-(j-1)}\kappa},$$

one deduces from (7.5)

$$\log A_j \le \frac{p_0 - 2^{-j}\kappa}{p_0 - 2^{1-j}\kappa} \log A_{j-1} + O\big(j2^{-j}\big).$$

Hence the limit of the $A_j$ exists, and one has

$$A_\infty = \lim_{j\to\infty} A_j \le cA^{p_0/(p_0-\kappa)},$$

while the $\alpha_j$ also converge; their limit is $1/2p_0$.

Finally, we see that the maximum norm of the curvature, $X_\infty$, satisfies the following inequality:

$$X_\infty(t) \le cA^{p_0/(p_0-\kappa)} t^{-1/2p_0},$$

which completes the proof of the first part of the theorem.

Just as with Theorem 7.2, we can still say something in the borderline case $p = \kappa \ge 1$. In this situation the inequality (7.3) implies that

$$\frac{dX_{2\kappa}}{dt} \le \left( -\frac{\lambda}{4\kappa} X_\kappa^{4\kappa} + 2c\kappa \right) X_{2\kappa}^{1+4\kappa}.$$

So, if $X_\kappa(t) \le \varepsilon < 8c\kappa^2/\lambda$, then we get an estimate $X_{2\kappa}(t) \le At^{-1/4\kappa}$, for small $t > 0$, with which we can start the iteration procedure again. As a result we get the second part of Theorem 7.3. □

## 8. The initial value problem in $W_p^2\Omega(M)$

In this section we use the estimates from Theorems 7.2 and 7.3 to show solvability of the initial value problems (1.1) for initial curves belonging to $W_p^2\Omega(M)$, i.e., the subset of $\Omega(M)$ which consists of all curves whose curvatures are $p$ integrable:

$$W_p^2\Omega(M) = \big\{\gamma \in \Omega(M) | k_\gamma \in L_p(ds)\big\}.$$

Any curve $\gamma \in W_p^2\Omega(M)$ has a Hölder continuous tangent (of exponent $1 - 1/p$). Hence one easily verifies that the set

$$\omega_K = \left\{\gamma \in W_p^2\Omega(M) \| \text{length}(\gamma) \le K, \|k_\gamma\|_{L_p} \le K\right\}$$

with the induced topology of $\Omega(M)$, i.e., the $C^1$ topology, is compact.

The main result in this section is:

THEOREM 8.1. *Let* $V$: $S^1(M) \times \mathbf{R} \to \mathbf{R}$ *satisfy* $\mathrm{V}_1 \cdots \mathrm{V}_5$ *and let* $\kappa < p < \infty$ *be given. Then the initial value problem* (1.1) *has a unique maximal solution* $\gamma$: $[0, t_{\text{Max}}) \to W_p^2\Omega(M)$, *for any initial curve* $\gamma_0 \in W_p^2\Omega(M)$.

*In fact, for each* $K > 0$ *there is a* $t_K > 0$ *such that* $t_{\text{Max}} > t_K$ *whenever* $\gamma_0 \in \omega_K$. *The map* $\phi$: $\omega_K \times [0, t_K] \to \Omega(M)$, *defined by* $\phi(\gamma_0, t) = \gamma(t)$, *where* $\gamma(t)$ *is the solution of* (1.1) *with initial value* $\gamma_0$, *is continuous.*

*Proof.* Uniqueness of the solution was established in Section 3, so that we only have to prove the existence and continuity part of the theorem.

Let $\gamma_0 \in W_p^2\Omega(M)$ be given. Then we approximate $\gamma_0$ by smooth regular curves $\gamma_n$, with $\|k_{\gamma_n}\|_{L_p}$ uniformly bounded, say $\gamma_n \in \omega_K$ for some large enough $K$. We can also approximate the function $V$ by smooth functions $V_n$ which satisfy the same hypotheses $\mathrm{V}_1 \cdots \mathrm{V}_5$, all with the same constants $\lambda$, $\mu$, $\hat{\mu}$, $\nu$ and $\kappa$.

