Power Concavity and Boundary Value Problems

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1. Introduction. This article presents an improved version of Korevaar's convexity maximum principle (1983, [4]), which is used to show that positive powers of solutions of various categories of boundary value problems are concave. For instance, it is shown that if $\Delta u + f(x) = 0$ for x in a bounded convex domain Ω in $\mathbf{R}^n$ for some $n \geq 2$, f is non-negative, f^β is concave in Ω for some $\beta \geq 1$, and $u = 0$ on $\partial\Omega$, then u^α is concave in Ω for $0 < \alpha \leq \beta/(1 + 2\beta)$. The upper bound for α is shown to be sharp.

More generally, it is shown for any $\alpha \in (0,1]$ that if u is a solution of $\Delta u + b(x,u) = 0$ in Ω such that $u = 0$ on $\partial\Omega$, then u^α is concave in $\bar{\Omega}$ whenever $b(x,u) \geq 0$ for all x and u, $t^{\alpha-1}b(x,t)$ is decreasing with respect to t, and $t^{(3\alpha-1)/\alpha}b(x,t^{1/\alpha})$ is jointly concave with respect to (x,t). Similar results are obtained for the equation $\Delta u = e^u$.

2. Definitions and preliminary results. Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$. For k a non-negative integer or $+\infty$, $C(\Omega)$, $C(\bar{\Omega})$, $C^k(\Omega)$ and $C^k(\bar{\Omega})$ will denote respectively the set of continuous functions on Ω, the set of continuous functions on $\bar{\Omega}$, the set of functions in $C(\Omega)$ whose derivatives of order less than or equal to k are continuous, and the set of functions in $C(\bar{\Omega})$ whose derivatives in Ω of order less than or equal to k have continuous extensions to $\bar{\Omega}$.

If Ω is a bounded convex domain in $\mathbf{R}^n$ and $u : \bar{\Omega} \to \mathbf{R}$ is a bounded function, then the *convexity function* c for u on $\bar{\Omega}$ is defined on $\bar{\Omega} \times \bar{\Omega} \times [0,1]$ by:

$$c(y,z,\lambda) = (1 - \lambda)u(y) + \lambda u(z) - u((1 - \lambda)y + \lambda z) \qquad \text{for } y, z \in \bar{\Omega}, \lambda \in [0,1].$$

Write $\bar{c} = \sup\{c(y,z,\lambda) : (y,z,\lambda) \in \bar{\Omega} \times \bar{\Omega} \times [0,1]\}$. Then $\bar{c}$ is a real number since c is bounded. Also, $\bar{c} \geq 0$ (which follows, for instance, by putting $\lambda = 0$), and $\bar{c} = 0$ if and only if u is concave in $\bar{\Omega}$.

In order to present the results of this paper coherently, it is convenient to introduce the concept of α-concavity. When α is a positive real number, a non-negative function u defined on a convex subset of $\mathbf{R}^n$ for any $n \geq 1$ is said to be α-concave when its α^{th} power, u^α, is concave. Then α-concavity can be extended in a natural way to all extended real numbers α in $[-\infty,+\infty]$ as follows: u is said to be *α-concave* for $\alpha = +\infty$ when u is constant, for $0 < \alpha < +\infty$ when u^α is

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concave, for $\alpha = 0$ when $\log u$ is concave, for $-\infty < \alpha < 0$ when u^α is convex, and for $\alpha = -\infty$ when the upper level sets of u, $\{x \in \Omega : u(x) > t\}$, are convex for all real constants t. (Here, $\log u$, and u^α for $-\infty < \alpha < 0$, are taken to mean $-\infty$ and $+\infty$ respectively when $u = 0$, and the usual extended definitions of concavity and convexity are then applied to the extended real valued functions that result.)

The naturalness of this generalisation of concavity, which is similar to a definition used by Brascamp and Lieb ([1], p. 373), will be evident from its properties given below.

The definition of (-1)-concavity can be extended from positive functions to general real functions as follows: If S is a convex set, then $b : S \to \mathbf{R}$ will be said to be *harmonic concave* when, for all $(y,z,\lambda) \in S \times S \times [0,1]$,

$$b((1-\lambda)y + \lambda z) \geq b(y)b(z)((1-\lambda)b(z) + \lambda b(y))^{-1} \qquad \text{if } (1-\lambda)b(z) + \lambda b(y) > 0$$

and

$$b((1-\lambda)y + \lambda z) \geq 0 \qquad \text{if } b(y) = b(z) = 0.$$

It is readily seen that a positive function b is harmonic concave if and only if $1/b$ is convex (that is, b is (-1)-concave). But it is also true that any concave function, whether it is non-negative or not, is harmonic concave, since if b is concave and $(1-\lambda)b(z) + \lambda b(y) > 0$ then

$$\begin{aligned} b((1-\lambda)y + \lambda z) &- b(y)b(z)((1-\lambda)b(z) + \lambda b(y))^{-1} \\ &\geq (1-\lambda)b(y) + \lambda b(z) - b(y)b(z)((1-\lambda)b(z) + \lambda b(y))^{-1} \\ &= \lambda(1-\lambda)(b(y) - b(z))^2((1-\lambda)b(z) + \lambda b(y))^{-1} \\ &\geq 0. \end{aligned}$$

The inequality for $b(y) = b(z) = 0$ is clearly satisfied by any concave function b. Harmonic concavity is used in preference to (-1)-concavity in Theorem 3.1 to allow the fullest possible generality.

Some basic properties of α-concavity are summarised here. They are given only brief justification since they are mostly elementary. Property 4 is probably the most useful for applications, since equation (2.1) provides a simple way of checking whether a function is α-concave.

Property 1. Let Ω be a convex set in $\mathbf{R}^n$ for some $n \geq 1$. A function u on Ω is α-concave if and only if for all y and z in Ω and $\lambda \in [0,1]$,

$$u((1-\lambda)y + \lambda z) \geq g_\alpha(\lambda, u(y), u(z)),$$

where $g_\alpha(\lambda,s,t)$ is defined for $\lambda \in [0,1]$ and s, $t \geq 0$ by:

$$g_\alpha(\lambda,s,t) = \begin{cases} \max(s,t) & \text{for } \alpha = +\infty \\ ((1-\lambda)s^\alpha + \lambda t^\alpha)^{1/\alpha} & \text{for } 0 < \alpha < +\infty \\ s^{1-\lambda}t^\lambda & \text{for } \alpha = 0 \\ st((1-\lambda)t^{-\alpha} + \lambda s^{-\alpha})^{1/\alpha} & \text{for } -\infty < \alpha < 0 \\ \min(s,t) & \text{for } \alpha = -\infty \end{cases}$$

where 0^0 is taken to mean 1. This equivalent definition for α-concavity, and the fact that $g_\alpha(\lambda,s,t)$ is monotone increasing with respect to α, are discussed by Brascamp and Lieb ([1], p. 373). It is straightforward to show that for each $\lambda \in [0,1]$ and $s,\ t \geq 0$,

$$g.(\lambda,s,t) \in C([-\infty,+\infty]) \cap C^\infty(\mathbf{R})$$

with respect to the usual two-point compactification topology on $[-\infty,+\infty]$ (but not analytic when either s or t is equal to zero, since, for instance, $g_\alpha(\lambda,0,t) = \lambda^{1/\alpha}t$ for $\alpha > 0$, and equals zero for $\alpha \leq 0$).

