

# Convex Hypersurfaces of Constant Curvature in Hyperbolic Space

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## Abstract

We show that for a very general and natural class of curvature functions, the problem of finding a complete strictly convex hypersurface in  $\mathbb{H}^{n+1}$  satisfying  $f(\kappa) = \sigma \in (0, 1)$  with a prescribed asymptotic boundary  $\Gamma$  at infinity has at least one solution which is a “vertical graph” over the interior (or the exterior) of  $\Gamma$ . There is uniqueness for a certain subclass of these curvature functions which includes the curvature quotients  $(\frac{H_n}{H_l})^{\frac{1}{n-l}}$ ,  $l = n - 2$ , or  $n - 1$ . For smooth simple  $\Gamma$ , as  $\sigma$  varies between 0 and 1, these hypersurfaces foliate the two components of the complement of the hyperbolic convex hull of  $\Gamma$ .

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## 1 Introduction

In this paper we return to our earlier study [7] of complete locally strictly convex hypersurfaces of constant curvature in hyperbolic space  $\mathbb{H}^{n+1}$  with a prescribed asymptotic boundary at infinity. Given  $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$  and a smooth symmetric function  $f$  of  $n$  variables, we seek a complete hypersurface  $\Sigma$  in  $\mathbb{H}^{n+1}$  satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1.1}$$

$$\partial\Sigma = \Gamma \tag{1.2}$$

where  $\kappa[\Sigma] = (\kappa_1, \dots, \kappa_n)$  denotes the *positive* hyperbolic principal curvatures of  $\Sigma$  and  $\sigma \in (0, 1)$  is a constant.

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We will use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{\sum_{i=1}^{n+1} dx_i^2}{x_{n+1}^2}. \quad (1.3)$$

Thus  $\partial_\infty \mathbb{H}^{n+1}$  is naturally identified with  $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$  and (1.2) may be understood in the Euclidean sense. For convenience we say  $\Sigma$  has compact asymptotic boundary if  $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$  is compact with respect to the Euclidean metric in  $\mathbb{R}^n$ .

The function  $f$  is assumed to satisfy the fundamental structure conditions in

$$K_n^+ := \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} : \quad (1.4)$$

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \text{ in } K_n^+, \quad 1 \leq i \leq n, \quad (1.5)$$

$$f \text{ is a concave function in } K_n^+, \quad (1.6)$$

and

$$f > 0 \text{ in } K_n^+, \quad f = 0 \text{ on } \partial K_n^+. \quad (1.7)$$

In addition, we shall assume that  $f$  is normalized

$$f(1, \dots, 1) = 1 \quad (1.8)$$

and satisfies the following more technical assumptions

$$f \text{ is homogeneous of degree one} \quad (1.9)$$

and

$$\lim_{R \rightarrow +\infty} f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \quad (1.10)$$

for some fixed  $\varepsilon_0 > 0$  and  $\delta_0 > 0$ , where  $B_{\delta_0}(\mathbf{1})$  is the ball of radius  $\delta_0$  centered at  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ .

All these assumptions are satisfied by  $f = (H_n/H_l)^{\frac{1}{n-l}}$ ,  $0 \leq l < n$ , where  $H_l$  is the normalized  $l$ -th elementary symmetric polynomial ( $H_0 = 1$ ,  $H_1 = H$  and  $H_n = K$  the mean and extrinsic Gauss curvatures, respectively). See [2] for proof of (1.5) and (1.6). For (1.10) one easily computes that

$$\lim_{R \rightarrow +\infty} f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) = \left(\frac{n}{l}\right)^{\frac{1}{n-l}}.$$

Moreover, if  $g^k$ ,  $k = 1, \dots, N$  satisfy (1.5)-(1.10), then so does the ‘‘concave sum’’  $f = \sum_{k=1}^N \alpha_k g^k$  or ‘‘concave product’’  $f = \prod_{k=1}^N (g^k)^{\alpha_k}$  where  $\alpha_k > 0$ ,  $\sum_{k=1}^N \alpha_k =$

1.

Since  $f$  is symmetric, by (1.6), (1.8) and (1.9) we have

$$f(\lambda) \leq f(\mathbf{1}) + \sum f_i(\mathbf{1})(\lambda_i - 1) = \sum f_i(\mathbf{1})\lambda_i = \frac{1}{n} \sum \lambda_i \text{ in } K_n^+ \quad (1.11)$$

and

$$\sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1 - \lambda_i) \geq f(\mathbf{1}) = 1 \text{ in } K_n^+. \quad (1.12)$$

In this paper all hypersurfaces in  $\mathbb{H}^{n+1}$  we consider are assumed to be connected and orientable. If  $\Sigma$  is a complete hypersurface in  $\mathbb{H}^{n+1}$  with compact asymptotic boundary at infinity, then the normal vector field of  $\Sigma$  is chosen to be the one pointing to the unique unbounded region in  $\mathbb{R}_+^{n+1} \setminus \Sigma$ , and the (both hyperbolic and Euclidean) principal curvatures of  $\Sigma$  are calculated with respect to this normal vector field.

As in our earlier work [11, 10, 5, 7, 6], we will take  $\Gamma = \partial\Omega$  where  $\Omega \subset \mathbb{R}^n$  is a smooth domain and seek  $\Sigma$  as the graph of a function  $u(x)$  over  $\Omega$ , i.e.

$$\Sigma = \{(x, x_{n+1}) : x \in \Omega, x_{n+1} = u(x)\}.$$

Then the coordinate vector fields and upper unit normal are given by

$$X_i = e_i + u_i e_{n+1}, \quad \mathbf{n} = u\nu = u \frac{(-u_i e_i + e_{n+1})}{w},$$

where  $w = \sqrt{1 + |\nabla u|^2}$ . The first fundamental form  $g_{ij}$  is then given by

$$g_{ij} = \langle X_i, X_j \rangle = \frac{1}{u^2} (\delta_{ij} + u_i u_j) = \frac{g_{ij}^e}{u^2}. \quad (1.13)$$

To compute the second fundamental form  $h_{ij}$  we use

$$\Gamma_{ij}^k = \frac{1}{x_{n+1}} \{-\delta_{jk} \delta_{in+1} - \delta_{ik} \delta_{jn+1} + \delta_{ij} \delta_{kn+1}\} \quad (1.14)$$

to obtain

$$\nabla_{X_i} X_j = \left( \frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}} \right) e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}. \quad (1.15)$$

Then

$$\begin{aligned} h_{ij} &= \langle \nabla_{X_i} X_j, u\nu \rangle = \frac{1}{uw} \left( \frac{\delta_{ij}}{u} + u_{ij} - \frac{u_i u_j}{u} + 2 \frac{u_i u_j}{u} \right) \\ &= \frac{1}{u^2 w} (\delta_{ij} + u_i u_j + uu_{ij}) = \frac{h_{ij}^e}{u} + \frac{\nu^{n+1}}{u^2} g_{ij}^e. \end{aligned} \quad (1.16)$$

The hyperbolic principal curvatures  $\kappa_i$  of  $\Sigma$  are the roots of the characteristic equation

$$\det(h_{ij} - \kappa g_{ij}) = u^{-n} \det(h_{ij}^e - \frac{1}{u} (\kappa - \frac{1}{w}) g_{ij}^e) = 0.$$

Therefore,

$$\kappa_i = u \kappa_i^e + \nu^{n+1}. \quad (1.17)$$

The relations (1.16) and (1.17) are easily seen to hold for parametric hypersurfaces.

