REAL VARIABLES: PROBLEM SET 1

BEN ELDER

1. Problem 1.1a

First let's prove that limsup E_k consists of those points which belong to infinitely many E_k. From equation 1.1:

$$limsup\ E_k = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right)$$

For limsup E_k, the intersection means that $\forall\ j$, any point x in limsup E_k is in $\bigcup_{k=j}^{\infty} E_k$. If x is not in infinitely many of the sets E_k, then we can take the last set which contains x, call this set E_F, so that $\forall\ k > F$, $x \notin E_k$. Then x cannot be in limsup E_k because it is not in the union $\bigcup_{k=j}^{\infty} E_k$ for $j > F$ and therefore not in the intersection of all j's. We have a contradiction so x must be in infinitely many of the sets E_k.

Suppose now that x is an element of infinitely many sets E_k. Then for any j, $x \in \bigcup_{k=j}^{\infty} E_k$ because otherwise x could only be in j - 1 sets. So, since x is in $\bigcup_{k=j}^{\infty} E_k$ for any j, then

$$x \in \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right) \implies x \in limsup\ E_k$$

So limsup E_k consists of exactly the points which are in infinitely many of the sets E_k.

Next let's prove that liminf E_k consists of those points which belong to all E_k from some k on. Again from equation 1.1:

$$liminf\ E_k = \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k \right)$$

Let y be in liminf E_k. Suppose that for any $j \in \mathbb{Z}^+$, $\exists\ k_0$ such that $k_0 > j$ and $y \notin E_{k_0}$. Then for any j, $y \notin V_j = \bigcap_{k=j}^{\infty} E_k$. Since y is not in the set V_j for any value of j, it is not in the intersection of the V_j, and by definition is not in liminf E_k, a contradiction. Then every element of liminf E_k must be in all E_k from some k on.

Suppose now that y is an element of all E_k from some k on, say $k = k_0$. Then for all $j > k_0$, we know that $y \in \bigcap_{k=j}^{\infty} E_k$. Since y is in some of these sets, it is also in the union of all of these sets:

$$y \in \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} E_k\right) \implies y \in liminf\ E_k$$

2. Problem 1.3

a) Use the De Morgan laws.

$$C(limsup\ E_k) = C\left[\bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right)\right] = \bigcup_{j=1}^{\infty} C\left(\bigcup_{k=j}^{\infty} E_k\right) = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} CE_k\right) = liminf\ CE_k$$

b) Suppose $E_k \nearrow E$. Then $E \supset E_{k+1} \supset E_k \quad \forall\ k$. So since every $x \in limsup\ E_k$ must be in infinitely many E_k, $x \in E$. Then $limsup\ E_k \subset E$. Now take a point $y \in E$. We know that $\exists\ k_0$ such that $\forall\ k > k_0$, $y \in E_k$. Then $y \in E_{k_0} \subset E_{k_0+1} \subset E_{k_0+2}....$ Then y is in infinitely many of the sets E_k and by problem 1a, it is in $limsup\ E_k$. Then $limsup\ E_k \supset E$ so $limsup\ E_k = E$.

If a point is in $liminf\ E_k$ then it is in all but finitely many E_k, so it is in infinitely many E_k, and is in $limsup\ E_k$. So $liminf\ E_k \subset limsup\ E_k$. Then $liminf\ E_k \subset E$.

Now if $y \in E$, $\exists\ k_0$ such that $\forall\ k > k_0$, $y \in E_k$. Then $\forall\ j > k_0$, $y \in V_j = \bigcap_{k=j}^{\infty} E_k$. Since y is in some of these sets V_j, it is in the union of all of these sets, and $y \in liminf\ E_k$. Then $E \subset liminf\ E_k \implies \boxed{E = liminf\ E_k = limsup\ E_k}$ as desired.

Now suppose $E_k \searrow E$. Then $E \subset E_{k+1} \subset E_k \quad \forall\ k$. Any x that is in infinitely many of the E_k is in $limsup\ E_k$. Because the E_k decrease monotonically, and E is contained in every E_k, every $x \in E$ is contained in infinitely many of the E_k. So $\forall\ x \in E$, we know that $x \in limsup\ E_k$. Then $E \subset limsup\ E_k$.