Property 2. If u is α-concave, then u is β-concave for all $\beta \leq \alpha$. This monotonicity property of α-concavity follows from the monotonicity of g_α with respect to α, which in turn follows from Jensen's inequality.

Property 3. For any $(-\infty)$-concave function u, define the *concavity number* of u by $\alpha(u) = \sup\{\beta \in \mathbf{R} : u \text{ is } \beta\text{-concave}\}$. Then u is $\alpha(u)$-concave. This continuity property of α-concavity follows from the continuity of g_α with respect to α. The results of this paper may be regarded as calculations of the concavity number for solutions of boundary value problems.

Property 4. For $\alpha \in (-\infty,+\infty)$, a positive C^2 function on a convex domain Ω is α-concave if and only if

$$u(x)u_{\theta\theta}(x) + (\alpha - 1)u_\theta(x)^2 \leq 0 \tag{2.1}$$

for all $x \in \Omega$ and $\theta \in S$, where $S = \{\phi \in \mathbf{R}^n : |\phi| = 1\}$ is the set of "directions" in $\mathbf{R}^n$, and $u_\theta(x)$ and $u_{\theta\theta}(x)$ denote the first and second derivatives of u with respect to x in the direction θ—namely, $\theta_i \partial u/\partial x_i$ and $\theta_i\theta_j\partial^2 u/\partial x_i \partial x_j$. (The summation convention will apply to all roman subscripts unless the context indicates otherwise.)

To prove (2.1) requires little more than the observations that a C^2 function v in a convex domain is convex if and only if $v_{\theta\theta}(x)$ is non-negative for all x and θ, and that

$$(u^\alpha)_{\theta\theta} = \alpha u^{\alpha-2}(uu_{\theta\theta} + (\alpha - 1)u_\theta^2) \qquad \text{for } \alpha \neq 0$$

and

$$(\log u)_{\theta\theta} = u^{-2}(uu_{\theta\theta} - u_\theta^2).$$

Property 5. It follows from Property 4 that if u is a positive C^2 function on a convex domain Ω, and $\alpha \in (-\infty,+\infty)$, then u is α-concave if and only if

$$\alpha \leq 1 - \sup\{u(x)u_{\theta\theta}(x)u_\theta(x)^{-2} : x \in \Omega, \theta \in S \text{ and } u_\theta(x) \neq 0\}. \tag{2.2}$$

In fact, this is also true for $\alpha = +\infty$, since the set in (2.2) is empty if and only if u is constant, in which case the supremum of the set is $-\infty$, and the right-hand side equals $+\infty$. Hence for any $(-\infty)$-concave positive C^2 function,

$$\alpha(u) = 1 - \sup\{u(x)u_{\theta\theta}(x)u_\theta(x)^{-2} : x \in \Omega, \theta \in S \text{ and } u_\theta(x) \neq 0\}. \tag{2.3}$$

This provides an effective means of calculating $\alpha(u)$ when u is given explicitly, and demonstrates the essential simplicity and naturalness of the definition of α-concavity.

Property 6. For any α, if f is a pointwise limit of a sequence of non-negative α-concave functions, then f is α-concave. This follows from Property 1 and the continuity of $g_\alpha(\lambda,s,t)$ with respect to s and t.

Property 7. For any $\alpha \geq 1$, the set of α-concave functions on a convex set Ω is a convex cone. That is, for $\alpha \geq 1$, the α-concave functions are closed under addition and positive scalar multiplication. This follows from a well-known extension of Minkowski's inequality which states that if one defines

$$\|v\|_p = \left(\sum_{i=1}^m v_i^p\right)^{1/p} \qquad \text{for } v \in \mathbf{R}^m,$$

then

$$\|(1-\lambda)v + \lambda w\|_p \geq (1-\lambda)\|v\|_p + \lambda\|w\|_p$$

for all $p \in (0,1]$, $\lambda \in [0,1]$ and v, $w \in \mathbf{R}^m$. From this, putting $m = 2$, $v = (f(y),g(y))$ and $w = (f(z),g(z))$, one obtains

$$\begin{aligned}
(1-\lambda)(f(y)^p + g(y)^p)^{1/p} &+ \lambda(f(z)^p + g(z)^p)^{1/p} \\
&= (1-\lambda)\|v\|_p + \lambda\|w\|_p \\
&\leq \|(1-\lambda)v + \lambda w\|_p \\
&= (((1-\lambda)f(y) + \lambda f(z))^p + ((1-\lambda)g(y) + \lambda g(z))^p)^{1/p} \\
&\leq (f((1-\lambda)y + \lambda z)^p + g((1-\lambda)y + \lambda z)^p)^{1/p}
\end{aligned}$$

whenever f and g are non-negative concave functions. But this inequality means precisely that $(f^p + g^p)^{1/p}$ is concave. Writing $p = \alpha^{-1}$ for $\alpha \in [1,\infty)$ and replacing f^p and g^p with f and g respectively then shows that $f + g$ is α-concave whenever f and g are α-concave. Property 7 readily follows.

Property 8. If α and $\beta \in [0,+\infty]$, f is α-concave, and g is β-concave, then the pointwise product $f \cdot g$ is γ-concave, where $\gamma \in [0,+\infty]$ satisfies $\gamma^{-1} = \alpha^{-1} + \beta^{-1}$, with the understanding that $0^{-1} = +\infty$ and $(+\infty)^{-1} = 0$. The result is obvious when α or β is infinite, and clear when α and β are both zero. In the other cases, $\alpha + \beta \in (0,+\infty)$, and by defining $\lambda = \alpha/(\alpha + \beta)$ one sees that it will be sufficient to show that $f^{1-\lambda} \cdot g^{\lambda}$ is concave whenever f and g are concave and $\lambda \in [0,1]$. But $f^{1-\lambda} \cdot g^{\lambda}$ is the pointwise limit of $((1 - \lambda)f^p + \lambda g^p)^{1/p}$ as $p \rightarrow 0^+$, which is concave for $p > 0$ by Property 7. Hence $f^{1-\lambda} \cdot g^{\lambda}$ is concave and Property 8 follows.