One beautiful consequence of (1.16) is the following result of [7].

**Theorem 1.1.** *Let  $\Sigma$  be a complete locally strictly convex  $C^2$  hypersurface in  $\mathbb{H}^{n+1}$  with compact asymptotic boundary at infinity. Then  $\Sigma$  is the (vertical) graph of a function  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ ,  $u > 0$  in  $\Omega$  and  $u = 0$  on  $\bar{\Omega}$ , for some domain  $\Omega \subset \mathbb{R}^n$ :*

$$\Sigma = \{(x, u(x)) \in \mathbb{R}_+^{n+1} : x \in \Omega\}$$

such that

$$\{\delta_{ij} + u_i u_j + u u_{ij}\} > 0 \text{ in } \Omega. \quad (1.18)$$

That is, the function  $u^2 + |x|^2$  is strictly convex.

According to Theorem 1.1, our assumption that  $\Sigma$  is a graph is completely general and the asymptotic boundary  $\Gamma$  must be the boundary of some bounded domain  $\Omega$  in  $\mathbb{R}^n$ .

Problem (1.1)-(1.2) then reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \text{ in } \Omega \subset \mathbb{R}^n \quad (1.19)$$

with the boundary condition

$$u = 0 \text{ on } \partial\Omega. \quad (1.20)$$

We seek solutions of equation (1.19) satisfying (1.18). Following the literature we call such solutions *admissible*. By [2] condition (1.5) implies that equation (1.19) is elliptic for admissible solutions. Our goal is to show that the Dirichlet problem (1.19)-(1.20) admits smooth admissible solutions for all  $0 < \sigma < 1$ , which is optimal.

Our main result of the paper may be stated as follows.

**Theorem 1.2.** *Let  $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$  where  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^n$ . Suppose  $\sigma \in (0, 1)$  and that  $f$  satisfies conditions (1.5)-(1.10) in  $K_n^+$ . Then there exists a complete locally strictly convex hypersurface  $\Sigma$  in  $\mathbb{H}^{n+1}$  satisfying (1.1)-(1.2) with uniformly bounded principal curvatures*

$$|\kappa[\Sigma]| \leq C \text{ on } \Sigma. \quad (1.21)$$

Moreover,  $\Sigma$  is the graph of an admissible solution  $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$  of the Dirichlet problem (1.19)-(1.20). Furthermore,  $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega})$  and

$$\begin{aligned} u|D^2u| &\leq C \text{ in } \Omega, \\ \sqrt{1 + |Du|^2} &= \frac{1}{\sigma} \text{ on } \partial\Omega \end{aligned} \quad (1.22)$$

For Gauss curvature,  $f(\lambda) = (H_n)^\frac{1}{n}$ , Theorem 1.2 was proved by Rosenberg and Spruck [11].

Equation (1.19) is singular where  $u = 0$ . It is therefore natural to approximate the boundary condition (1.20) by

$$u = \epsilon > 0 \quad \text{on } \partial\Omega. \quad (1.23)$$

When  $\epsilon$  is sufficiently small, we showed in [7] that the Dirichlet problem (1.19),(1.23) is solvable for all  $\sigma \in (0, 1)$ .

**Theorem 1.3.** *Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$  and  $\sigma \in (0, 1)$ . Suppose  $f$  satisfies (1.5)-(1.10) in  $K_n^+$ . Then for any  $\epsilon > 0$  sufficiently small, there exists an admissible solution  $u^\epsilon \in C^\infty(\bar{\Omega})$  of the Dirichlet problem (1.19), (1.23). Moreover,  $u^\epsilon$  satisfies the a priori estimates*

$$\sqrt{1 + |Du^\epsilon|^2} \leq \frac{1}{\sigma} + C\epsilon, \quad u^\epsilon |D^2u^\epsilon| \leq C \quad \text{on } \partial\Omega, \quad (1.24)$$

and

$$u^\epsilon |D^2u^\epsilon| \leq \frac{C}{\epsilon^2} \quad \text{in } \Omega. \quad (1.25)$$

where  $C$  is independent of  $\epsilon$ .

*Remark 1.4.* The global gradient estimate, Lemma 3.4 of [7] is not correct as stated. This may be corrected using the convexity argument of [11] or using Corollary 3.3 of section 3 of this paper. Theorem 1.3 above as well as Theorem 1.2 of [7] remain valid. However no a priori uniqueness can be asserted. In Theorem 1.6 we prove a uniqueness result for a special class of curvature functions.

Our main technical difficulty in proving Theorem 1.2 is that the estimate (1.25) does not allow us to pass to the limit. In [7] we were able to obtain a global estimate independent of  $\epsilon$  for the hyperbolic principal curvatures for  $\sigma^2 > \frac{1}{8}$ . In this paper we obtain such estimates for all  $\sigma \in (0, 1)$  by proving a maximum principle for the largest hyperbolic principal curvature.

**Theorem 1.5.** *Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$  and  $\sigma \in (0, 1)$ . Suppose  $f$  satisfies (1.5)-(1.10) in  $K_n^+$ . Then for any admissible solution  $u^\epsilon \in C^\infty(\bar{\Omega})$  of the Dirichlet problem (1.19), (1.23),*

$$\max_{\mathbf{x} \in \Sigma^\epsilon} \kappa_{\max}(\mathbf{x}) \leq C(1 + \max_{\mathbf{x} \in \partial\Sigma^\epsilon} \kappa_{\max}(\mathbf{x})) \quad (1.26)$$

where  $\Sigma^\epsilon = \text{graph } u^\epsilon$  and  $C$  is independent of  $\epsilon$ .

By Theorem 1.5, the hyperbolic principal curvatures of the admissible solution  $u^\epsilon$  given in Theorem 1.3 are uniformly bounded above independent of  $\epsilon$ . Since  $f(\kappa[u^\epsilon]) = \sigma$  and  $f = 0$  on  $\partial K_n^+$ , the hyperbolic principal curvatures admit a uniform positive lower bound independent of  $\epsilon$  and therefore (1.19) is uniformly elliptic on compact subsets of  $\Omega$  for the solution  $u^\epsilon$ . By the interior estimates of Evans and Krylov, we obtain uniform  $C^{2,\alpha}$  estimates for any compact subdomain

of  $\Omega$ . The proof of Theorem 1.2 is now routine.

