3.2 The Gauss map

given a local parametization $X: U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ near $p \in S$,

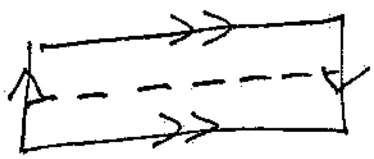
we can choose a unit normal vector

$$N(q) = \frac{X_u \times X_v}{|X_u \times X_v|}(q), \quad q \in X(U),$$

which defines a differentiable map (assume X is smooth) $N: X(U) \to S^2 \subset \mathbb{R}^3$

Thus N is a "differentiable field of unit normal vectors" on $V = X(U)$

Remark Not all surfaces admit a differentiable field (even continuous field) of unit normal vectors on the whole surface.

The standard example is the Mobiüs band which is topologically a rectangle  with the opposite edges identified with the opposite orientation (a twist). Note that without the twist we just obtain a finite piece of a circular cylinder. Going around the middle curve back to where you started, N has the opposite orientation, so is not continuous

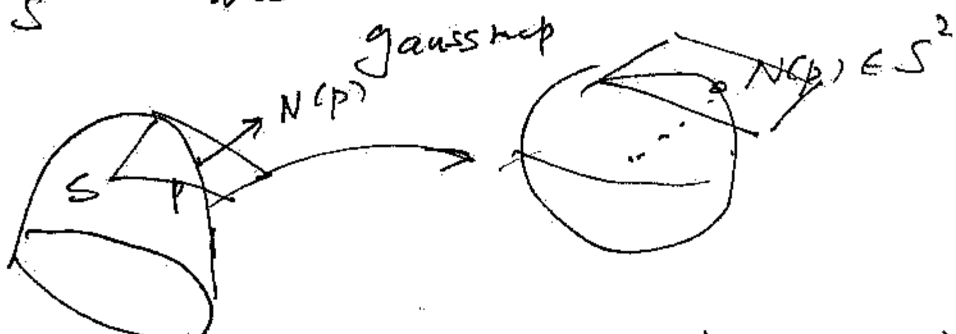
Def'n: We say <u>S is orientable</u> if there exists a (global) differentiable field of unit normal vectors N; the choice of N (there are two) is called an <u>orientation</u>. Locally S is always orientable near a pt $p \in S$.

An orientation N on S induces an orientation on each T_pS as follows: a basis $\{v, w\}$ of T_pS is said to be <u>positive</u> if

$$\langle v \times w, N \rangle > 0$$

<u>Defn</u> (Gauss map) Let $S \subset \mathbb{R}^3$ be a regular surface. The map $N: S \to \mathbb{R}^3$ which take values in $S^2 = \{(x,y,z): x^2+y^2+z^2=1\}$ is called the <u>Gauss</u> <u>map</u>

Recall that $dN_p : T_pS \to T_{N(p)}S^2$ is a linear map. Since T_pS and $T_{N(p)}S^2$ are the same vector space



we map think of $dN_p : T_p(S) \to T_pS$.

<u>Concretely</u>, Let $\alpha(t) : (-\varepsilon, \varepsilon) \to S$
be regular with $\alpha(0) = p$. Then
$N(t) := N(\alpha(t))$ is a curve in S^2 passing
through $N(p)$ and $N'(0) = dN_p(\alpha'(0))$
is a tangent vector in $T_p S$). It measures
how N turns away from $N(p)$ locally

<u>Example 1</u> For P: $ax + by + cz + d = 0$
$$N = \frac{(a, b, c)}{\sqrt{a^2+b^2+c^2}}$$ is constant

so $dN \equiv 0$

<u>Example 2</u> For S^2 let $\alpha(t) = (x(t), y(t), z(t))$,
then (since $x^2(t) + y^2(t) + z^2(t) = 1$)
$$2(xx' + yy' + zz') = 0$$
which says $\langle \alpha(t), \alpha'(t) \rangle$. If

We choose the orientation $N=(-x,-y,-z)$ (the inner normal to S^2) and if $X=(x,y,z)$ is the position vector of S^2,

$$\Rightarrow \quad dN_p(v) = -v \quad \forall p \in S^2$$

<u>Example 3</u> (cylinder) Let $C=\{(x,y,z);\ x^2+y^2=1\}$. Then if $N=(-x,-y,0)$, we have (since $x^2+y^2=1$)

$$dN(x'(t),y'(t),z'(t)) = N'(t) = (-x'(t),-y'(t),0),$$

and $dN(v) = 0\cdot v$ if v is a tangent vector parallel to the z axis.