Property 9. Let α, $\beta \in [0,+\infty]$, and f and g be respectively α- and β-concave functions on bounded convex subsets Ω_1 and Ω_2 of $\mathbf{R}^n$ with $n \geq 1$. Then the convolution $f * g$ defined by

$$(f * g)(x) = \int_{\Omega_1 \cap (x-\Omega_2)} f(t)g(x - t)dt$$

is γ-concave in $\Omega_1 + \Omega_2$ for $\gamma^{-1} = n + \alpha^{-1} + \beta^{-1}$, interpreted as in Property 8. This follows from Property 8 and [1] (Corollary 3.5).

Note added in revision. After this paper had been submitted, it was pointed out to the author by V. Jeyakumar (University of Melbourne) that the algebraic properties given here for α-concave functions are closely paralleled by properties of α-convex functions shown by Lindberg [6], who also proves a very general form of Lemma A2.

3. A convexity maximum principle and two concavity theorems. Theorem 3.1 is a convexity maximum principle which is an improved version of one proved by Korevaar ([4], Theorem 1.2). The improvement lies entirely in assumption (ii), which specifies harmonic concavity rather than concavity. The applications in this article cannot be obtained from the unimproved convexity maximum principle.

Theorem 3.1. *Assumptions:*

$n \geq 2$, and Ω is a bounded convex domain in $\mathbf{R}^n$.

For $x \in \Omega$, $u \in C^2(\Omega)$ satisfies the equation

$$a_{ij}(Du(x))u_{ij}(x) + b(x,u(x),Du(x)) = 0$$

where for all $p \in \mathbf{R}^n$, $(a_{ij}(p))$ is a real symmetric positive semidefinite matrix.

(i) For all $x \in \Omega$ and $p \in \mathbf{R}^n$, $b(x,\cdot,p)$ is strictly decreasing.

(ii) For all $p \in \mathbf{R}^n$, $b(\cdot,\cdot,p) : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is harmonic concave.

Assertion:

If $\bar{c} > 0$, then $\bar{c}$ is not attained in $\Omega \times \Omega \times [0,1]$.

Proof. c is the convexity function defined in Section 2. Suppose its supremum, $\bar{c}$, is positive, and that $c(y,z,\lambda) = \bar{c}$ at some $(y,z,\lambda) \in \Omega \times \Omega \times [0,1]$. Then $y \neq z$ and $\lambda \in (0,1)$. Let $D = (D_y,D_z)$ denote the gradient on $\Omega \times \Omega$, where $D_y = (\partial/\partial y_1,\ldots\partial/\partial y_n)$ and so forth, and write D^2 for the corresponding

Hessian. Then $Dc(y,z,\lambda) = 0$, and $D^2c(y,z,\lambda)$ is negative semidefinite. Let $x = (1 - \lambda)y + \lambda z$. Then

$$0 = D_y c = (1 - \lambda)Du(y) - (1 - \lambda)Du(x),$$

so that $Du(y) = Du(x)$. Similarly, $Du(z) = Du(x)$, and hence (a_{ij}) has the same value, A say, at each of x, y and z. Define the $2n \times 2n$ matrix B by:

$$B = \begin{bmatrix} s^2A & stA \\ stA & t^2A \end{bmatrix}$$

for $s, t \in \mathbf{R}$. Then B is positive semidefinite because A is, and so by Lemma A1 (see appendix), $\mathrm{Tr}(BD^2c) \leq 0$. That is:

$$\alpha s^2 + 2\beta st + \gamma t^2 \leq 0 \tag{3.1}$$

where $\alpha = \mathrm{Tr}(AD_y^2c)$, $\beta = \mathrm{Tr}(AD_yD_zc)$ and $\gamma = \mathrm{Tr}(AD_z^2c)$. For $w = x$, y and z, put $Q_w = a_{ij}u_{ij}(w)$. Then

$$\alpha = (1 - \lambda)Q_y - (1 - \lambda)^2Q_x$$
$$\beta = -\lambda(1 - \lambda)Q_x$$
$$\gamma = \lambda Q_z - \lambda^2Q_x.$$

The non-positivity of the quadratic form in (3.1) implies that $\alpha \leq 0$, $\gamma \leq 0$ and $\beta^2 - \alpha\gamma \leq 0$. That is:

$$Q_x \geq (1 - \lambda)^{-1}Q_y \tag{3.2}$$

$$Q_x \geq \lambda^{-1}Q_z \tag{3.3}$$

and

$$\begin{aligned} 0 &\geq \lambda^2(1 - \lambda)^2Q_x^2 - ((1 - \lambda)^2Q_x - (1 - \lambda)Q_y)(\lambda^2Q_x - \lambda Q_z) \\ &= \lambda(1 - \lambda)((1 - \lambda)Q_xQ_z + \lambda Q_xQ_y - Q_yQ_z). \end{aligned}$$

Since $\lambda(1 - \lambda) > 0$, this can be written as:

$$Q_x((1 - \lambda)Q_z + \lambda Q_y) \leq Q_yQ_z. \tag{3.4}$$

If $(1 - \lambda)Q_z + \lambda Q_y \geq 0$, then by (3.2),

$$Q_x((1 - \lambda)Q_z + \lambda Q_y) \geq Q_yQ_z + \lambda(1 - \lambda)^{-1}Q_y^2,$$

which, taken together with (3.4), implies that $Q_y = 0$. Similarly, $Q_z = 0$. It can be concluded then that either $(1 - \lambda)Q_z + \lambda Q_y < 0$ or $Q_y = Q_z = 0$. In the former case, (3.4) gives

$$Q_x \geq Q_yQ_z((1 - \lambda)Q_z + \lambda Q_y)^{-1}, \tag{3.5}$$

whereas when $Q_y = Q_z = 0$, (3.2) (or (3.3)) gives $Q_x \geq 0$. But $Q_w = -b(w,u(w),Du(w))$ for $w = x$, y and z. So

$$Q_x = -b(x,u(x),Du(x))$$

$$< -b((1-\lambda)y+\lambda z,(1-\lambda)u(y)+\lambda u(z),Du(x))$$

$$\text{(by (i), as } u(x) < (1-\lambda)u(y)+\lambda u(z))$$

$$\leq \begin{cases} Q_yQ_z((1-\lambda)Q_z+\lambda Q_y)^{-1} & \text{if } (1-\lambda)Q_z+\lambda Q_y < 0 \\ 0 & \text{if } Q_y = Q_z = 0 \end{cases}$$

$$\text{(by (ii))}.$$

This inequality contradicts (3.5). So $\bar{c}$ cannot be attained in $\Omega \times \Omega \times [0,1]$, and the theorem is verified. □

Remark 3.1.1. It is clear from the proof of Theorem 3.1 that assumptions (i) and (ii) need only hold for (x,u) in the convex hull of the graph of u. A problem for which this is an essential observation is dealt with in Theorem 5.3.

Theorem 3.2. *Assumptions:*

$n \geq 2$, and Ω is a bounded convex domain in $\mathbf{R}^n$.

(i) $u \in C(\bar{\Omega})$.

