<u>Remark</u> Even though dN is self adjoint $a_{21} \neq a_{12}$ - in the basis $\{X_u, X_v\}$

Moreover,

$$H = -\left(\frac{a_{11}+a_{22}}{2}\right) = \frac{1}{2}\frac{(EN+GL-2FM)}{EG-F^2}$$

$$K = \det a_{ij} = \frac{LN-M^2}{EG-F^2} = \frac{\det II}{\det I}$$

<u>Remark</u> In more modern notation, the first fundamental is denoted by (g_{ij}) and the second fundamental form by (b_{ij}). Thus $2H = \sum_{ij} g^{ij} b_{ij}$ (where $(g^{ij}) = (g_{ij})^{-1}$)

$$K = \frac{\det(b_{ij})}{\det(g_{ij})}$$

<u>Example</u> Consider the torus of revolution parametrized by

$$X(u,v) = \big((a+r\cos u)\cos v,\ (a+r\cos u)\sin v,\ r\sin u\big)$$

$$0 < u < 2\pi, \quad 0 < v < 2\pi.$$

Then $X_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$

$$X_v = (-(a+r\cos u)\sin v,\ (a+r\cos u)\cos v,\ 0)$$

$$X_{uu} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u)$$

$$X_{uv} = (r\sin u\sin v, -r\sin u\cos v, 0)$$

$$X_{vv} = (-(a+r\cos u)\cos v, -(a+r\cos u)\sin v, 0)$$

Hence, $E = r^2$, $F = 0$, $G = (a+r\cos u)^2$

$$\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{X_u \times X_v}{\sqrt{EG-F^2}}$$

$$L = \frac{\langle X_u \times X_v, X_{uu}\rangle}{\sqrt{EG-F^2}} \underset{\uparrow \text{ justify}}{=} \frac{(X_u, X_v, X_{uu})}{\sqrt{EG-F^2}}$$

$$\overset{\text{exercise}}{=} \frac{r^2(a+r\cos u)}{r(a+r\cos} = r.$$

Similarly (exercise) $M = \dfrac{(X_u, X_v, X_{uv})}{r(a+r\cos u)} = 0$

$$N = \frac{(X_u, X_v, X_{vv})}{r(a+r\cos} = \cos u(a+r\cos u)$$

Hence,

$$K = \frac{LN-M^2}{EG-F^2} = \frac{r\cos u(a+r\cos u)}{r^2(a+r\cos u)^2}$$

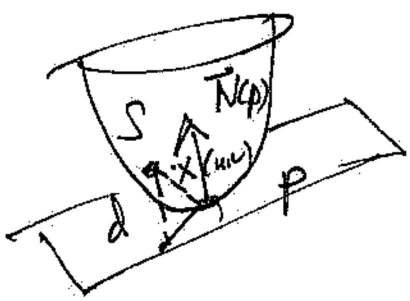
$$= \frac{\cos u}{r(a+r\cos u)}$$

Therefore $K=0$ on the parallels $u = \pi/2, 3\pi/2$;

$K<0$ on $\{\pi/2 < u < 3\pi/2\}$ and

$K>0$ on $\{0 < u < \pi/2\} \cup \{3\pi/2 < u < 2\pi\}$

<u>Proposition</u> Near an elliptic pt p of S, $\exists$ nbhd V of p s.that all pts of V lie on the same side of the tangent plane



(surface is locally convex)