On the other hand, if w is a tangent vector parallel to the xy plane,

$dN(w) = -w$. Thus

v, w are eigenvectors of dN with

eigenvalues $0, -1$ respectively

Remark

We will soon see that dN

is a "self-adjoint" linear map

so has real eigenvalue

A map $A: V \to V$ is self adjoint

if $\langle Av, w \rangle = \langle v, Aw \rangle$

$\forall\, v, w \in V$. In say

with respect to say an orthonormal

basis $\{e_i\}$, this means A

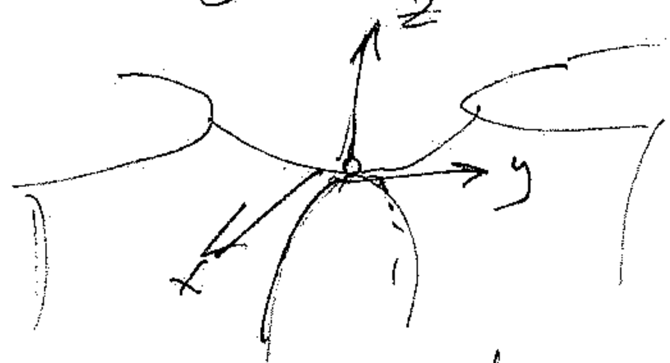
is represented by a symmetric

matrix (a_{ij}) since

$$a_{ij} = \langle Ae_i, e_j \rangle = \langle e_i, Ae_j \rangle$$
$$= a_{ji}.$$

<u>Example 4</u> (hyperbolic paraboloid)

Let $S: z = y^2 - x^2$, $p = (0,0,0)$



be the graphical representation. and

$$X(u,v) = (u, v, v^2 - u^2)$$
$$X_u = (1, 0, -2u) = e_1 \quad \text{at } p$$
$$X_v = (0, 1, 2v) = e_2 \quad " \quad "$$

Then $N = \left(\dfrac{u}{\sqrt{u^2+v^2+1/4}}, \dfrac{-v}{\sqrt{u^2+v^2+1/4}}, \dfrac{1}{2\sqrt{u^2+v^2+1/4}} \right)$

(the "upward unit normal")

Hence if $\alpha(t) = X(u(t), v(t))$ is a curve in S with $\alpha(0) = p$,

then $\alpha'(0) = (u'(0), v'(0), 0) \Rightarrow$

$$N'(0) = (2u'(0), -2v'(0), 0).$$

So $dN_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0)$

Therefore the vectors $(1,0,0)$, $(0,1,0)$ are eigenvectors of dN_p

with eigenvalues $2, -2$ respectively.

<u>Proposition</u> $dN_p : T_pS \to T_pS$ is a self adjoint linear map.

<u>Pf</u> We need to show
$$\langle dN_p(w_1), w_2 \rangle = \langle w_1, dN_p(w_2) \rangle$$

We use the basis $\{X_u, X_v\}$ associated to the parametrization $X(u,v)$. Then
$$\alpha(t) = X(u(t), v(t)) \qquad \alpha(0) = p$$

$$dN_p(\alpha'(0)) = dN_p(u'(0)X_u + v'(0)X_v)$$
$$= \frac{d}{dt} N(u(t), v(t))\Big|_{t=0}$$

$$= u'(0)\, N_u + v'(0)\, N_v$$

Hence $\quad dN_p(X_u) = N_u$
$\qquad\quad\;\; dN_p(X_v) = N_v$

Thus we need to show

$$(*) \quad \langle N_u, X_v \rangle = \langle X_u, N_v \rangle$$

We know $\quad \langle N, X_u \rangle = 0 = \langle N, X_v \rangle$

and so differentiating gives

$$\langle N_v, X_u \rangle + \langle N, X_{uv} \rangle = 0 \quad \text{and}$$

$$\langle N_u, X_v \rangle + \langle N, X_{vu} \rangle = 0$$

and so $(*)$ holds. //

Now that we know dN_p is self-adjoint we can associate a bilinear form $B(v,w) := -\langle dN_p(v), w\rangle = B(w,v)$

and then a quadratic form Q on T_pS by $Q(v) = \langle dN_p(v), v\rangle$

Def'n $II_p(v) := -Q(v) = -\langle dN_p(v), v\rangle$ is called the <u>second fundamental form</u> of S at p.

