Real Variables: Solutions to Homework 2

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Exercise 0.1. Chapter 2, # 1: Let $f(x) = x\sin(1/x)$ for $x \in (0,1]$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$ but $V[f;0,1] = +\infty$.

Proof. To see that f is bounded it is enough to realize that $|\sin(x)| \leq 1$ for $x \in [0,1]$, so

$$|f(x)| = |x\sin(1/x)| \leq 1.$$

To see that f is continuous, because it is a product of continuous functions on the interval $(0,1]$, it is sufficient to consider the limit as $\delta \to 0$ of $f(\delta)$ is $f(0)$:

$$0 \leq \lim_{\delta\to 0} f(\delta) \leq \lim_{\delta\to 0} |f(\delta)| = \lim_{\delta\to 0} |\delta\sin(1/\delta)| \leq \lim_{\delta\to 0} \delta = 0.$$

So we have that

$$\lim_{\delta\to 0} f(\delta) = 0 = f(0).$$

To see that f is not of bounded variation we will in fact prove something much more general:

Theorem 0.2. *Take $a, b > 0$, then define function f*

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & x \in (0,1] \\ 0 & x = 0 \end{cases}.$$

f is of bounded variation only if $a > b$.

Proof. Consider the partition defined by $\Gamma := \{x_n\} = \left\{\left(n\pi + \frac{\pi}{2}\right)^{-1/b}\right\}$. The motivation for defining such a quantity is

$$\sin(x_n^{-b}) = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases},$$

so

$$f(x_n) = \begin{cases} x_n^a & n \text{ even} \\ -x_n^a & n \text{ odd} \end{cases}.$$

Now,

$$\begin{aligned} \sum_{n=1}^{m} |f(x_n) - f(x_{n-1})| &= \sum_{n=1}^{m} \left|(-1)^n \left[x_n^a + x_{n-1}^a\right]\right| = \sum_{n=1}^{m} \left[x_n^a + x_{n-1}^a\right] \\ &= 2\sum_{n=1}^{m-1} x_n^a + x_m + x_0 \geq \sum_{n=1}^{m-1} x_n^a = \sum_{n=1}^{m-1} \left(n\pi + \frac{\pi}{2}\right)^{-a/b}. \end{aligned}$$

Here we see that

$$\lim_{m\to\infty}\sum_{n=1}^{m-1}\left(n\pi+\frac{\pi}{2}\right)^{-a/b}<\infty \iff a>b.$$

□

In our particular example we have $a = b$ and therefore we immediately know that $V[f;0,1]=+\infty$. □

Exercise 0.3. Chapter 2, # 4: Let $\{f_k\}$ be a sequence of functions of bounded variation on $[a,b]$. If $V[f_k;a,b]\le M<+\infty$ for all k and $f_k\to f$ point wise on $[a,b]$, show that f is of bounded variation and that $V[f;a,b]\le M$. Give an example of a convergent series of functions of bounded variation whose limit is not of bounded variation.

Proof. Begin by fixing a partition $\Gamma=\{x_i\}_{i=0}^k$ of the interval $[a,b]$. We know

$$V[f_n;a,b]=\sup_\Gamma\sum_{i=1}^k\left|f_n(x_i)-f_n(x_{i-1})\right|\le M$$

for all n. Furthermore, because $f_k\to f$ pointwise,

$$V[f;a,b]=\sup_\Gamma\sum_{i=1}^k\left|f(x_i)-f(x_{i-1})\right|=\sup_\Gamma\lim_{n\to\infty}\sum_{i=1}^k\left|f_n(x_i)-f_n(x_{i-1})\right|\le M.$$

□

As for the example of a convergent series of functions of bounded variation whose limit is not of bounded variation, taking a hint from problem 1, consider a function

$$f_n(x)=\begin{cases}x^a\sin(x^{-b}) & x\in[\frac{1}{n\pi},1]\\ 0 & x=0\end{cases}.$$

with $a\le b$. For any given n, f_n is of bounded variation but

$$f(x)=\lim_{n\to\infty}f_n(x)=\begin{cases}x^a\sin(x^{-b}) & x\in(0,1]\\ 0 & x=0\end{cases}.$$

we have show to not be of bounded variation.

Exercise 0.4. Chapter 2, # 5: Suppose that f is finite on $[a,b]$ and is of bounded variation on every interval $[a+\epsilon,b]$, $\epsilon>0$, with $V[f;a+\epsilon,b]\le M<+\infty$. Show that $V[f;a,b]<+\infty$. Is $V[f;a,b]\le M$? If not what additional assumptions will make it so?