(ii) $u(z) - u(y) < \limsup_{t\to 0^+} t^{-1}(u(y+t(z-y)) - u(y))$ for all $y \in \partial\Omega$ and $z \in \bar{\Omega}$ such that $[y,z]$ (the straight line segment joining y to z) is not a subset of $\partial\Omega$.

The restriction of u to Ω satisfies all of the assumptions of Theorem 3.1.

Assertion:

u is concave in $\bar{\Omega}$.

Proof. Suppose u is not concave. Then $\bar{c} > 0$. By (i), c is continuous on the compact set $\bar{\Omega} \times \bar{\Omega} \times [0,1]$, and so $\bar{c}$ must be attained at some $(y,z,\lambda) \in \bar{\Omega} \times \bar{\Omega} \times [0,1]$. Theorem 3.1 implies that $(y,z,\lambda) \notin \Omega \times \Omega \times [0,1]$. But $c(y,z,\lambda) = 0$ when $\lambda = 0$ or 1. So the remaining possibility is that (y,z,λ) is in $\partial\Omega \times \bar{\Omega} \times (0,1)$ (or $\bar{\Omega} \times \partial\Omega \times (0,1)$).

Suppose $y \in \partial\Omega$, $z \in \bar{\Omega}$ and $\lambda \in (0,1)$. If $[y,z]$ is a subset of $\partial\Omega$ then $c(y,z,\lambda) = 0$. Otherwise, by (ii), there exists $t \in (0,\lambda)$ such that

$$u((1-t)y+tz) > (1-t)u(y)+tu(z).$$

Write $y' = (1-t)y+tz$ and $\mu = (\lambda - t)/(1-t)$, so that $(1-\mu)y' + \mu z = (1-\lambda)y+\lambda z$. Then

$$\begin{aligned} c(y',z,\mu) &= (1-\mu)u(y') + \mu u(z) - u((1-\mu)y'+\mu z) \\ &> (1-\mu)(1-t)u(y) + ((1-\mu)t+\mu)u(z) - u((1-\mu)y'+\mu z) \\ &= (1-\lambda)u(y)+\lambda u(z) - u((1-\lambda)y+\lambda z) \\ &= c(y,z,\lambda). \end{aligned}$$

So c does not attain its maximum at this choice of (y,z,λ). The conclusion is the same if $(y,z) \in \bar{\Omega} \times \partial\Omega$. Thus $\bar{c} = 0$, and u is concave in $\bar{\Omega}$. □

Theorem 3.2 specifies some conditions under which concavity may be proved for a function u. Theorem 3.3 gives the corresponding conditions for proving α-concavity for $\alpha \in (0,1]$. When Theorem 3.1 is applied to the case $\alpha = 0$, the results obtained are identical to those obtained by Korevaar ([5], Theorem 2.5) for the unimproved convexity maximum principle. When $\alpha \notin [0,1]$, no applications seem to be possible.

Theorem 3.3 can be derived for $\alpha \in [1/3,1/2)$ by way of a concavity maximum principle for u^α directly (that is, with u^α in place of u in the convexity function in Theorem 3.1), but the calculations are more difficult and the method does not seem to extend below $\alpha = 1/3$.

Theorem 3.3. *Assumptions:*

$n \geq 2$, *and* Ω *is a bounded convex domain in* $\mathbf{R}^n$.

$u \in C(\bar{\Omega}) \cap C^2(\Omega)$, $u|_{\partial\Omega} = 0$, *and* $u|_\Omega > 0$.

$0 < \alpha \leq 1$.

For $x \in \Omega$, u *satisfies*

$$\Delta u(x) + b(x,u(x),Du(x)) = 0,$$

where $b\colon \Omega \times (0,\infty) \times \mathbf{R}^n \to (0,\infty)$ *satisfies (i) and (ii):*

(i) for all $x \in \Omega$ *and* $p \in \mathbf{R}^n$, $t^{\alpha-1}b(x,t,t^{1-\alpha}p)$ *is a strictly decreasing function of* t.

(ii) for all $p \in \mathbf{R}^n$, $s^{(3\alpha-1)/\alpha}b(x,s^{1/\alpha},s^{(1-\alpha)/\alpha}p)$ *is a jointly concave function of* $(x,s) \in \Omega \times (0,\infty)$.

(iii) For all $y \in \partial\Omega$ *and* $z \in \Omega$, $\limsup_{t\to 0^+} t^{-1/\alpha}u(y + t(z-y)) > u(z)$.

Assertion:

u^α *is a concave function in* $\bar{\Omega}$.

Proof. It will be shown that, under the assumptions of Theorem 3.3, the function $v = u^\alpha$ satisfies the requirements of Theorem 3.2. A routine calculation shows that v satisfies

$$\Delta v + \beta(x,v,Dv) = 0,$$

where

$$\beta(x,s,p) = \alpha^{-1}(1-\alpha)s^{-1}|p|^2 + \alpha s^{(\alpha-1)/\alpha}b(x,s^{1/\alpha},\alpha^{-1}s^{(1-\alpha)/\alpha}p).$$

Thus the requirements of Theorem 3.1 are met if β (in place of b) satisfies (i) and (ii) of that theorem. These can be proved from the corresponding assumptions of Theorem 3.3 as follows:

(i) For any x and p, $\alpha^{-1}(1-\alpha)s^{-1}|p|^2$ is non-increasing with respect to s. Writing t for $s^{1/\alpha}$, the second term of β becomes $\alpha t^{\alpha-1}b(x,t,\alpha^{-1}t^{1-\alpha}p)$, which is assumed strictly decreasing with respect to t. Hence $\beta(x,\cdot,p)$ is strictly decreasing for all x and p.

(ii) The function $\beta(\cdot,\cdot,p)$ must be shown to be harmonic concave for all p. Since β is positive, this is equivalent to showing that β^{-1} is convex. Now $\beta(x,s,p)^{-1} = \alpha s^2\gamma(x,s,p)^{-1}$, where

$$\gamma(x,s,p) = (1-\alpha)s|p|^2 + \alpha^2 s^{(3\alpha-1)/\alpha}b(x,s^{1/\alpha},\alpha^{-1}s^{(1-\alpha)/\alpha}p).$$

γ is positive and, by assumption (ii) on b, jointly concave with respect to (x,s) for all p. It then follows by Lemma A2 (see appendix) that β^{-1} is convex, as required.

It remains to verify condition (ii) of Theorem 3.2, but this follows immediately from (iii) of Theorem 3.3 upon noting that $v|_{\partial\Omega} = 0$, and $t^{-1}v = (t^{-1/\alpha}u)^{\alpha}$. □

In the remainder of this paper, b will be assumed to be independent of p.