<u>Defn</u> Let C be a regular curve in S passing through $p \in S$.

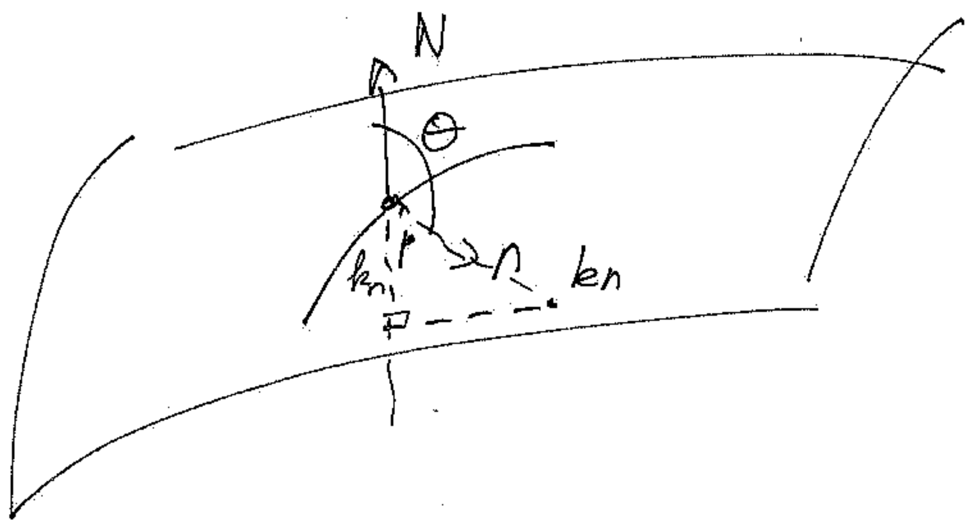
Let k = the curvature of C at p, where

$\cos\theta = \langle n, N \rangle$ and

n is the normal to C

N is " " " S

The number $k_n = k\cos\theta$ is called the <u>normal curvature</u> of $C \subset S$ at p



Let C be given by $\alpha(s)$ and let $N(s) = N|_C$. Then

$\langle N(s), \alpha'(s)\rangle = 0$ (since $\alpha' \in T_{\alpha(s)}S$,

so $\langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle$

Hence $\Pi_p(\alpha'(0)) := -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle$

$= -\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle$

$= \langle N, k n\rangle(p) = k_n(p)$

We have shown

Proposition 2 (Meusnier) $k_n(p)$ depends only on the direction (tangent line) of C at p.

<u>Def'n</u> (normal section):

given $v_p \in T_pS$, $|v| = 1$, let $\Pi(v_p, N_p)$ be the plane spanned by v, N. Then $\Pi \cap S$ is called a <u>normal section</u>

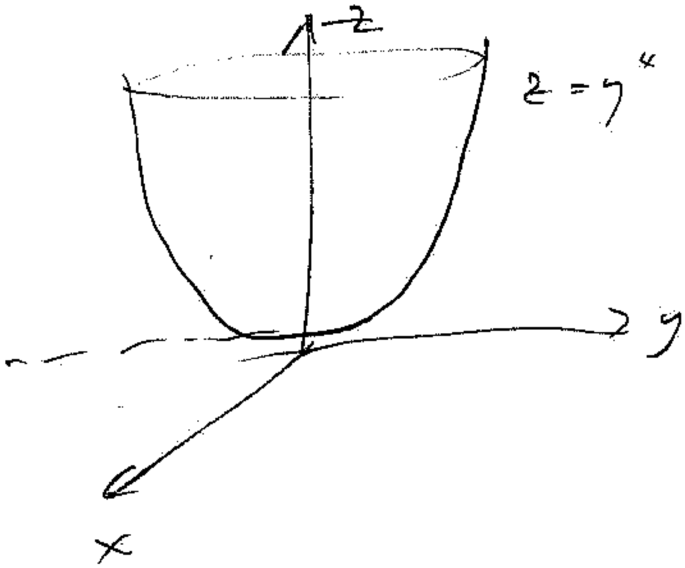
<u>Corollary</u> Let $\alpha(s)$ be a unit speed curve lying in $\Pi \cap S$ with $\alpha(0) = p$, $\alpha'(0) = v$.