<u>Pf</u>. Let $X(u,v)$ be a local par. for S near p with $X(0,0) = p$. Then the distance d from $q = X(u,v)$ to T_pS is given by $d = \langle X(u,v) - p, \vec{N}(p) \rangle$

$$\overset{\text{Taylor}}{=} \Big\langle X(0,0) + u X_u(0,0) + v X_v(0,0) + \frac{1}{2}\big(u^2 X_{uu}(0,0) + 2uv X_{uv}(0,0) + v^2 X_{vv}(0,0)\big) + \underbrace{\bar{R}}_{\text{error}}, \vec{N}(p) \Big\rangle$$

$$= \frac{1}{2}\left(u^2 \langle X_{uu}, \vec{N}\rangle(p) + 2uv\langle X_{uv}, \vec{N}\rangle + v^2\langle X_{vv}, \vec{N}\rangle\right)$$

$$+ R$$

$$= \frac{1}{2}\left(u^2 L(p) + 2uv\, M(p) + v^2 N(p)\right) + R$$

$\swarrow$ error of order ≥ 3

$$= \frac{1}{2}\, II_p(w) + R$$

where $w = u\, X_u(p) + v\, X_v(p)$

Since $II_p(w)$ has a sign since p elliptic (for example positive for a suitable orientation $\Rightarrow II_p(w) \geq \varepsilon_0 (u^2+v^2)$ for $\varepsilon_0 > 0$),

$$d \geq \varepsilon_0 (u^2+v^2) - C\,(u^2+v^2)^{3/2} > 0$$

for u^2+v^2 small enough. //

We next derive local expressions for the asymptotic and curvature directions. Recall $\alpha(t) = X(u(t), v(t))$ has an asymptotic direction as tangent iff $\mathrm{II}(\alpha'(t)) = 0$, i.e

$$b_{11}u'^2 + 2b_{12}u'v' + b_{22}v'^2 = 0 \qquad (*)$$

in "modern notation"

So the differential equation for an asymptotic curve is that $(*)$ holds $a \leq t \leq b$.

~~For example~~ Similarly, a (connected) curve $\alpha(t) = X(u(t), v(t))$ is a line of curvature iff

$$dN(\alpha'(t)) = \lambda(t)\, \alpha'(t) \quad \text{or}$$

since $\quad dN\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix},$

$$a_{11} u' + a_{12} v' = \lambda u' \qquad v'$$

$$a_{21} u' + a_{22} v' = \lambda v' \qquad u'$$

Eliminating λ gives

$$-a_{21} u'^2 + (a_{11} - a_{22}) u'v' + a_{12} v'^2 = 0$$

Hence (in "modern notation")

$$(g_{11} b_{12} - g_{12} b_{22}) u'^2 + (g_{11} b_{22} - g_{22} b_{11}) u'v' + (g_{12} b_{22} - g_{22} b_{12}) v'^2 = 0$$

which can be written as

$$\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix} = 0$$

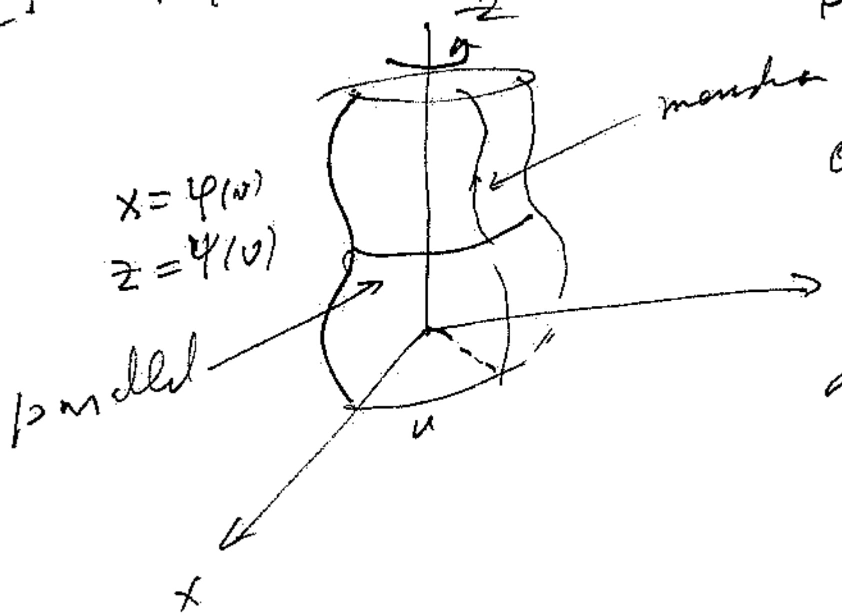
Note that the principal directions are orthogonal <u>at a non-umbilic pt.</u>