Remark 3.3.1. For a C^2 function $b:\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$, let b_t and b_{tt} denote the first and second partial derivatives of $b(x,t)$ with respect to t, and for $\theta \in S = \{\phi \in \mathbf{R}^n : |\phi| = 1\}$ write b_θ and $b_{\theta\theta}$ to denote the quantities

$$\theta_i \partial b/\partial x_i \quad \text{and} \quad \theta_i\theta_j \partial^2 b/\partial x_i \partial x_j$$

respectively—that is, the first and second partial derivatives of $b(x,t)$ with respect to x in the direction θ. Similar notations will apply to $\beta(x,s,p)$. (The summation convention will not apply to the subscripts s, t and θ.)

If b is C^1, condition (i) of Theorem 3.3 will hold if $(1-\alpha)b - tb_t$ is always positive. If b is C^2, condition (ii) is equivalent to the simultaneous non-positivity of the quantities β_{ss}, $\beta_{\theta\theta}$ and $\beta_{s\theta}^2 - \beta_{ss}\beta_{\theta\theta}$, or in terms of b:

(1) $(1-2\alpha)(1-3\alpha)b + (5\alpha - 1)tb_t + t^2 b_{tt} \le 0$. (That is, $s^{(3\alpha-1)/\alpha}b(x,s^{1/\alpha})$, or equivalently $s^{(1-2\alpha)/\alpha}b(x,s^{-1/\alpha})$, is concave with respect to s. See Remark 4.2.2.)

(2) $b_{\theta\theta} \le 0$ for all directions θ.

(3) $((1-3\alpha)b_\theta + tb_{t\theta})^2 \le b_{\theta\theta}((1-2\alpha)(1-3\alpha)b + (5\alpha-1)tb_t + t^2 b_{tt})$ for all directions θ and all $t > 0$.

Remark 3.3.2. For $\alpha = 1$, Theorem 3.1 is stronger than Theorem 3.3. For example, $b(x,u) = e^{-u}$ satisfies the requirements of the former but not the latter.

Remark 3.3.3. In the statement of Theorem 3.3, Δu may be replaced by $a_{ij}(u^{\alpha-1}Du)u_{ij}$, where (a_{ij}) is a matrix depending only on $u^{\alpha-1}Du$.

4. Applications to particular boundary value problems.

The general concavity theorems of the previous section are applied in this section to particular boundary value problems. Theorem 4.1 deals with the case that $b(x,u) = f(x)$ is independent of u. It was proved for $n = 2$ and constant f by Makar-Limanov (1971, [7]). Theorem 4.2 gives conditions which imply α-concavity when $b(x,u)$ is independent of the variable $x \in \mathbf{R}^n$. It should be remembered that an α-concavity result for any $\alpha \in [-\infty,+\infty]$ implies $(-\infty)$-concavity and therefore guarantees the convexity of the upper level sets.

Theorem 4.1. *Assumptions:*

$n \ge 2$, and Ω is a bounded convex domain in $\mathbf{R}^n$.
$u \in C(\bar{\Omega}) \cap C^2(\Omega)$, $u|_{\partial\Omega} = 0$.
$\beta \ge 1$.
For $x \in \Omega$, u satisfies $\Delta u(x) + f(x) = 0$, for some non-negative β-concave function $f:\Omega \to \mathbf{R}$.

Assertions:

(i) u is α-concave in $\bar{\Omega}$ with $\alpha = \beta/(1 + 2\beta)$.

(ii) If f is a positive constant in Ω, then $\sqrt{u}$ is concave. (That is, if f is $(+\infty)$-concave, then u is $(1/2)$-concave.)

Remark 4.1.1. The boundary value problem in Theorem 4.1 has a unique solution for all f, since f is concave and therefore bounded and locally Lipschitz continuous in Ω. ([2], Theorem 4.3.)

Remark 4.1.2. Any non-negative function f on Ω which is β-concave for some $\beta \geq 1$ is either identically zero in Ω or else positive at all points of Ω. Thus it may be assumed without loss of generality that f and u are both positive in Ω.

Proof of Theorem 4.1. Assertion (ii) follows readily from (i), since if f is a positive constant then f is β-concave for all $\beta \geq 1$, and so by part (i), u is α-concave for all $\alpha < 1/2$. Then by Property 3 of α-concavity, u is $(1/2)$-concave.

In applying Theorem 3.3 to prove assertion (i), it must be shown that b defined by $b(x,t) = f(x)$ satisfies assumptions (i), (ii) and (iii) of that theorem.

$t^{\alpha-1}b(x,t) = t^{\alpha-1}f(x)$ is decreasing with respect to t because $\alpha < 1$ and $f(x) > 0$, and so (i) is satisfied. (ii) is satisfied if $s^{(3\alpha-1)/\alpha}f(x)$ is jointly concave in (x,s). But $s^{(3\alpha-1)/\alpha}$ is $(\alpha/(3\alpha - 1))$-concave for $\alpha \geq 1/3$ (meaning $(+\infty)$-concave for $\alpha = 1/3$) and f is β-concave. So by Property 8, the product is γ-concave, with $\gamma = ((3\alpha - 1)/\alpha + \beta^{-1})^{-1} = 1$. That is, the product is concave, as required.

For (iii) it must be shown that

$$\limsup_{t\to 0^+} t^{-1/\alpha}u(y + t(z - y)) > u(z) \qquad \text{for all } y \in \partial\Omega \text{ and } z \in \Omega.$$

The left-hand side is infinite for sets Ω having an interior sphere property, since by the Hopf boundary point lemma, the interior normal derivative of u is positive everywhere on $\partial\Omega$. A general convex Ω can be approximated by a sequence of convex domains Ω_i included in Ω having the interior sphere property. If u_i is defined on $\bar{\Omega}_i$ by

$$\Delta u_i + f(x) = 0 \qquad \text{in } \Omega_i,$$

and

$$u_i = 0 \qquad \text{on } \partial\Omega_i,$$

then each u_i is α-concave for $\alpha = \beta/(1 + 2\beta)$, and hence $u = \lim_{i\to\infty} u_i$ is α-concave. □

Theorem 4.2. *Assumptions:*

$n \geq 2$, and Ω is a bounded convex domain in $\mathbf{R}^n$.

Ω satisfies an interior sphere condition.

$u \in C(\bar{\Omega}) \cap C^2(\Omega)$, $u|_{\partial\Omega} = 0$, $u|_{\Omega} > 0$.

$0 < \alpha < 1$.

For $x \in \Omega$, u satisfies

$$\Delta u(x) + h(u(x)) = 0,$$

where $h\colon(0,\infty) \to (0,\infty)$ *is a function such that (i) and (ii) hold for all* $t > 0$:

(i) $t^{\alpha-1}h(t)$ *is strictly decreasing with respect to* t

(ii) $t^{(3\alpha-1)/\alpha}h(t^{1/\alpha})$, *or equivalently* $t^{(1-2\alpha)/\alpha}h(t^{-1/\alpha})$, *is concave with respect to* t, (*or* $(1-2\alpha)(1-3\alpha)h + (5\alpha-1)th_t + t^2h_{tt} \leq 0$ *for twice differentiable* h).