Every point $y \in liminf\ E_k$ is in all but finitely many E_k. Since the E_k decrease monotonically, all $y \in liminf\ E_k$ are in all E_k. The set of all points which are in all E_k is the set E. A sequence of sets E_k decreases to $\bigcap_k E_k$ if $E_k \supset E_{k+1} \quad \forall\ k$. So for any j, we see that $E = V_j = \bigcap_{k=j}^{\infty} E_k$. Then $E = liminf\ E_k$.

Suppose there is a point $x \in limsup\ E_k$. Then x is in infinitely many of the E_k. Since the E_k decrease monotonically, x is in all of the E_k. As seen in the last paragraph, all points x which are in all of the E_k are in E. So $\forall\ x \in limsup\ E_k$, $x \in E$. Then $E \supset limsup\ E_k$. Finally, we get $\boxed{E = limsup\ E_k = liminf\ E_k}$.

3. Problem 1.8

If E is relatively open with respect to an interval I, then $E = I \cap G$ for some set G that is open in $\mathbb{R}^n$. By theorem 1.11, G can be written as the countable union of nonoverlapping cubes. Then $G = \cup_i Q_i$ where each Q is a cube. So $E = (\cup_i Q_i) \cap I = \cup_i (Q_i \cap I)$.

It is clear that the intersection of any cube with an interval is an interval. Since there were a countable number of cubes, there are a countable number of intervals J_i. Since the cubes were nonoverlapping, the intervals J_i must be nonoverlapping. Then E is the union of a countable number of nonoverlapping intervals.

4. Problem 1.10

First let's prove that in $\mathbb{R}^1$ any bounded infinite sequence $\{x_n\}$ has a limit point.

We start by proving that bounded sequences have monotone subsequences. Suppose that there exists a point x_{k1} such that there is no k for which $k > k1$ and $x_k > x_{k1}$. So x_{k1} is higher than every subsequent point in the sequence. Further suppose that there are an infinite number of such points, $x_{k1},\ x_{k2},\ x_{k3}, \ldots$. Then we can take these points as a subsequence of $\{x_n\}$. This subsequence is monotone decreasing. Now suppose that there are only a finite number of such points $x_{k1}, x_{k2}, \ldots,\ x_{km}$. If there exist some such points, choose the point $x_{km+1} = y_0$. If there are no such points, start with the first point of the sequence $x_0 = y_0$. Next choose some subsequent point y_1 which is greater than the initial point. There must be such a point, because the initial point is not the highest point in the rest of the sequence, and there are no such points after it in the sequence. We can then continue choosing points such that $y_n < y_{n+1}$ and obtain a monotone increasing subsequence $\{y_n\} \subset \{x_n\}$. So any bounded sequence in $\mathbb{R}^1$ has a monotone subsequence.

Now we need to show that in $\mathbb{R}^1$, monotone bounded sequences converge. Without loss of generality, assume we have a monotone increasing, bounded sequence $\{x_n\}$. Then by the Least Upper Bound Axiom, there exists a least upper bound in $\mathbb{R}^1$, call this bound L. Then $\forall\ \epsilon > 0,\quad \exists\ N$ such that $\forall\ n > N,\ x_n > L - \epsilon$, because otherwise $L - \epsilon$ would be an upper bound for $\{x_n\}$ less than L, a contradiction of the L.U.B. axiom. $L - \epsilon < x_n < L \implies |x - L| < \epsilon$. As $\epsilon \to 0$, $x_n \to L$, and the sequence converges to L. So we have proved the Bolzano-Weierstrauss theorem for $\mathbb{R}^1$.

Now to extend the theorem to $\mathbb{R}^n$. Take a bounded sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^n$. Use the theorem in one dimension to choose a subsequence $\{\mathbf{x}_n^1\}$ whose elements' first coordinates converge. We are guaranteed to have such a sequence because the first coordinates of each element of $\{\mathbf{x}_n\}$ form a bounded sequence in $\mathbb{R}^1$. Next choose a subsequence of this subsequence $\{\mathbf{x}_n^2\} \subset \{\mathbf{x}_n^1\}$ whose elements' second coordinates converge. Continue this n times until we have a subsequence whose elements converge in every coordinate, and therefore in $\mathbb{R}^n$.