So a necessary and sufficient condition for the coordinate curves to be lines of curvature <u>near a non-umbilic pt</u> is that

$$g_{12} = b_{12} = 0$$

Example (surfaces of revolution)

Let $X(u,v) = (\varphi(v)\cos u,\ \varphi(v)\sin u,\ \psi(v))$



$0 < u < 2\pi$

$a < v < b$

$\varphi(v) > 0$

Then $g_{11} = \varphi^2$, $g_{12} = 0$, $g_{22} = \varphi'^2 + \psi'^2$

Assume (for convenience) that v is arc length so $g_{22} = 1$. Then (check)

$$b_{11} = \frac{(X_u, X_v, X_{uu})}{\sqrt{g}} = \frac{1}{\varphi}\begin{vmatrix} -\varphi\sin u & \varphi'\cos u & -\varphi\cos u \\ \varphi\cos u & \varphi'\sin u & -\varphi\sin u \\ 0 & \psi' & 0 \end{vmatrix}$$

$$= -\varphi\psi'$$ and

$b_{12} = 0$ $\quad b_{22} = \psi'\varphi'' - \psi''\varphi'$

Since $g_{12} = b_{12} = 0$ the parallels (v = constant) and the meridians (u = constant) are lines of curvature

Note $K = \dfrac{\det II}{\det I} = -\dfrac{\varphi\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi^2}$

$$= -\frac{\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi} \qquad (\varphi > 0)$$

and $\varphi'^2 + \psi'^2 = 1 \implies$
so $\varphi'\varphi'' = -\psi'\psi''$

$$K = -\frac{(\psi'^2\varphi'' + \varphi'^2\varphi'')}{\varphi} = -\frac{\varphi''}{\varphi}$$

We can also compute the principal curvatures easily using $g_{12} = b_{12} = 0$. For than

$$K = \frac{b_{11}\, b_{22}}{g_{11}\, g_{22}} = k_1 k_2, \quad 2H = \frac{g_{22} b_{11} + g_{11} b_{22}}{g_{11} g_{22}} = \frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} = k_1 + k_2$$

and so the principle curvatures

are $\frac{b_{11}}{g_{11}}$ and $\frac{b_{22}}{g_{22}}$

In the case of a surface of revolution,

$$\frac{b_{11}}{g_{11}} = -\frac{\varphi\psi'}{\varphi^2} = -\frac{\psi'}{\varphi}, \quad \frac{b_{22}}{g_{22}} = \frac{\psi'\varphi'' - \psi''\varphi'}{1}$$

and $2H = \frac{1}{\varphi}\left\{-\psi' + \varphi(\psi'\varphi'' - \psi''\varphi')\right\}$

$$K = -\frac{\psi'}{\varphi}(\psi'\varphi'' - \psi''\varphi').$$

<u>Example</u> (graph of a function)

Let $X(u,v) = (u, v, f(u,v))$, $(u,v) \in U$

(we put $u = x$ $v = y$)

Then (check)

$$g_{ij} = \delta_{ij} + f_i f_j$$

$$g^{ij} = \delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}$$

$$b_{ij} = \frac{f_{ij}}{\sqrt{1+|\nabla f|^2}}$$

$$N = \frac{(-f_u, -f_v, 1)}{\sqrt{1+|\nabla f|^2}}$$

<u>Notation</u>

δ_{ij} = "Kronecker delta"

$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$f_i = f_u$ if $i=1$
$\quad f_v$ if $i=2$

$$K = \frac{\det(f_{ij})}{(1+|\nabla f|^2)^2} = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$$

$$2H = \frac{(1+h_v^2)h_{uu} - 2h_u h_v h_{uv} + (1+h_u^2)h_{vv}}{(1+|\nabla f|^2)^{3/2}}$$

Let $S(v) = -DN(v)$ be the "shape operator" of S, a regular surface.