From Section 3 we know that for each of these smooth initial curves there exists a maximal solution of the corresponding initial value problem (1.1), with $V = V_n$. By Theorem 7.1 we know that these solutions exist as long as their curvature remains bounded, and by Theorem 7.3 we know that their curvature remains bounded as long as the $L_p$ norm of their curvature does not blow up. Finally, from Theorem 7.2 we get a lower bound for the time it takes this $L_p$ norm to blow up (if it does this at all). This estimate only depends on the constants in the hypotheses on $V$, $R^*$, the length of the initial curve and its curvature's $L_p$ norm, so that it is independent of $n$. Therefore the approximating solutions $\gamma_n(t)$ exist on some common time interval $[0, t_0]$, and their curvatures are uniformly Hölder continuous on any interval $[\delta, t_0]$ $(\delta > 0)$, by Theorem 7.1 and its proof. This allows us to pass to a convergent subsequence, whose limit will be a classical solution $\gamma$: $(0, t_0] \to \omega_{K_1}$ of (1.1), for some $K_1 > K$. We complete the existence proof by showing that $\gamma$ has an initial value, and that this initial value is $\gamma_0$.

Choose an immersion $\sigma$: $S^1 \times [-1, 1] \to M$ and a function $u \in W_p^2(S^1)$, with $|u(x)| < 1$, such that $x \to \sigma(x, u(x))$ parametrises the curve $\gamma_0$. Given any $\varepsilon > 0$ one can find $n_\varepsilon$ such that the curves $\sigma^*(\gamma_n)$ with $n \ge n_\varepsilon$ are graphs $y = u_n(x)$ for certain $u_n \in W_p^2(S^1)$ with $\|u - u_n\|_{L_\infty} < \varepsilon/2$. Using Theorem 5.1 one finds a $t_\varepsilon > 0$, independent of $n$, such that $\sigma^*(\gamma_n(t)) \subset \mathbf{N}_\varepsilon(\sigma^*(\gamma_0))$ for

$0 \le t \le t_\varepsilon$. When the limit $n \to \infty$, this leads to $\sigma^*(\gamma(t)) \subset \mathrm{N}_\varepsilon(\sigma^*(\gamma_0))$ for $0 \le t \le t_\varepsilon$, which shows that the only possible limit point in $\omega_{K_1}$ of $\gamma(t)$ as $t \to 0$ is $\gamma_0$. Since $\omega_{K_1}$ is compact, this implies that $\gamma(t)$ converges to $\gamma_0$ as $t \to 0$.

A slightly modified version of this argument also shows that the solution $\gamma(t) \in \omega_{K_1}$ depends continuously on the initial data $\gamma_0 \in \omega_K$, and time $t \in [0, t_K]$, so that the proof is complete. □

The results of the previous section also allow one to say something about the rate at which the curvature blows up, if it does blow up at all.

THEOREM 8.2. *Let* V *satisfy* $\mathrm{V}_1 \cdots \mathrm{V}_5$, *and assume* $\kappa < p < \infty$. *If* $\gamma\colon [0, t_{\mathrm{Max}}) \to W_p^2\Omega(M)$ *is a maximal solution of* (1.1), *which blows up in finite time* $t_{\mathrm{Max}} < \infty$, *then there is a constant* $c_p < \infty$ *such that*

$$\|k(t, \cdot)\|_{L_p} \ge c_p (t_{\mathrm{Max}} - t)^{(2p)^{-1} - (2\kappa)^{-1}}.$$

*In particular, for every* $\varepsilon > 0$ *there is a* $c_\varepsilon$ *such that*

$$\|k(t, \cdot)\|_{L_\infty} \ge c_\varepsilon (t_{\mathrm{Max}} - t)^{-(2\kappa)^{-1} + \varepsilon}$$

*holds.*

*Proof.* The first inequality follows directly from Theorem 7.2, and the second follows from the first by the fact that $\|k\|_{L_p} \le L(t)^{1/p} \|k\|_{L_\infty}$. □

The example of the circle in the Euclidean plane, which shrinks according to its curvature, shows that these estimates are nearly sharp. The radius of this shrinking circle is $r(t) \propto (t_{\mathrm{Max}} - t)^{1/2}$, so that the $L_p$ norm of its curvature is proportional to $(t_{\mathrm{Max}} - t)^{1/2p - 1/2}$. Since $V(\mathbf{t}, k) = k$, we can let the constant $\kappa$ have its minimal value $\kappa = 1$ which yields that Theorem 8.2 is sharp for $p < \infty$.