Assertions:

u is α*-concave in* $\bar{\Omega}$.

In particular, if $h(t) = kt^\gamma$ *for some constant* $k > 0$ *and* $\gamma \in (0,1)$, *then* u *is* $(1/2)(1-\gamma))$*-concave.*

Proof. The boundary condition,

$$u(z) < \limsup_{t\to 0^+} t^{-1/\alpha}u(y + t(z-y)) \qquad \text{for all } y \in \partial\Omega \text{ and } z \in \Omega,$$

is satisfied, since by the Hopf boundary point lemma, the right-hand side is infinite when $0 < \alpha < 1$. Hence by Theorem 3.3, u is α-concave.

It remains to show that the function $h(t) = kt^\gamma$ has the required properties. Let $\alpha = (1/2)(1-\gamma)$. Then $\gamma = 1 - 2\alpha$ and:

$$t^{\alpha-1}h(t) = kt^{-\alpha}, \qquad \text{which is decreasing,}$$

and

$$t^{(1-2\alpha)/\alpha}h(t^{-1/\alpha}) = k, \qquad \text{which is a concave function.} \qquad \square$$

Remark 4.2.1. The case $\gamma = 0$ is the torsion problem (Theorem 4.1 with f constant), whereas the case $\gamma = 1$ corresponds to the fundamental solution of $\Delta u + \lambda u = 0$, for both of which the formula $\alpha = (1/2)(1-\gamma)$ gives the correct concavity number. Brascamp and Lieb ([1], Theorem 6.1) showed that the fundamental solution of $\Delta u + \lambda u = 0$ (as a special case of a much more general equation) on a bounded convex domain in $\mathbf{R}^n$ for $n \geq 2$ is 0-concave.

Remark 4.2.2. The equivalence of the two conditions in assumption (ii) follows from the fact that any positive function $g(t)$ is concave for $t > 0$ if and only if $tg(t^{-1})$ is concave for $t > 0$.

Remark 4.2.3. The comment should be made here that if h is bounded then condition (ii) of Theorem 4.2 cannot be satisfied for $\alpha > 1/2$. So the theorem cannot give results for such α. In fact, no α-concavity results are possible for some sets for $\alpha > 1/2$ when Δu is bounded and $u = 0$ on the boundary, since then u is bounded by a multiple of the torsion function (the solution for b constant) which therefore constrains the behaviour of u near $\partial\Omega$. In particular, near a boundary point where the set is locally congruent to a cone of narrow enough aperture, u is bounded by a multiple of $d(x)^2$, where $d(x)$ denotes the distance from x to the boundary. (This can be verified by a simple barrier argument.) Hence (by Lemma 6.1) u is at best $(1/2)$-concave for such sets.

Remark 4.2.4. When Theorem 3.3 is applied to an equation of the form

$\Delta u + u^{\gamma}f(x) = 0$ with zero Dirichlet data on the boundary of a bounded convex domain satisfying an interior sphere condition, the result obtained is that u is α-concave with $\alpha = (1 - \gamma)\beta/(1 + 2\beta)$ when f is β-concave, $\beta \geq 1$, and $\gamma \in (0,1)$. This result holds also for $\gamma = 0$ or 1.

5. Liouville's problem. In this section, Theorem 3.1 is applied to Liouville's problem: $\Delta\phi = e^{\phi}$ in a domain Ω, with $\phi(x) \to +\infty$ as $x \to \partial\Omega$. The existence and uniqueness of a solution to this problem are known for bounded convex Ω [9]. In $\mathbf{R}^2$, ϕ is the velocity potential for the path of a point vortex in an otherwise irrotational perfect fluid enclosed in the domain Ω. (See [9] for background.) Lemma 5.1 and Theorem 5.2 show that the velocity potential is convex. Hence the point vortex moves along the boundary of a convex set, and there is a unique interior point at which a free vortex can remain stationary.

Lemma 5.1. *Assumptions:*

$n \geq 2$, and Ω is a bounded convex domain in $\mathbf{R}^n$.

$\partial\Omega$ is C^2 and Ω is uniformly convex (in the sense that for some $\varepsilon > 0$, the principal curvatures of $\partial\Omega$ are everywhere greater than ε).

$u \in C^2(\bar{\Omega})$, $u|_{\partial\Omega} = 0$, $u|_{\Omega} > 0$, and $u_\nu|_{\partial\Omega} > 0$. ($u_\nu$ denotes interior normal derivative of u.)

For $x \in \Omega$, $\phi = -2\log(u)$ satisfies

$$\Delta\phi(x) = \exp(\phi(x)).$$

Assertion:

ϕ is convex.

Proof. Let $v = -\phi$. Then v satisfies:

$$\Delta v + \exp(-v) = 0 \qquad \text{in } \Omega.$$

After putting $b(x,t) = \exp(-t)$, and noting that b is strictly decreasing and 0-concave, Theorem 3.1 may be applied to the equation $\Delta v + b(x,v) = 0$ to show that the convexity function for v does not attain its maximum in $\Omega \times \Omega \times [0,1]$. A result of Korevaar ([5], Lemmas 2.1 and 2.4) implies that the convexity function for v is non-positive in a neighbourhood of $\partial(\Omega \times \Omega) \times [0,1]$ under the conditions of Lemma 5.1. Hence v is concave in $\bar{\Omega}$, and so ϕ is convex in $\bar{\Omega}$. □

Theorem 5.2. *Assumptions:*

Ω is a bounded convex domain in $\mathbf{R}^2$.

$\phi \in C^2(\Omega)$.

$\phi(x) \to +\infty$ as $d(x,\partial\Omega) \to 0$.

For $x \in \Omega$, ϕ satisfies

$$\Delta\phi(x) = \exp(\phi(x)).$$

Assertion:

ϕ is convex.

Proof. An expression is given in [9] (p. 324) for ϕ in terms of any conformal mapping f from Ω to the open unit disc in $\mathbf{R}^2$ with centre at 0, and its complex derivative f':

$$\phi = -2\log(u), \qquad \text{where } u(z) = k|f'(z)|^{-1}(1 - |f(z)|^2) \text{ and } k = 1/\sqrt{8}.$$

But f has a well-known representation in terms of the Green's function G for Ω with pole at $f^{-1}(0)$. If $g = 2\pi G$, then $f = \exp(-g - ih)$ for some harmonic function h in Ω conjugate to g. Thus $u = 2k|\nabla g|^{-1}\sinh(g)$. When its discontinuity at the pole of g is removed, the function u is analytic in Ω. Some straightforward calculations show that the resulting expression for ϕ is indeed the solution of Liouville's problem.