<u>Lemma</u> If v, w are linearly indep't vectors in T_pS, then

i) $S(v) \times S(w) = K(p)\,(v \times w)$

ii) $S(v) \times w + v \times S(w) = 2H(p)\,(v \times w)$

<u>Pf</u> Write $S(v) = av + bw$
$S(w) = cv + dw$

Then $K = \det S = ad - bc$
$2H = \operatorname{trace} S = a + d$

Compute

$$S(v) \times S(w) = (av + bw) \times (cv + dw)$$

$$= (ad - bc)\, v \times w = K\, v \times w$$

proving i) .

$$S(v) \times w + v \times S(w) = (a+d)(v \times w)$$

$$= 2H\,(v \times w) \qquad //$$

We use the last lemma i) to give Gauss' original interpretation of K.

<u>Proposition</u> Let $p \in S$ be a point where $K(p) \neq 0$ and let V be a connected open nbhd of p where $K \neq 0$. Then

$$K(p) = \lim_{A \to 0} A'/A$$

where A is the area of a small region $B \subset V$ containing p, A' is the area of the Gaussian image of B by the Gauss map $N : S \to S^2$ and the regions B shrink to p ("nicely")

<u>Pf</u> Let $B = X(U)$ so

$$A = \iint_U |X_u \times X_v| \, du \, dv$$

Then $A' = \iint_U |N_u \times N_v| \, du\, dv$

$$= \iint_U K |X_u \times X_v| \, du\, dv$$

Hence (formally)

$$\lim_{|A| \to 0} \frac{A'}{A} = \lim_{|U| \to 0} \frac{A'/|U|}{A/|U|} =$$

$$\lim_{|U| \to 0} \frac{\frac{1}{|U|} \iint_U K |X_u \times X_v| \, du\, dv}{\frac{1}{|U|} \iint_U |X_u \times X_v| \, du\, dv}$$

$$= K(p) \frac{|X_u \times X_v|(p)}{|X_u \times X_v|(p)} = K(p)$$

//.

3.5 Minimal surfaces

Def'n: A regular surface S is called a minimal surface if $H(p) = 0$ for all $p \in S$.

Classically, minimal surfaces arose from studying surfaces which (locally) minimize area among all surfaces with the same boundary ("soap bubbles", We now sketch a "variational" argument that computes the "first variation" of an arbitrary regular surface S (locally). Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular coordinate patch. Let $D \subset U$ be a nice subdomain and let $h : \bar{D} \to \mathbb{R}$ be a C^1 function

Def'n A "normal variation" of $X(\bar{D})$

is a one-parameter family

$X^t(u,v) = \varphi(u,v,t) =$

$X(u,v) + t\, h(u,v)\, N(u,v)$

where $(u,v) \in \bar{D}$, $t \in (-\varepsilon, \varepsilon)$.

Note that if h vanishes on ∂D

X^t and X agree on ∂D so

"both have the same boundary".

We now compute the area of

$X^t(D)$.

$$X^t_u = X_u + t h N_u + t h_u N$$
$$X^t_v = X_v + t h N_v + t h_v N$$

$$g_{11}^t = \langle X^t_u, X^t_u \rangle =$$

$$g_{11} + 2th\langle X_u, N_u\rangle + t^2 \left(h^2 \langle N_u, N_u \rangle + h_u^2 \right)$$

$$g_{12}^t = g_{12} + th\left(\langle X_u, N_v\rangle + \langle X_v, N_u \rangle\right) + t^2\left(h^2 \langle N_u, N_v \rangle + h_u h_v \right)$$

$$g_{22}^t = g_{22} + 2th\langle X_v, N_v\rangle + t^2 \left(h^2 \langle N_v, N_v \rangle + h_v^2 \right)$$