Proof. We know that $V[f;a+\epsilon,b]\le M$ with $\epsilon>0$ varying between 0 and $b-a$. By definition of variation,

$$|f(b)-f(a+\epsilon)|\le V[f;a+\epsilon,b]\le M \text{ for all } \epsilon\in(0,b-a].$$

We can rearrange this to say

$$\sup_{x\in(a,b]} |f(x)| \le |f(b)| + M.$$

Now fix a partition $\Gamma = \{x_i\}_{i=0}^k$ of the interval $[a = x_0, b = x_k]$. Then we have

$$\begin{aligned}
\sum_{i=1}^{k} \big|f(x_i) - f(x_{i-1})\big| &\le \big|f(x_1) - f(x_0)\big| + V[f; x_1, x_k] \\
&\le \sup_{x\in(a,b]} |f(x_1)| + |f(a)| + M \\
&= |f(b)| + M + |f(a)| + M.
\end{aligned}$$

Thus we have shown that

$$V[f; a, b] = \sup_{\Gamma} \sum_{i=1}^{k} \big|f(x_i) - f(x_{i-1})\big| \le |f(b)| + |f(a)| + 2M.$$

It is clear from this that $V[f; a, b]$ is *not always* bounded by M. We claim that in order for this to be true one thing need to happen, f needs to be a continuous function at a to insure that there is no jump at $f(a)$ which would break the M-bound. To see that this does it, again fix a partition $\Gamma = \{x_i\}_{i=0}^k$ of the interval $[a = x_0, b = x_k]$. Pick some $x_\ell \in [x_0, x_1]$. Then

$$\begin{aligned}
\sum_{i=1}^{k} \big|f(x_i) - f(x_{i-1})\big| &\le \big|f(x_\ell) - f(x_0)\big| + \big|f(x_1) - f(x_\ell)\big| + \sum_{i=2}^{k} \big|f(x_i) - f(x_{i-1})\big| \\
&\le \big|f(x_\ell) - f(x_0)\big| + \big|f(x_1) - f(x_\ell)\big| + V[f; x_\ell, x_k] \\
&\le \big|f(x_\ell) - f(x_0)\big| + V[f; x_\ell, x_k] \\
&\le \big|f(x_\ell) - f(x_0)\big| + M.
\end{aligned}$$

Now, taking the limit as $\ell \to 0$, because f is continuous at a, we see

$$\lim_{\ell\to 0} \big|f(x_\ell) - f(x_0)\big| + M = M.$$

Taking the supremum we find:

$$V[f; a, b] = \sup_{\Gamma} \sum_{i=1}^{k} \big|f(x_i) - f(x_{i-1})\big| \le M,$$

and we are done. □

Exercise 0.5. Chapter 2, # 6: Let $f(x) = x^2 \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Show that $V[f, 0, 1] < +\infty$.

Proof. f is differentiable on $[0, 1]$ of bounded derivative. Then by Exercise 5, $V(f) < +\infty$ □

Exercise 0.6. Chapter 2, # 7: Suppose that f is of bounded variation on$[a, b]$. If f is continuous on $[a, b]$, show that $V(x), N(x)$ and $P(x)$ are continuous on $[a, b]$.

Proof. First note that it is sufficient to prove that if f is continuous on $[a, b]$ then so too is V because by Theorem 2.6 in the text,

$$P = \frac{1}{2}[V + f(b) - f(a)] \text{ and } N = \frac{1}{2}[V - f(b) + f(a)]$$

so if V is continuous, by this decomposition so are P and N.

To see that f continuous implies V continuous consider $c \in [a, b]$. We need a notation for left and right hand limits, denote the limit from above as $\lim_{x\to c^+} V[f; x, b]$ with $x \in (c, b]$ and the limit from below as $\lim_{y\to c^-} V[f; x, b]$ with $y \in [a, c)$. First let us work with limits from the right hand side. We see pretty trivially that

$$V[f; c, b] \geq \lim_{x\to c^+} V[f; x, b].$$

In fact, we can say something even stronger by directly applying the result from Exercise 0.4 (Chapter 2, # 5). Since

$$V[f; x, b] \leq \lim_{x\to c^+} V[f; x, b].$$

we find that

$$\lim_{x\to c^+} V[f; x, b] \geq V[f; c, b]$$

and therefore, combining this with out earlier result gives

$$\lim_{x\to c^+} V[f; x, b] = V[f; c, b].$$

Now, to show that V is continuous from the right, we need to show that

$$\lim_{x\to c^+} V[f; a, x] = V[f; a, c].$$

We go on to calculate:

$$\begin{aligned}\lim_{x\to c^+} V[f; a, x] &= V[f; a, b] - V[f; a, x]\\ &= V[f; a, b] - V[f; a, c]\\ &= V[f; a, c].\end{aligned}$$