## 9. Blow-up in the scale invariant case

Consider a maximal solution $\gamma\colon [0, t_{\mathrm{Max}}) \to \Omega(M)$ of (1.1), and define for any $\varepsilon > 0$ and $t \in (0, t_{\mathrm{Max}})$

$$\alpha_\varepsilon(t) = \sup_{|s_1 - s_0| < \varepsilon} \left| \int_{s_0}^{s_1} k(s, t)\, ds \right|.$$

Thus $\alpha_\varepsilon(t)$ is the largest angle $\mathbf{t}(s_0)$ and $\mathbf{t}(s_1)$ can make (measured after one parallel transports $\mathbf{t}(s_1)$ from $T_{\gamma(s_1)}(M)$ to $T_{\gamma(s_0)}(M)$) for any $s_0$ and $s_1$ with $|s_0 - s_1| \le \varepsilon$.

THEOREM 9.1. *Assume that* $V$ *satisfies* $V_1$, $V_2$, $V_3$ *and* $V_5^*$. *Let* $\gamma\colon [0, t_{\mathrm{Max}}) \to \Omega(M)$ *be a maximal solution with* $t_{\mathrm{Max}} < \infty$; *then for any* $\varepsilon > 0$,

$$\limsup_{t \to t_{\mathrm{Max}}} \alpha_\varepsilon(t) \geq \pi.$$

*In addition, there is a constant* $c > 0$ *such that*

$$\|k_{\gamma(t)}\|_{L_\infty} \geq c(t_{\mathrm{Max}} - t)^{-1/2}.$$

*Proof.* We argue by contradiction: Suppose that for some $\varepsilon > 0$ there is an $\alpha_0 < \pi$ such that $\alpha_\varepsilon(t) \leq \alpha_0$ holds for $t$ close enough to $t_{\mathrm{Max}}$.

Introduce an arclength parametrisation $\gamma\colon (0, t_{\mathrm{Max}}) \times \mathbf{R} \to M$ of the family of curves $\gamma(t)$. Thus, for $0 < t < t_{\mathrm{Max}}$, $\gamma(t, \cdot\,)$ is an $L(t)$ periodic function of $s \in \mathbf{R}$. Since the supremum norm of the curvature must blow up as $t \to t_{\mathrm{Max}}$, we can find a sequence of points $(t_n, s_n)$ such that

$$|k(t, s)| \leq |k(t_n, s_n)| \qquad (s \in \mathbf{R}, 0 < t < t_n)$$

holds for $n = 1, 2, \ldots$ .

Put $\sigma_n = |k(t_n, s_n)|^{-1}$, and define a new, rescaled, version of the old metric by $g^n = \sigma_n^{-1} g$. Then the family of curves

$$\gamma^n(t) = \gamma(t_n + \sigma_n^2 t) \qquad \left(-\frac{t_n}{\sigma_n^2} < t \leq 0\right)$$

satisfies (1.1), with $V$ replaced by $V^n$, where

$$V^n(\mathbf{t}, k) = \sigma_n V\left(\mathbf{t}, \frac{k}{\sigma_n}\right).$$

The curvature of the $\gamma^n$ satisfies $|k_n(t, s)| \leq 1$ for all $t \leq 0$. One easily verifies that the rescaled speed functions, $V^n$, all satisfy conditions $V_1, V_2, V_3$ with the same constants $\lambda$ and $\mu$. In fact, one can even replace $\mu$ by $\mu_n = \sigma_n \mu$. Similarly, the $V^n$ also satisfy a stronger version of $V_5^*$, namely

$$|\nabla^{\mathrm{h}} V^n| + |k|\,|\nabla^{\mathrm{v}} V^n| \leq \nu\left(\sigma_n^2 + |k|^2\right). \tag{9.1}$$

It follows that the normal velocities of the curves $\gamma^n(t)$ with $-\sigma_n^{-2} t_n < t \leq 0$ satisfy $|v^{\perp n}| \leq \lambda + \mu$. By the same arguments as in the proof of Theorem 7.1, it follows from the uniform boundedness of $k_n$ that the curvatures $k_n$ are uniformly Hölder continuous. The normal velocities must therefore also be uniformly Hölder continuous.