Recall $\langle X_u, N_u \rangle = -\langle X_{uu}, N \rangle = -b_{11}$

(since $\langle X_u, N \rangle = 0$)

and similarly $\langle X_u, N_v \rangle = -b_{12}$,

$\langle X_v, N_v \rangle = -b_{22}$

Therefore,

$$|X^t_u \times X^t_v|^2 = \det(g^t_{ij}) = \det g_{ij}$$
$$- 2th\left(\sum g^{ij} b_{ij}\right)(\det g_{ij}) + O(t^2)$$
$$= (\det g_{ij})(1 - 4thH) + O(t^2)$$
$$= |X_u \times X_v|^2 (1 - 4thH) + O(t^2),$$

and so

$$A(t) := A(X^t(D)) = \int_D \sqrt{\det g^t_{ij}}\, du\, dv$$
$$= \int_D \sqrt{1 - 4tH + O(t^2)}\, \sqrt{\det g_{ij}}\, du\, dv.$$

We conclude

$$A'(0) = -\int 2hH\sqrt{\det g_{ij}}\,du\,dv$$

Proposition 1. Let $X: U \to \mathbb{R}^3$ be a (regular) coordinate patch of S. Then $X(U)$ is a minimal surface if and only if $A'(0)=0$ for all normal variations of $X(\bar{D})$, $D \subset U$.

Pf If $X(U)$ is minimal $H \equiv 0$ so $A'(0)=0$. (trivial part)

Conversely if $H(q) \neq 0$ for some $q \in D$, choose $h(q) = H(q)$ and $h \equiv 0$ outside a small nhbd of q

(with $hH \geq 0$ on $\{h \neq 0\}$)
for example take $h = \eta H$ with $\eta \geq 0$, $\eta(q) = 1$ η vanishing outside a small nhbd of q)

Then $A'(0) < 0$ #.

<u>Defn</u> 1. We call $\vec{H} = H\vec{N}$ the mean curvature vector of S.

Def'n 2 We call coordinates such that

$$g_{11} = g_{22}, \quad g_{12} = 0$$

isothermal (the metric is then looks like $ds^2 = \lambda^2(u,v)(du^2+dv^2)$

Proposition 2 Let $X(u,v)$ be a regular coordinate patch parametrized by isothermal coordinates. Then

$$\Delta X := X_{uu} + X_{vv} = 2\lambda^2 \vec{H}$$

(where $g_{11} = g_{22} = \lambda^2$ and Δ is the famous Laplace operator)

<u>Pf</u> We have

$$X_u^2 = X_v^2 = \lambda^2$$

so

$$X_u \cdot X_v = 0$$

$$X_{uu} \cdot X_u = X_{uv} \cdot X_v = -X_u \cdot X_{vv}$$

Therefore $\Delta X \cdot X_u \equiv 0$ and similarly $\Delta X \cdot X_v \equiv 0$.

Hence ΔX is parallel to $\vec{N}$.

In isothermal coordinates (check)