Then $k_n(v) = \pm\ \kappa(\alpha)(0)$

curvature of the plane curve $\alpha(s)$

Pf (exercise)

Example Consider the surface S obtained by rotating $z = y^4$ about the z axis. Let $p = (0,0,0)$



Then $dN_p = 0$. For the curvature of the plane curve $z = y^4$ at p is 0 and T_pS = xy plane.

So $N(p)$ is parallel to the z axis.

Since any normal section at p

is a rotation of $z = y^4$ (so has curvature 0) we have

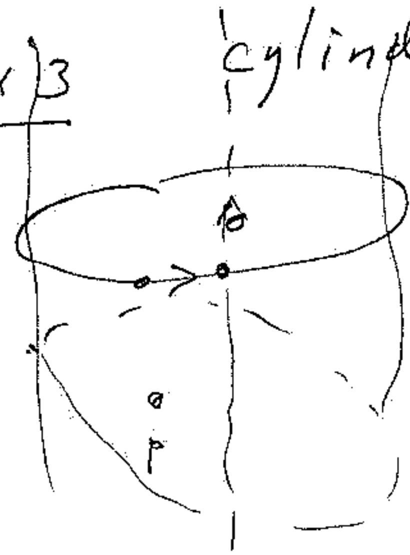
$k_n(p) = 0 \quad \forall v$ i.e.

$dN_p = 0$

Example, For P = plane, so
$k_n = 0 \Rightarrow dN_p \equiv 0$

Example 2 For S^2 all normal sections through $p \in S^2$ are circles of radius 1 so $k_n = 1$ if N is the outer normal and $k_n = -1$ if we choose the inner orientation

Ex 3 cylinder



normal sections are lines, circles, ellipses and (with the inner orientation)

$0 \le k_n \le 1$

min max

Returning to dN_p, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pS so that $dN_p(e_1) = -k_1 e_1$, $dN_p(e_2) = -k_2 e_2$. If we call $k_1 =$ maximum (curvature) and $k_2 =$ minimum (curvature), these are the maximum and minimum of II restricted to the unit circle of T_pS, i.e. the extremes of the normal curvature.

Def'n 1. $k_1 \geq k_2$ are called the principal curvatures of S and their eigenvectors are called the principal directions (at p).

2. A regular curve $C \subset S$ such that $\forall\, p \in C$ the tangent line

is a principal direction
is called <u>a line of curvature of S</u>

<u>Proposition</u> 3 (Rodrigues) A
regular curve $C \subset S$ is a line
of curvature iff

$$N'(t) = \lambda(t)\, \alpha'(t)$$

for any parametrization $\alpha(t)$ of C,
where $N(t) = N \circ \alpha\,(t)$ and
$\lambda(t)$ is a differentiable
function of t. (In this case,
$-\lambda(t)$ is the principal curvature)

Pf exercise.

To compute normal curvature, let $v \in T_p(S)$, $|v| = 1$. Then the principal directions $\{e_1, e_2\}$ form an orthonormal basis of T_pS, so

$$v = e_1 \cos\theta + e_2 \sin\theta,$$

where θ = angle from e_1 to v (in the orientation of T_pS). Hence

$$k_n = II_p(v) = -\langle dN_p(v), v\rangle$$

$$= -\langle dN_p(e_1\cos\theta + e_2\sin\theta),\ e_1\cos\theta + e_2\sin\theta\rangle$$

$$= \langle e_1 k_1 \cos\theta + e_2 k_2 \sin\theta,\ e_1\cos\theta + e_2\sin\theta\rangle$$

$$= k_1\cos^2\theta + k_2\sin^2\theta$$ (Euler's formula).