Let $P_n \in M$ denote the point $\gamma(t_n, s_n)$. In view of the upper bound for the displacement of the curve $\gamma(t)$, derived in Section 5, the curves $\gamma(t)$ stay in some bounded, and hence compact, subset of $M$. By passing to a subsequence, if necessary, we may assume that the $P_n$ converge to a point $P_* \in M$.

We consider the disk $D_n$ with radius $\sigma_n^{1/2}$ (in the original metric) and centre $P_n$. Its radius in the rescaled metric is $\sigma_n^{-1/2}$. As $n \to \infty$, this disk, equipped with the rescaled metric $g^n$, converges to the flat Euclidean plane. Looking at the portion of the curve $\gamma^n(t)$ which lies in the disk $D_n$, and using the uniform Hölder continuity of $k_n$ and $v^{\perp n}$, one can extract a subsequence of $\gamma^n(t)$ which converges to a family of curves $\gamma^*\colon (-\infty, 0] \times \mathbf{R} \to \mathbf{R}^2$.

These curves may be unbounded since we have no control over the length of the $n^{\text{th}}$ family of curves. Indeed we shall see that the assumption $\alpha_0 < \pi$ forces the limit curves to be unbounded.

As remarked before, the total absolute curvature of a curve does not change when one rescales the metric. Therefore the $\gamma^*(t)$ all have finite total absolute curvature.

Our assumption that $\alpha_\varepsilon(t) \le \alpha_0$ for $t$ close to $t_{\text{Max}}$ implies that, for any $t \le 0$ and $s_0, s_1 \in \mathbf{R}$,

$$\left| \int_{s_0}^{s_1} k_*(t, s)\, ds \right| \le \alpha_0.$$

Since $\alpha_0 < \pi$, this implies that for each $t \le 0$ the curve $\gamma^*(t)$ is the graph of a function $y = u(x)$, if the $x$ and $y$ axes are chosen in the right direction. In particular, the limit curves are indeed unbounded.

In addition to the qualitative property of being a graph in the right coordinates, one also gets the following quantitative result. If the coordinate axes are chosen properly, then $\gamma^*(t)$ is the graph of a Lipschitz continuous function, whose derivative satisfies

$$|u_x| \le \arctan\left( \frac{\pi - \alpha_0}{2} \right).$$

The direction in which one should choose the coordinate axes depends a priori on the time $t$. However, using the bound on the total absolute curvature, the fact that the curvature is uniformly Hölder continuous, and the bound on $v^\perp$, one shows that $t \to \gamma^*(t)$ is continuous in the uniform $C^1$ topology. Therefore, if at $t = t_0$, the curve $\gamma^*(t_0)$ is a graph with respect to a particular choice of coordinate axes, then for $t$ close to $t_0$, $\gamma^*(t)$ will also have this property. So, on short time intervals, the limit family $\gamma^*(t)$ may be represented as the graph of a function $y = u(t, x)$.

The velocity functions $V^n$ are all uniformly Lipschitz on bounded sets, due to $\mathrm{V}_1$ and (9.1). Therefore, after passing to a subsequence again, if necessary, we may assume that they converge to a Lipschitz continuous function $W\colon S^1(\mathbf{R}^2) \times \mathbf{R} \to \mathbf{R}$, which also satisfies $\mathrm{V}_1 \cdots \mathrm{V}_3$, and

$$|\nabla^{\mathrm{h}} W| + |k|\,|\nabla^{\mathrm{v}} W| \le \nu |k|^2.$$

The limit family of curves satisfies $v^{\perp *} = W(\mathbf{t}^*, k^*)$. Choosing coordinate axes, and representing the family $\gamma^*$ as a graph $y = u(t, x)$, locally in time, we find that $u$ has to satisfy