Finally we prove a uniqueness result and as an application prove a result about foliations. This latter result is relevant to the study of foliations of the complement of the convex core of quasi-fuchsian manifolds (see [8], [11], [13]).

**Theorem 1.6.** *Suppose  $f$  satisfies (1.5)-(1.10) in  $K_n^+$  and in addition,*

$$\sum f_i > \sum \lambda_i^2 f_i \text{ in } K_n^+ \cap \{0 < f < 1\}. \quad (1.27)$$

*Then the solutions given in Theorem 1.2 and Theorem 1.3 are unique. In particular uniqueness holds for  $f = (\frac{H_n}{H_l})^{\frac{1}{n-l}}$  with  $l = 1$  or  $l = 2$ .*

**Theorem 1.7.** *a. Let  $f$  satisfy the conditions of Theorem 1.6 and assume that  $\Gamma$  is smooth and connected. Then for each  $\sigma \in (0, 1)$  there are exactly two embedded strictly locally convex hypersurfaces satisfying (1.1), (1.2). Each surface is a graph of  $u^\sigma \in C^\infty(\Omega^\pm) \cap C^1(\bar{\Omega}^\pm)$  where  $\Omega^\pm$  are the components of the complement of  $\Gamma$ . Moreover the solution hypersurfaces  $\Sigma^\sigma = \text{graph } u^\sigma$  have uniformly bounded principal curvatures and foliate each component of  $\mathbb{H}^{n+1} \setminus \mathcal{CH}(\Gamma)$ , the complement of the hyperbolic convex hull of  $\Gamma$ .*

*b. Let  $f = \frac{H_n}{H_{n-1}}$  and  $\Gamma = \partial\Omega$  where  $\Omega$  is a simply connected Jordan domain. If  $n > 2$ , assume in addition that  $\Gamma$  is regular for Laplace's equation. Then the conclusions of part a. hold with  $u^\sigma(x) \in C^\infty(\Omega^\pm) \cap C^0(\bar{\Omega}^\pm)$  and the principal curvatures are uniformly bounded on compact subsets.*

*Remark 1.8.* i. Graham Smith pointed out to us that in the special case  $n = 3$ ,  $f = (\frac{K}{H})^{\frac{1}{2}}$  is his special Lagrangian curvature with angle  $\theta = \pi$  and interior curvature bounds follow from the geometric ideas of his paper [14]. Moreover in Lemma 7.4 of [13] he showed that special Lagrangian curvature with angle  $\theta \geq (n-1)\frac{\pi}{2}$  satisfies our uniqueness condition (1.27). Thus by Theorems 1.6 and 1.7, the existence of foliations of constant special Lagrangian curvature can be proven for  $\theta \geq (n-1)\frac{\pi}{2}$  for all  $n$ . This includes the special case  $f = K^{\frac{1}{2}}$  when  $n = 2$  and  $f = (\frac{K}{H})^{\frac{1}{3}}$  for  $n = 3$  mentioned above.

ii. Rosenberg and Spruck [11] proved part b of Theorem 1.7 for  $f = K^{\frac{1}{2}}$  in case  $n = 2$ . Here and also in [11], no claim is made about the higher regularity of the  $\mathcal{CH}(\Gamma)$ . In other words, the curvature estimates obtained in the proof of Theorem 1.7 (global for  $\Gamma$  smooth and interior for  $\Gamma$  Jordan) blow up as  $\sigma \rightarrow 0$ . We have not yet derived interior curvature estimates for the case  $f = \frac{H_n}{H_{n-2}}$  in the general case.

The organization of the paper is as follows. In section 2 we establish some basic identities on a hypersurface  $\Sigma$  satisfying (1.1) that will form the basis of the global gradient estimates derived in section 3 and the maximum principle for  $\kappa_{\max}$ , the largest principal curvature of  $\Sigma$ , which is carried out in section 4. Finally in section 5 we prove the uniqueness Theorem 1.6 and the foliation Theorem 1.7.

## 2 Formulas on hypersurfaces

In this section we will derive some basic identities on a hypersurface by comparing the induced hyperbolic and Euclidean metrics.

Let  $\Sigma$  be a hypersurface in  $\mathbb{H}^{n+1}$ . We shall use  $g$  and  $\nabla$  to denote the induced hyperbolic metric and Levi-Civita connection on  $\Sigma$ , respectively. As  $\Sigma$  is also a submanifold of  $\mathbb{R}^{n+1}$ , we shall usually distinguish a geometric quantity with respect to the Euclidean metric by adding a ‘tilde’ over the corresponding hyperbolic quantity. For instance,  $\tilde{g}$  denotes the induced metric on  $\Sigma$  from  $\mathbb{R}^{n+1}$ , and  $\tilde{\nabla}$  is its Levi-Civita connection.

Let  $\mathbf{x}$  be the position vector of  $\Sigma$  in  $\mathbb{R}^{n+1}$  and set

$$u = \mathbf{x} \cdot \mathbf{e}$$

where  $\mathbf{e}$  is the unit vector in the positive  $x_{n+1}$  direction in  $\mathbb{R}^{n+1}$ , and ‘ $\cdot$ ’ denotes the Euclidean inner product in  $\mathbb{R}^{n+1}$ . We refer  $u$  as the *height function* of  $\Sigma$ .

Throughout the paper we assume  $\Sigma$  is orientable and let  $\mathbf{n}$  be a (global) unit normal vector field to  $\Sigma$  with respect to the hyperbolic metric. This also determines a unit normal  $\nu$  to  $\Sigma$  with respect to the Euclidean metric by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote  $\nu^{n+1} = \mathbf{e} \cdot \nu$ .

Let  $(z_1, \dots, z_n)$  be local coordinates and

$$\tau_i = \frac{\partial}{\partial z_i}, \quad i = 1, \dots, n.$$

The hyperbolic and Euclidean metrics of  $\Sigma$  are given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle, \quad \tilde{g}_{ij} = \tau_i \cdot \tau_j = u^2 g_{ij},$$

while the second fundamental forms are

$$\begin{aligned} h_{ij} &= \langle D_{\tau_i} \tau_j, \mathbf{n} \rangle = -\langle D_{\tau_i} \mathbf{n}, \tau_j \rangle, \\ \tilde{h}_{ij} &= \nu \cdot \tilde{D}_{\tau_i} \tau_j = -\tau_j \cdot \tilde{D}_{\tau_i} \nu, \end{aligned} \quad (2.1)$$

where  $D$  and  $\tilde{D}$  denote the Levi-Civita connection of  $\mathbb{H}^{n+1}$  and  $\mathbb{R}^{n+1}$ , respectively. The following relations are well known (see (1.16), (1.17)):

$$h_{ij} = \frac{1}{u} \tilde{h}_{ij} + \frac{\nu^{n+1}}{u^2} \tilde{g}_{ij} \quad (2.2)$$

and

$$\kappa_i = u \tilde{\kappa}_i + \nu^{n+1}, \quad i = 1, \dots, n \quad (2.3)$$

where  $\kappa_1, \dots, \kappa_n$  and  $\tilde{\kappa}_1, \dots, \tilde{\kappa}_n$  are the hyperbolic and Euclidean principal curvatures, respectively. The Christoffel symbols are related by the formula