Suppose now that Ω is uniformly convex and that $\partial\Omega$ is $C^{3,\varepsilon}$ with $0 < \varepsilon < 1$. It follows from the Hopf boundary point lemma that $|\nabla g|$ is bounded below on $\partial\Omega$, and from Kellogg's theorem ([2], Theorem 6.19) that g is $C^{3,\varepsilon}$ in a neighbourhood of $\partial\Omega$. So $u \in C^{2,\varepsilon}(\bar{\Omega})$ and all of the requirements of Lemma 5.1 are met. Hence ϕ is convex in $\bar{\Omega}$.

The result follows immediately for general bounded convex sets in $\mathbf{R}^2$ by approximating Ω externally with uniformly convex sets with $C^{3,\varepsilon}$ boundaries. Comparison principles for Liouville's problem ([9], appendix A) guarantee that the approximate solutions converge uniformly on compact subsets of Ω to the exact solution. □

The uniqueness of the stationary point of a vortex in a bounded convex domain in $\mathbf{R}^2$—that is, the point where $\Delta\phi = 0$—follows from the convexity and analyticity of ϕ.

The question of whether some sort of convexity result can be obtained for the equation $\Delta\phi = e^{\phi}$ when ϕ is a prescribed (finite) constant on $\partial\Omega$ is neatly answered by the following theorem, discovered by Grant Keady [3] and generously communicated to the author for inclusion in this article. It is a problem for which an estimate of the gradient of the solution is required before Theorem 3.1 can be applied. (See remark 3.1.1).

Theorem 5.3. *Assumptions:*

$n \geq 2$, and Ω is a bounded convex domain in $\mathbf{R}^n$.

k is a real constant, and $\phi \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies

$$\Delta\phi(x) = \exp(\phi(x)) \qquad \text{for } x \in \Omega,$$

and

$$\phi(x) = k \qquad \text{for } x \in \partial\Omega.$$

Assertion:

$$k - \phi \text{ is } (1/2)\text{-concave in } \bar{\Omega}.$$

Proof. $k - \phi$ is positive in Ω (by the maximum principle for Δ in Ω). Let $u^2 = k - \phi$ in Ω, with $u \geq 0$. Then

$$\Delta u + b(x,u,Du) = 0 \qquad \text{in } \Omega,$$

with

$$b(x,u,p) = u^{-1}\left(|p|^2 + \frac{1}{2}\exp(k - u^2)\right).$$

Also, $u|_{\partial\Omega} = 0$ and $u|_{\Omega} > 0$. Thus Theorem 3.1 can be applied if b satisfies (i) and (ii) of that theorem. Clearly $b(x,\cdot,p)$ is decreasing for all $(x,p) \in \Omega \times \mathbf{R}^n$, so that (i) is satisfied. But $b(\cdot,\cdot,p)$ is not harmonic concave for all p. Indeed, because b is positive, twice differentiable and independent of x, b is harmonic concave with respect to u in a subinterval of $(0,\infty)$ for a fixed value of p if and only if $2b_u^2 - bb_{uu} \geq 0$ in that subinterval. (This follows from Property 4.) A short calculation gives:

$$2b_u^2 - bb_{uu} = u^{-2}\exp(k - u^2)\left(\frac{1}{2}(3 + 2u^2)\exp(k - u^2) + (3 - 2u^2)|p|^2\right).$$

This is not non-negative for all $(u,p) \in (0,\infty) \times \mathbf{R}^n$. However, $p = \nabla u$ can be bounded in terms of u by a maximum principle ([8], equation (2.12)), which states for the present problem (under the assumption that $\partial\Omega$ is $C^{2,\varepsilon}$ with $0 < \varepsilon < 1$) that

$$|\nabla\phi(x)|^2 \leq 2\int_{k-\phi(x)}^{k-\phi_m} \exp(k - t)\,dt,$$

where $\phi_m = \inf\{\phi(x) : x \in \Omega\}$. Then

$$2u^2|p|^2 = \frac{1}{2}|\nabla\phi|^2 \leq e^{\phi} = \exp(k - u^2).$$

For each $p \in \mathbf{R}^n$, this confines u to a subinterval of $(0,\infty)$ (because $u^{-2}\exp(k - u^2)$ is a monotonic function of u). For u in this interval, $2b_u^2 - bb_{uu}$ is then readily seen to be non-negative. So assumption (ii) of Theorem 3.1 is satisfied, and its conclusion follows. Since the Hopf boundary point lemma for u^2 on Ω guarantees that the normal derivative of u is infinite on $\partial\Omega$, Theorem 3.2 gives the conclusion that u is concave. That is, $k - \phi$ is $(1/2)$-concave.

To eliminate the need for a $C^{2,\varepsilon}$ boundary, approximation of Ω externally may again be used as in the proof of Theorem 5.2. □

Remark 5.3.1. The concavity number $1/2$ here is sharp. (See remark 4.2.3.)

6. A sharpness result. This section gives a proof (Theorem 6.2) that the number $\alpha = \beta/(1 + 2\beta)$ in Theorem 4.1 is sharp, as is the number $\alpha = 1/2$ when f is constant. This shows that at least some of the concavity numbers calculated in this paper are not mere artifacts of the method of proof. Lemma 6.1 is useful for showing that certain functions are not α-concave.

Lemma 6.1. *Assumptions:*

$n \geq 2$, and Ω is a convex domain in $\mathbf{R}^n$.
$u \in C(\bar{\Omega})$, $u|_{\partial\Omega} = 0$, and $u|_{\Omega} > 0$.

$\alpha > 0$, $y \in \partial\Omega$ and $z \in \Omega$.
$\liminf_{t\to 0^+} t^{-1/\alpha}u(y + t(z - y)) = 0.$

Assertion:
u is not α-concave in Ω.

Proof. Suppose u is α-concave. Then for $t \in (0,1)$, the concavity of u^α implies that

$$u^\alpha(y + t(z - y)) \geq (1 - t)u^\alpha(y) + tu^\alpha(z) = tu^\alpha(z),$$

so that $t^{-1/\alpha}u(y + t(z - y)) \geq u(z) > 0$, which contradicts assumption. □

Now let $n \geq 2$ and $x \in \mathbf{R}^n$, and write $x_n = x \cdot e_n$ and $x' = x - x_ne_n$, where $e_n = (0,\ldots 0,1)$ is the n^{th} unit vector in the standard basis for $\mathbf{R}^n$. Define an infinite open cone K for $a \in (0,1)$ by $K = \{x \in \mathbf{R}^n : |x'| < ax_n\}$.

Theorem 6.2. *Assumptions:*
Ω is a bounded convex domain in $\mathbf{R}^n$ and a subset of K.
$\beta \geq 1$, $0 \in \partial\Omega$ and $e_n \in \Omega$.

(6.1) *$u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$\Delta u(x) + f(x) = 0 \qquad \textit{for } x \in \Omega,$$

and

$$u(x) = 0 \qquad \textit{for } x \in \partial\Omega,$$

where $f(x) = k_1x_n^q - k_2x_n^{q-2}|x'|^2$ for $x \in \Omega$, $q = \beta^{-1}$,
$k_1 = 2(n - 1) - a^2(q + 1)(q + 2)$, and $k_2 = q(1 - q)$.