V is continuous from the right. To see that V is continuous from the left we need to show that

$$\lim_{y\to c^+} V[f; y, b] = V[f; c, b].$$

This follows from the argument for right hand continuity if one defines a function $g = f(b-y)$ and notices that

$$V[f; y, b] = V[g; 0, b - y].$$

$\square$

Exercise 0.7. Chapter 2, # 9: Let C be a curve with parametric equations $x = \phi(t)$ and $y = \psi(t)$ for $t \in [a, b]$.

1. If ϕ, ψ are of bounded variation and continuous, show that $L = \lim_{|\Gamma|\to 0} \ell(\Gamma)$.

   Proof. To phrase this somewhat differently, given an $\epsilon > 0$, there exists a $\delta > 0$ such that $|L - \ell(\Gamma)| < \epsilon$ for all Γ satisfying $|\Gamma| := \max_i(x_i - x_{i-1}) < \delta$. Since ϕ and ψ are of bounded variation, L is finite. Fix a partition $\Gamma_0 = \{x_i\}_{i=0}^k$ of the interval $[a, b]$ such that $\ell(\Gamma_0) > L - \epsilon$ Furthermore, since ϕ, ψ are continuous on the interval, we can find $\delta, \delta' > 0$ such that

   $$|x - x'| < \delta \implies |\phi(x) - \phi(x')| < \frac{\sqrt{\epsilon}}{2k}$$
   $$|x - x'| < \delta \implies |\psi(x) - \psi(x')| < \frac{\sqrt{\epsilon}}{2k}.$$

   Then for any Γ satisfying $|\Gamma| := \max_i(x_i - x_{i-1}) < \delta$, let $\Gamma' = \Gamma \cup \Gamma_0$, then clearly $\ell(\Gamma') > L - \epsilon$ And by triangle inequality

   $$|L - \ell(\Gamma)| \leq |L - \ell(\Gamma')| + |\ell(\Gamma') - \ell(\Gamma)| \leq \epsilon + 2\epsilon.$$

   □

2. If ϕ, ψ are continuously differentiable, show that $L = \int_a^b \sqrt{\phi'(t)^2 + \psi'(t)^2}dt$.

   Proof. We have that $\phi, \psi \in C^1([a, b])$. That is, their derivatives and they themselves are continuous on the interval $[a, b]$. Fix a partition $\Gamma = \{x_i\}_{i=0}^k$ of the interval $[a, b]$ then applying part 1 of this exercise we find

   $$\begin{aligned} L = \lim_{|\Gamma|\to 0} \ell(\Gamma) &= \lim_{k\to\infty} \sum_{i=1}^k \left([\phi(x_i) - \phi(x_{i-1})]^2 + [\psi(x_i) - \psi(x_{i-1})]^2 \right)^{1/2} \\ &= \int_a^b \sqrt{\phi'(x)^2 + \psi'(x)^2} dx. \end{aligned}$$

   □

Exercise 0.8. Chapter 2, # 10: If $a \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n \leq b$ is a finite sequence and $|s| < \infty$, write $\sum_k a_k e^{-s\lambda_k}$ as a Riemann-Stieltjes integral.

Proof. Let $\phi(x)$ be a step function. We construct it completely analogously to the text (remark # 3 pg. 23): Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$,

$$\phi(\lambda_i \pm) = \lim_{x \to \lambda \pm} \phi(x) \qquad \text{and} \qquad a_i = \phi(\lambda_i +) - \phi(\lambda_i -)$$

Then choose $f = e{-}sx$, so

$$\sum_{i=1}^{n} a_i e^{-s\lambda_i} = \int_a^b e^{-sx} d\phi.$$

□

Exercise 0.9. Chapter 2, # 14: Give an example which shows that for $c \in (a, b)$, $\int_a^c f d\phi$ and $\int_c^b f d\phi$ may both exist but $\int_a^b f d\phi$ may not.

Proof. Take functions f, ϕ to be

$$\phi(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 1 & x \in (0,1] \\ 0 & \text{otherwise.} \end{cases}$$

In this case, both

$$\int_{-1}^{0} f d\phi \qquad \text{and} \qquad \int_0^1 f d\phi$$

exist but

$$\int_{-1}^{1} f d\phi$$

does not exist. The reasoning for this is on pg. 29 of the text. □

References