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k - \frac{1}{u} (u_i \delta_{kj} + u_j \delta_{ik} - \tilde{g}^{kl} u_l \tilde{g}_{ij}). \quad (2.4)$$

It follows that for  $v \in C^2(\Sigma)$

$$\nabla_{ij}v = v_{ij} - \Gamma_{ij}^k v_k = \tilde{\nabla}_{ij}v + \frac{1}{u}(u_i v_j + u_j v_i - \tilde{g}^{kl} u_k v_l \tilde{g}_{ij}) \quad (2.5)$$

where (and in sequel)

$$v_i = \frac{\partial v}{\partial z_i}, \quad v_{ij} = \frac{\partial^2 v}{\partial z_i \partial z_j}, \quad \text{etc.}$$

In particular,

$$\nabla_{ij}u = \tilde{\nabla}_{ij}u + \frac{2u_i u_j}{u} - \frac{1}{u} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij} \quad (2.6)$$

and

$$\nabla_{ij} \frac{1}{u} = -\frac{1}{u^2} \tilde{\nabla}_{ij}u + \frac{1}{u^3} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij}. \quad (2.7)$$

Moreover,

$$\nabla_{ij} \frac{v}{u} = v \nabla_{ij} \frac{1}{u} + \frac{1}{u} \tilde{\nabla}_{ij}v - \frac{1}{u^2} \tilde{g}^{kl} u_k v_l \tilde{g}_{ij}. \quad (2.8)$$

In  $\mathbb{R}^{n+1}$ ,

$$\begin{aligned} \tilde{g}^{kl} u_k u_l &= |\tilde{\nabla}u|^2 = 1 - (\nu^{n+1})^2 \\ \tilde{\nabla}_{ij}u &= \tilde{h}_{ij} \nu^{n+1}. \end{aligned} \quad (2.9)$$

Therefore, by (2.3) and (2.7),

$$\begin{aligned} \nabla_{ij} \frac{1}{u} &= -\frac{\nu^{n+1}}{u^2} \tilde{h}_{ij} + \frac{1}{u^3} (1 - (\nu^{n+1})^2) \tilde{g}_{ij} \\ &= \frac{1}{u} (g_{ij} - \nu^{n+1} h_{ij}). \end{aligned} \quad (2.10)$$

We note that (2.8) and (2.10) still hold for general local frames  $\tau_1, \dots, \tau_n$ . In particular, if  $\tau_1, \dots, \tau_n$  are orthonormal in the hyperbolic metric, then  $g_{ij} = \delta_{ij}$  and  $\tilde{g}_{ij} = u^2 \delta_{ij}$ .

We now consider equation (1.1) on  $\Sigma$ . Let  $\mathcal{A}$  be the vector space of  $n \times n$  matrices and

$$\mathcal{A}^+ = \{A = \{a_{ij}\} \in \mathcal{A} : \lambda(A) \in K_n^+\},$$

where  $\lambda(A) = (\lambda_1, \dots, \lambda_n)$  denotes the eigenvalues of  $A$ . Let  $F$  be the function defined by

$$F(A) = f(\lambda(A)), \quad A \in \mathcal{A}^+ \quad (2.11)$$

and denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \quad (2.12)$$

Since  $F(A)$  depends only on the eigenvalues of  $A$ , if  $A$  is symmetric then so is the matrix  $\{F^{ij}(A)\}$ . Moreover,

$$F^{ij}(A) = f_i \delta_{ij}$$

when  $A$  is diagonal, and

$$F^{ij}(A) a_{ij} = \sum f_i(\lambda(A)) \lambda_i = F(A), \quad (2.13)$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \quad (2.14)$$

Equation (1.1) can therefore be rewritten in a local frame  $\tau_1, \dots, \tau_n$  in the form

$$F(A[\Sigma]) = \sigma \quad (2.15)$$

where  $A[\Sigma] = \{g^{ik}h_{kj}\}$ . Let  $F^{ij} = F^{ij}(A[\Sigma])$ ,  $F^{ij,kl} = F^{ij,kl}(A[\Sigma])$ .

**Lemma 2.1.** *Let  $\Sigma$  be a smooth hypersurface in  $\mathbb{H}^{n+1}$  satisfying equation (1.1). Then in a local orthonormal frame,*

$$F^{ij}\nabla_{ij}\frac{1}{u} = -\frac{\sigma\nu^{n+1}}{u} + \frac{1}{u}\sum f_i. \quad (2.16)$$

and

$$F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} = \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2. \quad (2.17)$$

*Proof.* The first identity follows immediately from (2.10), (2.13) and assumption (1.9). To prove (2.17) we recall the identities in  $\mathbb{R}^{n+1}$

$$\begin{aligned} (\nu^{n+1})_i &= -\tilde{h}_{ij}\tilde{g}^{jk}u_k, \\ \tilde{\nabla}_{ij}\nu^{n+1} &= -\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij}). \end{aligned} \quad (2.18)$$

By (2.2), (2.13), (2.14), and  $\tilde{g}^{ik} = \delta_{jk}/u^2$  we see that

$$\begin{aligned} F^{ij}\tilde{g}^{kl}\tilde{h}_{il}\tilde{h}_{kj} &= \frac{1}{u^2}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\ &= F^{ij}(h_{ik}h_{kj} - 2\nu^{n+1}h_{ij} + (\nu^{n+1})^2\delta_{ij}) \\ &= f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum f_i. \end{aligned} \quad (2.19)$$

As a hypersurface in  $\mathbb{R}^{n+1}$ , it follows from (2.3) that  $\Sigma$  satisfies

$$f(u\tilde{\kappa}_1 + \nu^{n+1}, \dots, u\tilde{\kappa}_n + \nu^{n+1}) = \sigma,$$

or equivalently,

$$F(\{\tilde{g}^{ik}(u\tilde{h}_{kj} + \nu^{n+1}\tilde{g}_{kj})\}) = \sigma. \quad (2.20)$$

Differentiating equation (2.20) and using  $\tilde{g}_{ik} = u^2\delta_{ik}$ ,  $\tilde{g}^{ik} = \delta_{ik}/u^2$ , we obtain

$$F^{ij}(u\tilde{\nabla}_k\tilde{h}_{ij} + u_k\tilde{h}_{ij} + (\nu^{n+1})_k u^2\delta_{ij}) = 0. \quad (2.21)$$