$$u_t = \sqrt{1 + u_x^2}\, W\left( x, u, \arctan(u_x), \frac{u_{xx}}{\left(1 + u_x^2\right)^{3/2}} \right) \tag{9.2}$$

$$= (\text{def}) F(x, u, u_x, u_{xx}),$$

where we have identified $S^1(\mathbf{R}^2)$ with $\mathbf{R}^2 \times S^1$. Thus, a unit tangent vector $\mathbf{t} \in S^1(\mathbf{R}^2)$ has three coordinates; two $(x, y)$ for its base point, and one angle $\theta \in S^1$ for its direction, $(\cos(\theta), \sin(\theta))$. One should compare this with the discussion in Section 3; (9.2) is a special case of (3.2).

Since $|V^n(\mathbf{t}, 0)| \leq \sigma_n \mu$, we have $W(\mathbf{t}, 0) \equiv 0$. Therefore there is a function $\Lambda(x, u, p, q)$ such that $F(x, u, p, q) = \Lambda(x, u, p, q) \cdot q$. The coefficient $\Lambda$ satisfies $(1 + p^2)\Lambda(x, u, p, q) \in (\lambda, \lambda^{-1})$ (compare with (3.3)). Differentiating (9.2) with respect to $x$, we find that $p = u_x$ is a weak solution of

$$p_t = (\Lambda p_x)_x \qquad (x \in \mathbf{R}). \tag{9.3}$$

On short time intervals, $u_x$ will remain uniformly bounded, so that $(1 + u_x^2)\Lambda \in (\lambda, \lambda^{-1})$ implies that (9.3) is uniformly parabolic. One consequence of this is that the supremum norm of $p = u_x$ cannot increase with time. In particular, if $\gamma^*(t_0)$ is a Lipschitz graph in some choice of $(x, y)$ axes, then any $\gamma^*(t)$ with $t_0 \leq t \leq 0$ will be a graph with respect to the same coordinates. By taking an arbitrary large negative number for $t_0$, and using the compactness of the set of orthogonal coordinate systems, one sees that the entire family $\gamma^*(t)$ $(t \leq 0)$, can be represented as the graph of one function $y = u(t, x)$.

The $x$ derivative of $u, p = u_x$, is a bounded weak solution of the strictly parabolic equation (9.3), which is defined for $-\infty < t \leq 0$ and $x \in \mathbf{R}$. By Moser's Harnack inequality ([Mo]) $u_x$ must be constant.

On the other hand, the curvature of the curve $\gamma^n(t)$, at the point $P_n$ and at time $t = 0$, is $\pm 1$. Since the curvatures of the $\gamma^n$ are uniformly Hölder continuous, they converge uniformly on compact sets to the curvature of $\gamma^*$, so that

$$\left. \frac{u_{xx}}{\left(1 + u_x^2\right)^{3/2}} \right|_{(x=0,\, t=0)} = \pm 1 \neq 0.$$

This clearly contradicts the fact that $u_x$ must be constant.

To get the rate at which the curvature blows up we note that the evolution of $v^\perp$ can be obtained by differentiating $v^\perp = V(\mathbf{t}, k)$, and using (4.2). The result is

$$v_t^{\perp} = V_k v_{ss}^{\perp} + \nabla_n^{\mathrm{v}}(V) v_s^{\perp} + \left(\nabla_n^{\mathrm{h}}(V) + V_k(R + k^2)\right) v^{\perp}$$

Using this equation and hypotheses $V_1 \cdots V_5^*$ one finds that

$$\frac{d}{dt} \| v^{\perp}(t, \cdot) \|_{L_\infty} \le c\left(1 + \| v^{\perp}(t, \cdot) \|_{L_\infty}^3\right),$$

which, after integration, shows that $\|v^{\perp}\|_{L_\infty}$ must blow up at least as fast as $(t_{\mathrm{Max}} - t)^{-1/2}$. By $V_2$ and $V_3$ the same must be true for $\|k_{\gamma(t)}\|_{L_\infty}$. □

## 10. Application to a nonlinear parabolic equation

The methods of the last section can be used to prove a global existence theorem for a certain class of nonlinear parabolic initial value problems. Let $f\colon S^1 \times \mathbf{R}^3 \to \mathbf{R}$ satisfy the following conditions:

($F_1$) $f(x, u, p, q)$ is a locally Lipschitz function of its four arguments,

$$(F_2) \qquad \lambda \le \frac{\partial f}{\partial q} \le \lambda^{-1},$$

$$(F_3) \qquad |f(x, u, p, 0)| \le \mu \qquad (x \in S^1, u, p \in \mathbf{R})$$

$$(F_4) \qquad |f_x| + |f_u| + |q f_p| \le \nu\left(1 + |q|^2\right),$$

and let a $u_0 \in W_p^2(S^1)$ be given, for some $p > 1$. Then we can consider the initial value problem

$$(10.1) \qquad \begin{aligned} u_t &= f(x, u, u_x, u_{xx}) \qquad && (x \in S^1, 0 < t < t_0) \\ u(0, x) &= u_0(x) && (x \in S^1) \end{aligned}$$

THEOREM 10.1. *If $f$ satisfies $F_1 \cdots F_4$, then the initial value problem* (10.1) *has a solution for any $u_0 \in W_p^2(S^1)$, and any $t_0 > 0$.*

In Part II we shall see that the same result holds if the initial data give merely a Lipschitz function, instead of $W_p^2(S^1)$.

*Proof.* Just as we found a bound for the total absolute curvature of solutions of (1.1) in Section 4, we obtain an estimate for the $L_1$ norm of $u_{xx}$, for any solution $u$ of (10.1), in terms of the constants $\lambda$, $\mu$, $t_0$ and the $L_1$ norm of $u_0''(x)$. We only observe the existence of the following equation for $q = u_{xx}$:

$$q_t = \frac{\partial}{\partial x}\left(f_q \frac{\partial q}{\partial x} + f_x + f_u p + f_p q\right),$$

which is analogous to (4.2), and leave the details to the reader.

The bound for the $L_1$ norm of $u_{xx}$ leads to an estimate for the supremum norm of $u_x$. Indeed, for any periodic function $u(\cdot)$ one has $\|u_x\|_{L_\infty} \le \|u_{xx}\|_{L_1}$, since $u_x$ must have a zero somewhere, from which one can start integrating $u_{xx}$ to estimate $u_x$ at other points.

In view of the bound on $u_x$ we may modify the function $f(x, u, p, q)$ in the region $p > \sup(\|u_{xx}(t, \cdot)\|_{L_1}: 0 < t < t_0)$ in any way we like, without affecting the solution $u$ of (10.1). In particular, we can change $f$ so that (10.1) has the form (9.2), i.e., so that it comes from specialising an equation of the form (1.1), where the manifold $M$ is the cylinder $S^1 \times \mathbf{R}$.

From Theorem 8.1 we know that (10.1), with the modified $f$, has a solution if $t_0$ is small enough. Let $t_{\mathrm{Max}}$ be the lifetime of the maximal solution of (10.1). Then Theorem (9.1) tells us that, if $t_{\mathrm{Max}} < \infty$,

$$\limsup_{t \to t_{\mathrm{Max}}} \alpha_\varepsilon(t) \ge \pi.$$

In the present situation, where $M = S^1 \times \mathbf{R}$ with the flat metric, one can describe the quantity $\alpha_\varepsilon(t)$ as follows. Consider two points on the graph of $y = u(t, \cdot)$, whose distance, measured along the graph, is less than $\varepsilon$; form their tangents, and compute the angle between these tangents. Then $\alpha_\varepsilon(t)$ is the lowest upper bound for the angles that arise in this way.

If one has an upper bound for $u_x$, say $|u_x| \le p_{\mathrm{Max}}$, then, interpreting $\alpha_\varepsilon(t)$ as above, one easily finds that

$$\alpha_\varepsilon(t) \le 2\arctan(p_{\mathrm{Max}}) < \pi.$$

This contradicts the conclusion of Theorem 9.1, at least if $t_{\mathrm{Max}} < \infty$. So the solution of (10.1) exists for any positive $t_0$. □

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(Part II of this paper will appear in the next issue.)