5. Problem 1.12

Take two sets A and B in $\mathbb{R}^2$. Let $A = \{y : y \geq e^{-x^2}\}$ and $B = \{y : y \leq -e^{-x^2}\}$. These two sets are disjoint, because both approach the x axis asymptotically from different directions at $\pm\infty$. The infimum of the distance is 0, because there is no $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all $x \in A$ and $y \in B$. These sets can be topologically embedded into any higher dimensional Euclidean space without changing these properties.

6. Problem 1.15

First let's prove "$\Rightarrow$". Assume that a bounded function f is Riemann integrable on an interval I. Then from equation 1.20, we know that $inf_\Gamma\ U_\Gamma = sup_\Gamma\ L_\Gamma = A$.

$$U_\Gamma = \sum_{k=1}^{N} [sup_{x \in I_k} f(\mathbf{x})] v(I_k), \quad L_\Gamma = \sum_{k=1}^{N} [inf_{x \in I_k} f(\mathbf{x})] v(I_k)$$

It is clear from the definition that $U_\Gamma \geq L_\Gamma\ \ \forall\, \Gamma$. Suppose that $\exists\, \epsilon > 0$ such that $U_\Gamma - L_\Gamma > \epsilon$ for all Γ.

$$\begin{aligned} \epsilon &< \sum_{k=1}^{N} [sup_{x \in I_k} f(\mathbf{x})] v(I_k) - \sum_{k=1}^{N} [inf_{x \in I_k} f(\mathbf{x})] v(I_k) \\ &= \sum_{k=1}^{N} \big(sup_{x \in I_k} f(\mathbf{x}) - inf_{x \in I_k} f(\mathbf{x})\big) v(I_k) \end{aligned}$$

Obviously as the number of intervals in the partition Γ goes to infinity, the volume of each partition goes to 0. By choosing the points $\{\xi_k\}$ to be the supremum (infimum) of each interval I_k, we see from the definition of the Riemann integral that both U_Γ (L_Γ) approaches its infimum (supremum) in the limit as $|\Gamma| \to 0$. So taking the limit as $|\Gamma| \to 0$, we get:

$$lim_{|\Gamma| \to 0} \left(\sum_{k=1}^{N} [sup_{x \in I_k} f(\mathbf{x})] v(I_k) \right) = lim_{|\Gamma| \to 0} U_\Gamma = A$$

$$lim_{|\Gamma| \to 0} \left(\sum_{k=1}^{N} [inf_{x \in I_k} f(\mathbf{x})] v(I_k) \right) = lim_{|\Gamma| \to 0} L_\Gamma = A$$

$$\implies lim_{|\Gamma| \to 0} (\epsilon) < lim_{|\Gamma| \to 0} \left(\sum_{k=1}^{N} [sup_{x \in I_k} f(\mathbf{x})] v(I_k) - \sum_{k=1}^{N} [inf_{x \in I_k} f(\mathbf{x})] v(I_k) \right)$$

$$\implies \epsilon < lim_{|\Gamma|\to 0}\left(\sum_{k=1}^{N}[sup_{x\in I_k} f(\mathbf{x})]v(I_k)\right) - lim_{|\Gamma|\to 0}\left(\sum_{k=1}^{N}[inf_{x\in I_k} f(\mathbf{x})]v(I_k)\right)$$

$$\implies \epsilon < lim_{|\Gamma|\to 0}\big(U_\Gamma\big) - lim_{|\Gamma|\to 0}\big(L_\Gamma\big) \implies \epsilon < A - A = 0$$

Since we chose $\epsilon > 0$, we have found our contradiction. Then if f is Riemann integrable on I, then $\forall\ \epsilon > 0,\ \ \exists\ \Gamma$ such that $0 \leq U_\Gamma - L_\Gamma < \epsilon$. We have proved "$\Rightarrow$".

Now prove "$\Leftarrow$". Assume that on an interval I and for a function f, $\forall\ \epsilon > 0,\ \ \exists\ \Gamma$ such that $0 \leq U_\Gamma - L_\Gamma < \epsilon$. Since $U_\Gamma \geq L_\Gamma\ \ \forall\ \Gamma$, and for the same Γ they can be made arbitrarily close to one another, it is clear that $inf_\Gamma\ U_\Gamma = sup_\Gamma\ L_\Gamma = A$ for some constant A. This is the equivalent definition of a function being Riemann integrable. So f is Riemann integrable on I $\iff\ \ \forall\ \epsilon > 0\ \ \exists$ a partition Γ such that $0 \leq U_\Gamma - L_\Gamma < \epsilon$.