That is,

$$\begin{aligned} F^{ij}\tilde{\nabla}_k\tilde{h}_{ij} + (\nu^{n+1})_k u \sum F^{ii} &= -\frac{u_k}{u}F^{ij}\tilde{h}_{ij} \\ &= -u_k F^{ij}(h_{ij} - \nu^{n+1}\delta_{ij}) \\ &= -u_k\left(\sigma - \nu^{n+1}\sum f_i\right). \end{aligned} \quad (2.22)$$

Finally, combining (2.8), (2.16), (2.18), (2.19), (2.22), and the first identity in (2.9), we derive

$$\begin{aligned}
F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} &= \nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} + \frac{|\tilde{\nabla}u|^2}{u}F^{ij}\tilde{h}_{ij} - \frac{\nu^{n+1}}{u^3}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\
&= \frac{\nu^{n+1}}{u}\left(\sum f_i - \nu^{n+1}\sigma\right) + \frac{|\tilde{\nabla}u|^2}{u}\left(\sigma - \nu^{n+1}\sum f_i\right) \\
&\quad - \frac{\nu^{n+1}}{u}\left(f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum f_i\right) \\
&= \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2.
\end{aligned} \tag{2.23}$$

This proves (2.17).  $\square$

### 3 The asymptotic angle maximum principle and gradient estimates

In this section we show that the upward unit normal of a solution tends to a fixed asymptotic angle on approach to the boundary. This implies a global gradient bound on solutions.

**Theorem 3.1.** *Let  $\Sigma$  be a smooth strictly locally convex hypersurface in  $\mathbb{H}^{n+1}$  satisfying equation (1.1). Suppose  $\Sigma$  is globally a graph:*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

where  $\Omega$  is a domain in  $\mathbb{R}^n \equiv \partial\mathbb{H}^{n+1}$ . Then

$$F^{ij}\nabla_{ij}\frac{\sigma - \nu^{n+1}}{u} \geq \sigma(1 - \sigma)\frac{(\sum f_i - 1)}{u} \geq 0 \tag{3.1}$$

and so,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \sup_{\partial\Omega} \frac{\sigma - \nu^{n+1}}{u} \text{ on } \Sigma. \tag{3.2}$$

Moreover, if  $u = \epsilon > 0$  on  $\partial\Omega$ , then there exists  $\epsilon_0 > 0$  depending only on  $\partial\Omega$ , such that for all  $\epsilon \leq \epsilon_0$ ,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \frac{\sqrt{1 - \sigma^2}}{r_1} + \frac{\epsilon(1 + \sigma)}{r_1^2} \text{ on } \Sigma \tag{3.3}$$

where  $r_1$  is the maximal radius of exterior tangent spheres to  $\partial\Omega$ .

*Proof.* Set  $\eta = \frac{\sigma - \nu^{n+1}}{u}$ . By (2.16) and (2.17) we have

$$F^{ij}\nabla_{ij}\eta = \frac{\sigma}{u}\left(\sum f_i - 1\right) + \frac{\nu^{n+1}}{u}\left(\sum f_i\kappa_i^2 - \sigma^2\right).$$

On the other hand,

$$\sum \kappa_i^2 f_i \geq \frac{(\sum \kappa_i f_i)^2}{\sum f_i} = \frac{\sigma^2}{\sum f_i}.$$

Hence,

$$F^{ij} \nabla_{ij} \eta \geq \frac{\sigma}{u} \left( \sum f_i - 1 \right) \left( 1 - \frac{\sigma \nu^{n+1}}{\sum f_i} \right) \geq \frac{\sigma(1-\sigma)}{u} \left( \sum f_i - 1 \right) \geq 0.$$

So (3.2) follows from the maximum principle, while (3.3) follows from (3.2) and the approximate asymptotic angle condition,

$$\eta \leq \frac{\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon(1+\sigma)}{r_1^2} \quad \text{on } \partial\Sigma$$

which is proved in Lemma 3.2 of [7].  $\square$

**Proposition 3.2.** *Let  $\Sigma$  be a smooth strictly locally convex graph*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

in  $\mathbb{H}^{n+1}$  satisfying  $u \geq \varepsilon$  in  $\Omega$ ,  $u = \varepsilon$  on  $\partial\Omega$ . Then

$$\frac{1}{\nu^{n+1}} \leq \max \left\{ \frac{\max_{\Omega} u}{u}, \max_{\partial\Omega} \frac{1}{\nu^{n+1}} \right\}. \quad (3.4)$$

*Proof.* Let  $h = \frac{u}{\nu^{n+1}} = uw$  and suppose that  $h$  assumes its maximum at an interior point  $x_0$ . Then at  $x_0$ ,

$$\partial_i h = u_i w + u \frac{u_k u_{ki}}{w} = (\delta_{ki} + u_k u_i + u u_{ki}) \frac{u_k}{w} = 0 \quad \forall 1 \leq i \leq n.$$

Since  $\Sigma$  is strictly locally convex, this implies that  $\nabla u = 0$  at  $x_0$  so the proposition follows immediately.  $\square$

Combining Theorem 3.1 and Proposition 3.2 gives

**Corollary 3.3.** *Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$  and  $\sigma \in (0, 1)$ . Suppose  $f$  satisfies (1.5)-(1.10) in  $K_n^+$ . Then for any  $\varepsilon > 0$  sufficiently small, any admissible solution  $u^\varepsilon \in C^\infty(\bar{\Omega})$  of the Dirichlet problem (1.19), (1.23) satisfies the apriori estimate*

$$|\nabla u^\varepsilon| \leq C \quad \text{in } \Omega \quad (3.5)$$

where  $C$  is independent of  $\varepsilon$ .

## 4 Curvature estimates

In this section we prove a maximum principal for the largest principal curvature of locally strictly convex graphs satisfying  $f(\kappa) = \sigma$ .

Let  $\Sigma$  be a smooth hypersurface in  $\mathbb{H}^{n+1}$  satisfying  $f(\kappa) = \sigma$ . For a fixed point  $\mathbf{x}_0 \in \Sigma$  we choose a local orthonormal frame  $\tau_1, \dots, \tau_n$  around  $\mathbf{x}_0$  such that  $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$ . The calculations below are done at  $\mathbf{x}_0$ . For convenience we shall write  $v_{ij} = \nabla_{ij} v$ ,  $h_{ijk} = \nabla_k h_{ij}$ ,  $h_{ijkl} = \nabla_{lk} h_{ij} = \nabla_l \nabla_k h_{ij}$ , etc.