Given a symmetric linear map $A : V \to V$ ($\dim V = 2$) and a basis $\{v_1, v_2\}$ of V, the invariants of A are

$$\det A = a_{11}a_{22} - a_{12}^2 , \quad \operatorname{trace}(A) = a_{11} + a_{22}$$

and do not depend on the choice of basis (exercise)

Applying this to $-dN_p$ with eigenvalues k_1, k_2, $\det(-dN_p) = k_1 k_2 =: K$ (the Gauss curvature of S at p) and $\operatorname{trace}(-dN_p) = k_1 + k_2 =: 2H$ (the mean curvature of S at p).

Remark K is indep't of the orientation of S but H changes sign if we reverse orientation.

Def'n 1. A point p of S is called

1. elliptic if $K > 0$
2. hyperbolic if $K < 0$
3. parabolic if $K = 0$ but $dN_p \neq 0$
4. planar (flat) if $dN_p = 0$

Def'n 2 A pt $p \in S$ is called an umbilic pt if $k_1 = k_2$ (so planar pts are umbilic)

Proposition 4 If all pts of S are umbilic and S connected, then S is contained in either a sphere or a plane

<u>Pf</u> Let $p \in S$ and $X(u,v)$ a local parametrization of S near p s.t. that $p \in V = X(U)$ is connected

Since each $q \in V$ is umbilic, for each vector $w = a_1 X_u + a_2 X_v$ in $T_q S$,

(*) $\quad dN_q(w) = \lambda(q)\, w \quad$ where $\lambda = \lambda(q)$ is differentiable in V.

<u>Claim</u> $\lambda \equiv$ constant in V.

For (*) $\Longleftrightarrow \quad a_1 N_u + a_2 N_v = \lambda(a_1 X_u + a_2 X_v)$

$\Longrightarrow$ a. $N_u = \lambda X_u$
b. $N_v = \lambda X_v$

(assume $X \in C^3$)

Differentiate a. w.r.t v, b w.r.t u $\Rightarrow$

$$\lambda_v X_u = \lambda_u X_v$$

$\Rightarrow \quad \lambda_u = \lambda_v \equiv 0$ in V. Since V connected, $\lambda \equiv$ constant

If $\lambda \equiv 0$ then $N_u = N_v = 0$

so $N \equiv N_0$ constant in V and

$\langle X(u,v), N_0 \rangle_u = \langle X(u,v), N_0 \rangle_v = 0$

so $\langle X(u,v), N_0 \rangle =$ constant

in V so $V \subseteq$ plane.

If $\lambda \neq 0$ then setting

$Y(u,v) := X(u,v) - \frac{1}{\lambda} N(u,v)$,

$Y_u = Y_v = 0$ i.e. $Y \equiv Y_0$ in V.

Hence $|X(u,v) - Y_0| = \frac{1}{\lambda}$ so

$V \subset B_{1/|\lambda|}(Y_0)$. This proves the

Proposition locally. To prove

it globally, we use the connected of S. Take $r \in S$

and a continuous curve $\alpha \subset S$ joining p to r, say $\alpha: [0,1] \to S$, $\alpha(0)=p$, $\alpha(1)=r$. For each $\alpha(t) \in S$ there is a nhbd $V_t \subset S$ containing $\alpha(t)$ contained in a sphere or plane, and s. that $\alpha^{-1}(V_t \cap \alpha[0,1])$ is open in $[0,1]$. Since $[0,1]$ compact, a finite # of these open sets cover $[0,1]$ (Heine-Borel). So $\alpha([0,1])$ is covered by a finite # of the $\{V_t\}$. The rest of the argument is an exercise. //

<u>Def'n</u> Let $p \in S$. An asymptotic direction of S at p is a direction $v \in T_pS$ s.that $k_n(v) = 0$. A curve $C \subset S$ is called an <u>asymptotic curve</u> if (say $C = \alpha(t)$, $a \leq t \leq b$)

$$k_n(\alpha'(t)) = 0, \quad a \leq t \leq b$$

<u>Remark</u> At an elliptic pt $p \in S$ there are no asymptotic directions.

Asymptotic directions may be interpreted in terms of the "<u>Dupin indicatrix</u>" at $p \in S$ $= \{ w \in T_pS : II_p(w) = \pm 1 \}$.