(6.2) $a^2 \leq (n - 1)/(2 + q^2)$.

Assertions:
(i) f is non-negative and β-concave.
(ii) u is not α-concave when $\alpha > \beta/(1 + 2\beta)$.
(iii) For $q = 0$, f is $(+\infty)$-concave (that is, constant), and u is not α-concave for $\alpha > 1/2$.

Remark 6.2.1. Problem (6.1) has a unique solution, as f is a bounded locally Lipschitz function in Ω.

Proof. (i) For $x \in K$, $|x'|^2 \leq a^2x_n^2$, and so by (6.2)

$$\begin{aligned} f(x) &\geq x_n^q(2(n - 1) - a^2(q + 1)(q + 2) - q(1 - q)a^2) \\ &\geq 2(n - 1)(1 - q)^2x_n^q/(2 + q^2) \\ &\geq 0. \end{aligned}$$

If $q = 1$ then $f^\beta = f = 2(n - 1 - 3a^2)x_n$, which is a non-negative concave function, as required for $\beta = 1$. Suppose that $q \in (0,1)$. Then f^β is concave if and only if $(k_2^{-1}f)^\beta$ is concave. So without loss of generality k_1 and k_2 may be replaced by $k = k_1 k_2^{-1}$ and 1 respectively. k_1 is non-negative for $q \leq 1$, and so k is also non-negative. It follows from Property 4 (section 2) that f is β-concave in Ω if and only if

$$qff_{\theta\theta} + (1 - q)f_\theta^2 \leq 0 \qquad \text{in } \Omega,$$

for all directions θ, since $\beta - 1 = q^{-1}(1 - q)$ and $q > 0$. This quantity may be calculated as follows:

$$f(x) = x_n^{q-2}(kx_n^2 - |x'|^2).$$

$$f_\theta(x) = x_n^{q-3}(kqx_n^2\theta_n - (q - 2)|x'|^2\theta_n - 2x_n x' \cdot \theta).$$

$$f_{\theta\theta}(x) = x_n^{q-4}(kq(q - 1)x_n^2\theta_n^2 - 4(q - 2)x_n\theta_n x' \cdot \theta - (q - 2)(q - 3)\theta_n^2|x'|^2 - 2x_n^2|\theta'|^2).$$

Hence

$$\frac{1}{2}x_n^{6-2q}(qff_{\theta\theta} + (1 - q)f_\theta^2) = (kqx_n^2 + (q - 2)|x'|^2)(2x_n\theta_n x' \cdot \theta - \theta_n^2|x'|^2)$$
$$+ x_n^2(2(1 - q)(x' \cdot \theta)^2 + q|x'|^2|\theta'|^2 - kqx_n^2|\theta'|^2).$$

But $x' \cdot \theta \leq |x'||\theta'|$, $x_n \geq 0$, and $\theta_n \leq |\theta_n|$, and so by virtue of (6.2),

$$\begin{aligned}(kqx_n^2 &+ (q - 2)|x'|^2) \\ &= (k_1x_n^2 - (2 - q)(1 - q)|x'|^2)/(1 - q) \\ &\geq x_n^2(2(n - 1) - a^2(q + 1)(q + 2) - (2 - q)(1 - q)a^2)/(1 - q) \\ &\geq (n - 1)x_n^2(2 - ((q + 1)(q + 2) + (2 - q)(1 - q))/(2 + q^2))/(1 - q) \\ &= 0.\end{aligned}$$

Hence

$$\frac{1}{2}x_n^{6-2q}(qff_{\theta\theta} + (1 - q)f_\theta^2) \leq -(kqx_n^2 + (q - 2)|x'|^2)(|\theta_n||x'| - |\theta'|x_n)^2,$$

which is non-positive, so that f is indeed β-concave.

(ii) Define a function $b:\bar{K} \to \mathbf{R}$ by $b(x) = x_n^q(a^2x_n^2 - |x'|^2)$. Then $b(x) \geq 0$ for all $x \in \bar{K}$, and direct calculation shows that for $x \in K$, $\Delta b(x) + f(x) = 0$. Since Ω is a subset of K, $b(x) \geq 0$ for all $x \in \partial\Omega$. But $\Delta b = \Delta u$ in Ω. So the comparison principle for Δ on Ω implies that $u(x) \leq b(x)$ for all $x \in \bar{\Omega}$. For $t \in (0,1]$, let $x = te_n$. Then $x \in \Omega$ and so $u(x) \leq b(x) = a^2t^{q+2}$. Hence

$$\limsup_{t \to 0^+} t^{-1/\alpha}u(x) = 0$$

if $-\alpha^{-1} + q + 2 > 0$; that is, if $\alpha > (q + 2)^{-1} = \beta/(1 + 2\beta)$. Then Lemma 6.1 with $y = 0$ and $z = e_n$ shows that u is not α-concave for such α.

(iii) Putting $q = 0$ in the expressions for f and b makes f a non-negative constant, $2(n - 1 - a^2)$, while $b(x) = a^2x_n^2 - |x'|^2$, so that u^α is not concave when $\alpha > 1/2$, using the same kind of argument as in the proof of part (ii). □

Appendix

Lemma A1. *If M and N are real symmetric positive semidefinite matrices, then* $\operatorname{Tr}(MN) \geq 0$.

Proof. There exists a real orthogonal matrix P such that $D = P^{-1}NP$ is diagonal. Then

$$\begin{aligned}\operatorname{Tr}(MN) &= \operatorname{Tr}(P^{-1}MNP)\\ &= \operatorname{Tr}(P^{-1}MPD)\\ &\geq 0,\end{aligned}$$

because the diagonal elements of $P^{-1}MP$ are all non-negative. □

Lemma A2. *If $n \geq 1$, K is a convex subset of $\mathbf{R}^n$, and g is a positive concave function on K, then the function on K with value $x_1^2 g(x)^{-1}$ is convex.*

Proof. Let y and z be in K and $0 \leq \lambda \leq 1$. Put $x = (1 - \lambda)y + \lambda z$. Then

$$\begin{aligned}&(1 - \lambda)y_1^2 g(y)^{-1} + \lambda z_1^2 g(z)^{-1} - x_1^2 g(x)^{-1}\\ &\qquad \geq (1 - \lambda)y_1^2 g(y)^{-1} + \lambda z_1^2 g(z)^{-1} - x_1^2((1 - \lambda)g(y) + \lambda g(z))^{-1}\\ &\qquad = \lambda(1 - \lambda)(y_1 g(z) - z_1 g(y))^2 g(y)^{-1} g(z)^{-1}((1 - \lambda)g(y) + \lambda g(z))^{-1}\\ &\qquad \geq 0.\end{aligned}$$

The function is therefore convex. □

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