Since  $\mathbb{H}^{n+1}$  has constant sectional curvature  $-1$ , by the Codazzi and Gauss equations we have  $h_{ijk} = h_{ikj}$  and

$$\begin{aligned} h_{iijj} &= h_{jjii} + (h_{ii} h_{jj} - 1)(h_{ii} - h_{jj}) \\ &= h_{jjii} + (\kappa_i \kappa_j - 1)(\kappa_i - \kappa_j). \end{aligned} \quad (4.1)$$

Consequently for each fixed  $j$ ,

$$F^{ii} h_{jjii} = F^{ii} h_{iijj} + (1 + \kappa_j^2) \sum f_i \kappa_i - \kappa_j \sum f_i - \kappa_j \sum \kappa_i^2 f_i. \quad (4.2)$$

**Theorem 4.1.** *Let  $\Sigma$  be a smooth strictly locally convex graph in  $\mathbb{H}^{n+1}$  satisfying  $f(\kappa) = \sigma$  and*

$$\nu^{n+1} \geq 2a > 0 \text{ on } \Sigma. \quad (4.3)$$

For  $\mathbf{x} \in \Sigma$  let  $\kappa_{\max}(\mathbf{x})$  be the largest principal curvature of  $\Sigma$  at  $\mathbf{x}$ . Then

$$\max_{\Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \max \left\{ \frac{4n}{a^3}, \max_{\partial \Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a} \right\}. \quad (4.4)$$

*Proof.* Let

$$M_0 = \max_{\mathbf{x} \in \Sigma} \frac{\kappa_{\max}(x)}{\nu^{n+1} - a}. \quad (4.5)$$

Assume  $M_0 > 0$  is attained at an interior point  $\mathbf{x}_0 \in \Sigma$ . Let  $\tau_1, \dots, \tau_n$  be a local orthonormal frame around  $\mathbf{x}_0$  such that  $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$ , where  $\kappa_1, \dots, \kappa_n$  are the principal curvatures of  $\Sigma$  at  $\mathbf{x}_0$ . We may assume  $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$ . Thus, at  $\mathbf{x}_0$ ,  $\frac{h_{11}}{\nu^{n+1} - a}$  has a local maximum. Therefore,

$$\frac{h_{11i}}{h_{11}} - \frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} = 0, \quad (4.6)$$

$$\frac{h_{11ii}}{h_{11}} - \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1} - a} \leq 0. \quad (4.7)$$

Using (4.2), we find after differentiating the equation  $F(h_{ij}) = \sigma$  twice that

**Lemma 4.2.** *At  $\mathbf{x}_0$ ,*

$$F^{ii} h_{11ii} = -F^{ij,rs} h_{ij1} h_{rs1} + \sigma(1 + \kappa_1^2) - \kappa_1 \sum f_i - \kappa_1 \sum \kappa_i^2 f_i. \quad (4.8)$$

By Lemma 2.1 we immediately derive

**Lemma 4.3.** *Let  $\Sigma$  be a smooth hypersurface in  $\mathbb{H}^{n+1}$  satisfying  $f(\kappa) = \sigma$ . Then in a local orthonormal frame,*

$$F^{ij} \nabla_{ij} \nu^{n+1} = \frac{2}{u} F^{ij} \nabla_i u \nabla_j \nu^{n+1} + \sigma(1 + (\nu^{n+1})^2) - \nu^{n+1} \left( \sum f_i + \sum f_i \kappa_i^2 \right). \quad (4.9)$$

Using Lemma 4.2 and Lemma 4.3 we find from (4.7)

$$\begin{aligned} 0 \geq & -F^{ij,rs} h_{ij1} h_{rs1} + \sigma \left( 1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a} \kappa_1 \right) \\ & + \frac{a\kappa_1}{\nu^{n+1} - a} \left( \sum f_i + \sum \kappa_i^2 f_i \right) - \frac{2\kappa_1}{\nu^{n+1} - a} F^{ij} \frac{u_i}{u} \nabla_j \nu^{n+1}. \end{aligned} \quad (4.10)$$

Next we use an inequality due to Andrews [1] and Gerhardt [3] which states

$$-F^{ij,kl} h_{ij1} h_{kl,1} \geq \sum_{i \neq j} \frac{f_i - f_j}{\kappa_j - \kappa_i} h_{ij1}^2 \geq 2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} h_{i11}^2. \quad (4.11)$$

Recall that (see (2.18))

$$\nabla_i \nu^{n+1} = \frac{u_i}{u} (\nu^{n+1} - \kappa_i).$$

Then at  $\mathbf{x}_0$ , we obtain from (4.6)

$$h_{11i} = \frac{\kappa_1}{\nu^{n+1} - a} \frac{u_i}{u} (\nu^{n+1} - \kappa_i). \quad (4.12)$$

Inserting this into (4.11) we derive

$$-F^{ij,kl} h_{ij1} h_{kl,1} \geq 2 \left( \frac{\kappa_1}{\nu^{n+1} - a} \right)^2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1})^2. \quad (4.13)$$

Note that we may write

$$\sum f_i + \sum \kappa_i^2 f_i = (1 - (\nu^{n+1})^2) \sum f_i + \sum (\kappa_i - \nu^{n+1})^2 f_i + 2\sigma \nu^{n+1}. \quad (4.14)$$

Combining (4.11), (4.13) and (4.14) gives

$$\begin{aligned} 0 \geq & \sigma \left( 1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a} \kappa_1 \right) \\ & + \frac{a\kappa_1}{\nu^{n+1} - a} \left( (1 - (\nu^{n+1})^2) \sum f_i + \sum (\kappa_i - \nu^{n+1})^2 f_i + 2\sigma \nu^{n+1} \right) \\ & + 2 \frac{\kappa_1}{\nu^{n+1} - a} \sum f_i \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1}) \\ & + 2 \frac{\kappa_1^2}{(\nu^{n+1} - a)^2} \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1})^2. \end{aligned} \quad (4.15)$$

Note that (assuming  $\kappa_1 \geq \frac{2}{a}$ ) all the terms of (4.15) are positive except possibly the ones in the sum involving  $(\kappa_i - \nu^{n+1})$  and only if  $\kappa_i < \nu^{n+1}$ .

Therefore define

$$\begin{aligned} J &= \{i : \kappa_i - \nu^{n+1} < 0, f_i < \theta^{-1} f_1\}, \\ L &= \{i : \kappa_i - \nu^{n+1} < 0, f_i \geq \theta^{-1} f_1\}, \end{aligned}$$

where  $\theta \in (0, 1)$  is to be chosen later. Since  $\sum u_i^2/u^2 = |\tilde{\nabla} u|^2 = 1 - (\nu^{n+1})^2 \leq 1$  and  $\kappa_1 f_1 \leq \sigma$ , we have

$$\sum_{i \in J} (\kappa_i - \nu^{n+1}) f_i \frac{u_i^2}{u^2} \geq -\frac{n}{\theta} f_1 \geq -\frac{n\sigma}{\theta \kappa_1}. \quad (4.16)$$