7. Problem 1.17

$$\omega(\delta) = sup\{|f(\mathbf{x}) - f(\mathbf{y})| : |\mathbf{x} - \mathbf{y}| < \delta\}$$

It is clear from the definition that the modulus of continuity must decrease monotonically but not necessarily strictly decrease as δ decreases. For instance, suppose that $\delta = \infty$. Then $\omega(\delta) = f_{max} - f_{min}$ where $f(\mathbf{a}) = f_{max}$ is the global maximum of f and $f(\mathbf{b}) = f_{min}$ is the global minimum. Then if we decrease δ, ω will remain constant until $\delta = |\mathbf{b} - \mathbf{a}|$, where it will begin decreasing, because the greatest possible difference the function has is no longer included in the modulus of continuity. Because there are fewer points within a distance δ of any given point as δ decreases, the maximum distance between that point and any other within a radius δ must either remain constant or decrease.

By definition, f is uniformly continuous if $\forall\ \mathbf{x},\ \mathbf{y} \in \mathbb{R}^n,\ \ \ \forall\ \epsilon > 0,\ \exists\ \delta$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{y}| < \delta$. If $\omega(\delta) \to 0$ as $\delta \to 0$, then $\omega(\delta) = sup\{|f(\mathbf{x}) - f(\mathbf{y})| : |\mathbf{x} - \mathbf{y}| < \delta\} \to 0 \implies |f(\mathbf{x}) - f(\mathbf{y})| \to 0\ \ \forall\ |\mathbf{x} - \mathbf{y}| < \delta$. This is exactly the definition of uniform continuity. Then these two conditions are completely equivalent.

8. Problem 1.18

F closed in $(-\infty,\ \infty)$ implies that $F^C \subset (-\infty,\ \infty)$ is open. By theorem 1.10, every open subset of the real line can be expressed as the countable union of disjoint, open intervals. Our desired function g has its values set on F by the requirement $g(x) = f(x) \mid x \in F$. So we just need to find a definition for g on the open set F^C. There are three possible types of intervals that make up F^C: $(-\infty, a)$, (b, c), and (d, ∞).

First consider the interval (b,c) $\subset F^C$ such that $b \in F$ and $c \in F$. Since f is defined on F, f(b) and f(c) must be both defined and finite. In order to satisfy our requirements, g needs to be continuous on this interval and have $lim_{x \to b+} \ g(x) = f(b)$ and $lim_{x \to c-} \ g(x) = f(c)$. The linear function:

$$g(x) = f(b) + \left(\frac{x-b}{c-b}\right)\big[f(c) - f(b)\big]$$

obviously satisfies these conditions. So then take this as the definition of g on all bounded open intervals in F^C. Next, what about unbounded open intervals? Consider (d, ∞). If this interval is in F^C, then again we need $lim_{x \to d+} \ g(x) = f(d)$. Let:

$$g(x) = f(d) \cdot e^{-(x-d)^2}$$

This function is continuous at x = d, as well as on the interval (d, ∞). Then we will define g by $g(x) = f(x) \ \ \forall \ x \in F, \quad g(x) = f(b) + \left(\frac{x-b}{c-b}\right)\big[f(c) - f(b)\big] \ \ \forall \ x \in (b, c)$ such that $(b, c) \subset F^C$ and $b, c \in F$, and if $(-\infty, a)$ or (d, ∞) are in F^C then $g(x) = f(a) \cdot e^{-(x-a)^2}$ or $g(x) = f(d) \cdot e^{-(x-d)^2}$ respectively. $|g|$ is monotone increasing on $(-\infty, a)$ and monotone decreasing (d, ∞). WLOG assume that for an arbitrary interval $(b, c) \subset F^C$ with $b, c \in F$, that $f(b) \leq f(c)$. Then on (b,c), $f(b) \leq g(x) \leq f(c)$. So $\forall \ x \in (-\infty, \infty)$, $|g(x)| \leq |f(x_0)|$ for some $x_0 \in F$. So if f is bounded, then so is g.