Let ξ, η be Cartesian coordinates for T_pS and (ρ, θ) polar coord. for T_pS, i.e. $w = \rho v$, $|w| = 1$ $v = e_1 \cos\theta + e_2 \sin\theta$

For $\rho \neq 0$, by <u>Euler's formula</u>

$$\pm 1 = II_p(w) = \rho^2 II_p(v)$$

$$= k_1 \rho^2 \cos^2\theta + k_2 \rho^2 \sin^2\theta$$

$$= k_1 \xi^2 + k_2 \eta^2$$

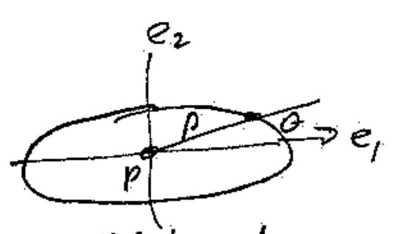
Hence the coordinates (ξ, η) of a pt on the Dupin indicatrix is

$$k_1 \xi^2 + k_2 \eta^2 = \pm 1$$

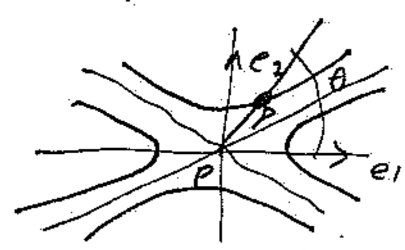
which is a union of conics in $T_p V$.

Notice $k_n(v) = II_p(v) = \pm 1/\rho^2$

For an elliptic pt, the Dupin indicatrix is an ellipse while for a hyperbolic pt is two hyperbolas with a common pair of asymptotic lines



elliptic pt



hyperbolic pt

3.3 The Gauss map in local coordinates

Let $X : U \subset R^2 \to S$ be a local parametization near $p \in X(U)$ with $N = \frac{X_u \times X_v}{|X_u \times X_v|}$ compatible with the orientation of S. Let $\alpha(t) = X(u(t), v(t))$ with $\alpha(0) = p$ be a curve on S

Then $\alpha' = u' X_u + v' X_v$ and (always evaluated at p)

$$dN(\alpha') = \underset{\text{("total deriv")}}{N'(u(t), v(t))} = u' N_u + v' N_v$$

Since N_u, N_v belong to T_pS (i.e. $\langle N_u, N \rangle = \langle N_v, N \rangle = 0$),

we may write:

$$N_u = a_{11} X_u + a_{21} X_v$$

(1) $$N_v = a_{12} X_u + a_{22} X_v$$

$$dN(\alpha') = (a_{11} u' + a_{12} v') X_u + (a_{21} u' + a_{22} v') X_v ,$$

i.e. $$dN\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

in the basis $\{X_u, X_v\}$.

On the other hand,

$$\begin{aligned} II(\alpha) &= -\langle dN(\alpha'), \alpha' \rangle \\ &= -\langle N_u u' + N_v v', u' X_u + v' X_v \rangle \\ &= L u'^2 + 2M u' v' + N v'^2 \end{aligned}$$

where

$$L = -\langle N_u, X_u\rangle = \langle X_{uu}, N\rangle$$

$$M = -\langle N_u, X_v\rangle = -\langle N_v, X_u\rangle = \langle X_{uv}, N\rangle$$

$$N = -\langle N_v, X_v\rangle = \langle X_{vv}, N\rangle$$

(Since $\langle X_u, N\rangle = \langle X_v, N\rangle = 0$).

From (1),

(2)
$$-M = \langle N_u, X_v\rangle = a_{11}F + a_{21}G$$
$$-M = \langle N_v, X_u\rangle = a_{12}E + a_{22}F$$
$$-L = \langle N_u, X_u\rangle = a_{11}E + a_{21}F$$
$$-N = \langle N_v, X_v\rangle = a_{12}F + a_{22}G,$$

i.e

(3) $$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

So,

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \frac{\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}}{EG-F^2}$$

Thus in the basis $\{X_u, X_v\}$, the coefficients (a_{ij}) of the matrix of dN_p are given by

$$a_{11} = \frac{FM-GL}{EG-F^2}$$

(4) $$a_{21} = \frac{FL-EM}{EG-F^2}$$ "Weingarten equations"

$$a_{12} = \frac{FN-GM}{EG-F^2}$$

$$a_{22} = \frac{FM-EN}{EG-F^2}.$$