Finally,

$$\begin{aligned} & \frac{2\kappa_1^2}{\nu^{n+1} - a} \sum_{i \in L} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1})^2 \\ & \geq \frac{2(1-\theta)\kappa_1}{\nu^{n+1} - a} \sum_{i \in L} (\kappa_i - \nu^{n+1})^2 f_i \frac{u_i^2}{u^2} \\ & = -2\kappa_1 \sum_{i \in L} f_i \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1}) - \frac{2\theta\kappa_1}{\nu^{n+1} - a} \sum_{i \in L} (\kappa_i - \nu^{n+1})^2 f_i \frac{u_i^2}{u^2} \quad (4.17) \\ & \quad + \frac{2\kappa_1}{\nu^{n+1} - a} \sum_{i \in L} f_i \frac{u_i^2}{u^2} (\kappa_i^2 - (a + \nu^{n+1})\kappa_i + a\nu^{n+1}) \\ & \geq -2\kappa_1 \sum_{i \in L} f_i \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1}) - \frac{2\theta}{a} \kappa_1 \sum_{i \in L} (\kappa_i - \nu^{n+1})^2 f_i - \frac{6\sigma}{a} \kappa_1. \end{aligned}$$

In deriving the last inequality in (4.17) we have used that  $\kappa_i f_i \leq \sigma$  for each  $i$  and that  $\nu^{n+1} \geq 2a$ . We now fix  $\theta = \frac{a^2}{4}$ . From (4.16) and (4.17) we see that the right hand side of (4.15) is strictly positive provided that  $\kappa_1 > \frac{4n}{a^2}$ , completing the proof of Theorem 4.1.  $\square$

## 5 Uniqueness and foliations

In this section we identify a class of curvature functions for which there is uniqueness. This implies that for these curvature functions and smooth asymptotic boundaries  $\Gamma$  which are Jordan, there is a foliation of each component of  $\mathbb{H}^{n+1} \setminus \mathcal{CH}(\Gamma)$  (the complement of the hyperbolic convex hull of  $\Gamma$ ) by solutions  $f(\kappa) = \sigma$  as  $\sigma$  varies between 0 and 1.

**Theorem 5.1.** *Let  $f(\kappa)$  satisfy (1.5)-(1.10) in the positive cone  $K_n^+$  and in addition satisfy*

$$\sum_i f_i > \sum \lambda_i^2 f_i \text{ in } K_n^+ \cap \{0 < f < 1\}. \quad (5.1)$$

Let  $\Sigma_i, i = 1, 2$  be strictly locally convex hypersurfaces (oriented up) in  $\mathbb{H}^{n+1}$  satisfying  $f(\kappa) = \sigma_i \in (0, 1), \sigma_1 \leq \sigma_2$ , with the same boundary in the horosphere  $x_{n+1} = \epsilon$  or with the same asymptotic boundary  $\Gamma = \partial\Omega$ . Then  $\Sigma_2$  lies below  $\Sigma_1$ , that is, if  $\Sigma_i$  are represented as graphs  $x_{n+1} = u_i(x)$  over  $\Omega \subset \mathbb{R}^n$ , then  $u_2 \leq u_1$  in  $\Omega$ .

*Proof.* We build on an idea of Schlenker [12]. Suppose for contradiction that  $\Sigma_2$  contains points in the unbounded region of  $\mathbb{R}_+^{n+1} \setminus \Sigma_1$  and let  $P$  be a point of  $\Sigma_2$  farthest from  $\Sigma_1$  (necessarily  $P$  is not a boundary point) where the maximal distance, say  $t^*$  is achieved. Then the local parallel hypersurfaces  $\Sigma_2^t$  to  $\Sigma_2$  obtained by moving a distance  $t$  (on the concave side of  $\Sigma_2$  near  $P$ ) are convex and contact  $\Sigma_1$  at a point  $Q$  in  $\Sigma_1$  when  $t = t^*$ . Moreover  $\Sigma_2^{t^*}$  locally lies below  $\Sigma_1$  by the maximality of the distance  $t^*$ . We claim that the distance function  $d(x, \Sigma_2)$  is smooth in a neighborhood of  $Q$ . To show this we need only show (see [9]) that  $P$  is the unique closest point to  $Q$  on  $\Sigma_2$ . If  $P'$  was a second point of  $\Sigma_2$  at distance  $t^*$  from  $Q$ , then the local parallel hypersurfaces  $\Sigma_2^t$  to  $\Sigma_2$  obtained by moving a distance  $t$  (on the concave side of  $\Sigma_2$  near  $P'$ ) are also convex and when  $t = t^*$ , contact  $\Sigma_1$  at  $Q$  and also locally lies below  $\Sigma_1$  by the previous argument. This is clearly impossible since  $\Sigma_1$  has a unique tangent plane at  $Q$ .

The principal curvatures of  $\Sigma_2^t$  at points along the normal geodesic emanating from any point of  $\Sigma_1$  (say near  $P$ ) are given by the ode (see [4]):

$$\kappa_i'(t) = 1 - \kappa_i^2.$$

In particular, if  $\kappa_i(0) < 1$ , then  $\kappa_i(0) \leq \kappa_i(t) < 1$  while if  $\kappa_i(0) > 1$ , then  $1 < \kappa_i(t) \leq \kappa_i(0)$ . Of course if  $\kappa_i(0) = 1$ , then  $\kappa_i(t) \equiv 1$ . Moreover by (5.1),

$$\frac{d}{dt}f(\kappa(t)) = \sum f_i - \sum \kappa_i^2 f_i > 0 \text{ in } K_n^+ \cap \{0 < f < 1\}. \quad (5.2)$$

It follows that the  $\Sigma_2^t$  satisfy  $f(\kappa) > \sigma_2$  and so are strict subsolutions of the equation  $f(\kappa) = \sigma_1$ . On the other hand at  $t = t^*$  we have  $\Sigma_2^{t^*}$  lies below  $\Sigma_1$  but touches  $\Sigma_1$  at  $Q$  violating the maximum principle.  $\square$

**Corollary 5.2.** *Let  $f(\kappa)$  satisfy (1.5)-(1.10) in the positive cone  $K_n^+$  and in addition satisfy (5.1). Let  $\Sigma_i, i = 1, 2$  be strictly locally convex graphs (oriented up) in  $\mathbb{H}^{n+1}$  over  $\Omega \subset \mathbb{R}^n$  satisfying  $f(\kappa) = \sigma \in (0, 1)$  with the same boundary in the horosphere  $x_{n+1} = \epsilon$  or with the same asymptotic boundary  $\Gamma = \partial\Omega$ . Then  $\Sigma_1 = \Sigma_2$ .*

**Example 5.3.** For  $l = n - 1$  or  $n - 2$ , let  $f = (\frac{H_n}{H_l})^{\frac{1}{n-l}}$  in the cone  $K_n^+ \subset \mathbb{R}^n$ . Then (see Lemma 2.14 of [15])

$$f_i = \frac{f}{n-l} \left( \frac{1}{\lambda_i} - (\log H_l)_i \right) = \frac{f}{n-l} \left( \frac{1}{\lambda_i} - \frac{H_{l-1;i}}{H_l} \right),$$

where  $H_{l-1;i} = H_{l-1}|_{\lambda_i=0}$ . Hence,

$$\sum f_i = \frac{f}{n-l} \left( n \frac{H_{n-1}}{H_n} - l \frac{H_{l-1}}{H_l} \right). \quad (5.3)$$