$$2H = \frac{b_{11} + b_{22}}{\lambda^2}, \quad \text{i.e.}$$

$$2H\lambda^2 = b_{11} + b_{22} = \Delta X \cdot N,$$

i.e. $\Delta X = 2H\lambda^2 N$. //

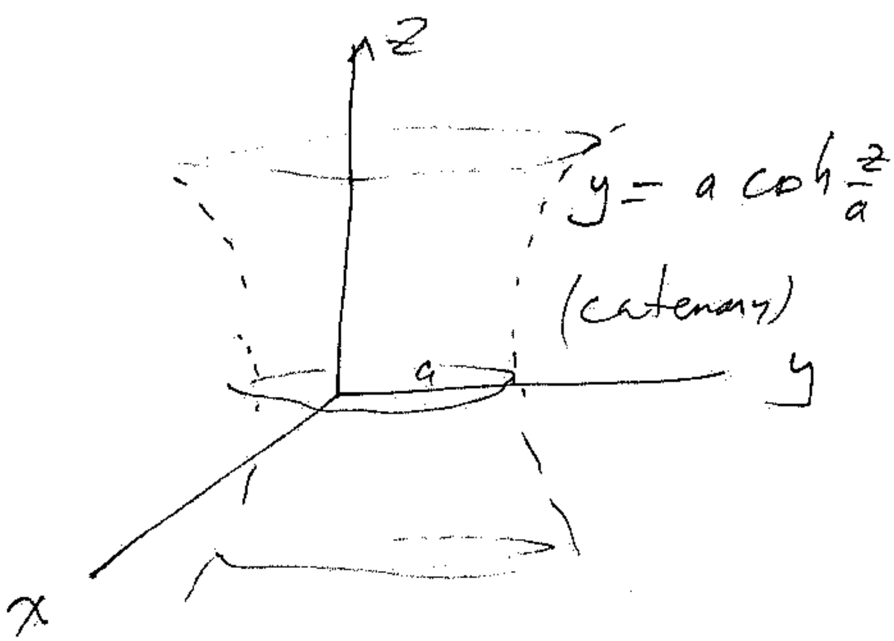
<u>Corollary</u> Let $X(U)$ be a regular parametrized surface in isothermal coordinates. Then $X(U)$ is minimal if and only if the coordinate function x, y, z are <u>harmonic</u> <u>functions</u> i.e.

$$\Delta x = \Delta y = \Delta z = 0$$

<u>Example</u> (catenoid) Consider

$$X(u,v) = (a \cosh v \cos u, a \cosh v \sin u, a$$

$$0 < u < 2\pi, \quad -\infty < v < \infty$$



This is the surface of revolution obtained by rotating the catenary $y = a \cosh \frac{z}{a}$ about the z axis. Then

(check) $g_{11} = g_{22} = a^2 \cosh^2 v$,
$g_{12} = 0$

and $\Delta X = 0$ (so the catenoid is a minimal surface)

You can read in do Carmo p 205 the proof that it is the unique minimal surface of revolution

__Example__ (helicoid) Recall

$$X(u,v) = (a \sinh v \cos u,\ a \sinh v \sin u,\ a u)$$

Then it is easily checked that

$$g_{11} = g_{22} = a^2 \cosh^2 v \ , \quad g_{12} = 0$$

and $\Delta X \equiv 0$.

Another interesting, classical minimal surface is

<u>Example</u> (Enneper's surface)

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

Then
$$X_u = (1 - u^2 + v^2,\ 2uv,\ 2u)$$
$$X_v = (2uv,\ 1 - v^2 + u^2,\ -2v)$$

and so
$$g_{11} = X_u^2 = (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2$$
$$= (1 + u^2 + v^2)^2$$

$$g_{22} = X_v^2 = 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2$$
$$= (1 + u^2 + v^2)^2$$

$$g_{12} = X_u \cdot X_v = (1 - u^2 + v^2) \cdot 2uv + 2uv(1 - v^2 + u^2) - 4uv = 0$$

Moreover,

$$X_{uu} = (-2u, 2v, 2)$$
$$X_{vv} = (2u, -2v, -2)$$

so $\Delta X = 0$.

You can find nice pictures of Enneper's surface online.

As a global surface it has "self-intersections"

Another famous example is Scherk's surface which can (locally) be represented as the graph $z = \ln \frac{\cos y}{\cos x}$ over the square $-\pi/2 < x, y < \pi/2$

(It can be extended as a doubly periodic "complete" minimal surface)

<u>Exercise</u> Check that in graphical coordinates $H = 0$