Similarly,

$$\sum \lambda_i^2 f_i = \frac{f}{n-l} \left( nH_1 - \frac{\sum \lambda_i^2 H_{l-1;i}}{H_l} \right).$$

Using

$$\frac{\sum \lambda_i^2 H_{l-1;i}}{H_l} = nH_1 - (n-l) \frac{H_{l+1}}{H_l},$$

we find

$$\sum \lambda_i^2 f_i = f \frac{H_{l+1}}{H_l}. \quad (5.4)$$

Combining (5.3) and (5.4) gives

$$\sum f_i - \sum \lambda_i^2 f_i = \frac{f}{n-l} \left( n \frac{H_{n-1}}{H_n} - l \frac{H_{l-1}}{H_l} - (n-l) \frac{H_{l+1}}{H_l} \right). \quad (5.5)$$

By the Newton-Maclaurin inequalities,

$$\frac{H_{n-1}}{H_n} \geq \frac{H_{l-1}}{H_l}$$

with equality if and only all the  $\lambda_i$  are equal. Hence,

$$\sum f_i - \sum \lambda_i^2 f_i \geq f \left( \frac{H_{n-1}}{H_n} - \frac{H_{l+1}}{H_l} \right). \quad (5.6)$$

Therefore if  $l = n - 1$ , we find

$$\sum f_i - \sum \lambda_i^2 f_i \geq 1 - f^2 > 0 \text{ in } K_n^+ \cap \{0 < f < 1\} \quad (5.7)$$

while if  $l = n - 2$  we similarly find

$$\sum f_i - \sum \lambda_i^2 f_i \geq \frac{H_{n-1}}{H_n} f(1 - f^2) \geq 1 - f^2 > 0 \text{ in } K_n^+ \cap \{0 < f < 1\}. \quad (5.8)$$

We now complete the proof of Theorem 1.7.

*Proof.* a. For  $\Gamma$  smooth and  $f(\kappa)$  satisfying the conditions of Theorem 5.1, we have by Theorem 1.2 and Theorem 5.1 a smooth ‘‘monotone decreasing’’ family of smooth solutions  $\Sigma^\sigma = \text{graph } u^\sigma(x)$ ,  $x \in \Omega$  of (1.1), (1.2). That is, if  $\sigma_1 < \sigma_2$ , then  $u^{\sigma_1} > u^{\sigma_2}$  in  $\Omega$ . Note also that if  $\Omega \subset B_\delta(0)$  then

$$u^\sigma < v^\sigma(x) := -\frac{\sigma\delta}{\sqrt{1-\sigma^2}} + \sqrt{\frac{\delta^2}{1-\sigma^2} - |x|^2} \text{ in } \Omega,$$

where  $v^\sigma(x)$  corresponds to the equidistant sphere solution of  $f(\kappa) = \sigma$ , which is a graph over  $B_\delta(0)$ . As  $\sigma \rightarrow 1$ ,  $v(x) \rightarrow 0$  uniformly and so the same holds for  $u^\sigma(x)$ .

We claim that as  $\sigma \rightarrow 0$ ,  $\Sigma^\sigma$  tends to the component  $S$  of  $\partial\mathcal{CH}(\Gamma)$  that is a graph over  $\Omega$ . To see this note that  $\Sigma^\sigma$  lies below  $S$  but also eventually lies above

any smooth strictly locally convex hypersurface  $S'$  by Theorem 5.1.

This completes the proof of Theorem 1.7 part a. In order to prove part b, it suffices by a standard approximation argument, to show that the graph solutions of  $f(\kappa) = \sigma$  have uniformly bounded principal curvatures on compact subdomains of  $\Omega$ , independent of the smoothness of  $\Gamma$ . We carry this out for the special curvature quotient  $f = \frac{H_n}{H_{n-1}}$  in Lemma 5.4 below, thus completing the proof of part b.  $\square$

**Lemma 5.4.** *Let  $\Sigma = \{\text{graph } u(x) : x \in \Omega\}$  be the unique strictly locally convex solution of  $\frac{H_n}{H_{n-1}}(\kappa) = \sigma \in (0, 1)$ . For any compact subdomain  $\Omega' \subset\subset \Omega$ , let  $\Sigma' = \{\text{graph } u(x) : x \in \Omega'\}$ . Then,*

$$\max_{\mathbf{x} \in \Sigma'} \kappa_{\max} \leq C,$$

where  $C$  depends only on  $\sigma$  and the (Euclidean) distance from  $\Omega'$  to  $\partial\Omega$ .

*Proof.* Fix a small constant  $\theta \in (0, 1)$  and set  $\phi = (u - \theta)_+$ . Recall from Lemma 2.1,

$$Lu = \frac{2}{u} F^{ij} u_i u_j + \sigma u \nu^{n+1} - u \sum f_i. \quad (5.9)$$

We modify the argument of section 4 by setting

$$M_0 = \max_{\mathbf{x} \in \Sigma} \phi \kappa_{\max}(x). \quad (5.10)$$

Then  $M_0 > 0$  is attained at an interior point  $\mathbf{x}_0 \in \Sigma$ . Let  $\tau_1, \dots, \tau_n$  be a local orthonormal frame around  $\mathbf{x}_0$  such that  $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$ , where  $\kappa_1, \dots, \kappa_n$  are the principal curvatures of  $\Sigma$  at  $\mathbf{x}_0$ . We may assume  $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$ . Thus, at  $\mathbf{x}_0$ ,  $\log \phi + \log h_{11}$  has a local maximum and so,

$$\frac{\phi_i}{\phi} + \frac{h_{11i}}{h_{11}} = 0, \quad (5.11)$$

$$\frac{\phi_{ii}}{\phi} + \frac{h_{11ii}}{h_{11}} \leq 0. \quad (5.12)$$

As in section 4 we obtain from (5.12) and (5.9) that at  $\mathbf{x}_0$ ,

$$0 \geq -\frac{u\kappa_1}{\phi} \sum f_i + \sigma(1 + \kappa_1^2) - \kappa_1 \sum f_i - \kappa_1 \sum \kappa_i^2 f_i. \quad (5.13)$$

From the calculations of Example 5.3,

$$1 \leq \sum f_i \leq n, \quad \sum \kappa_i^2 f_i = \sigma^2. \quad (5.14)$$

Hence from (5.13) and (5.14) we obtain  $\phi\kappa_1 \leq C$ . Choosing  $\theta$  so small that  $u \geq 2\theta$  on  $\Omega'$  completes the proof.  $\square$

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