2.2 Regular surfaces

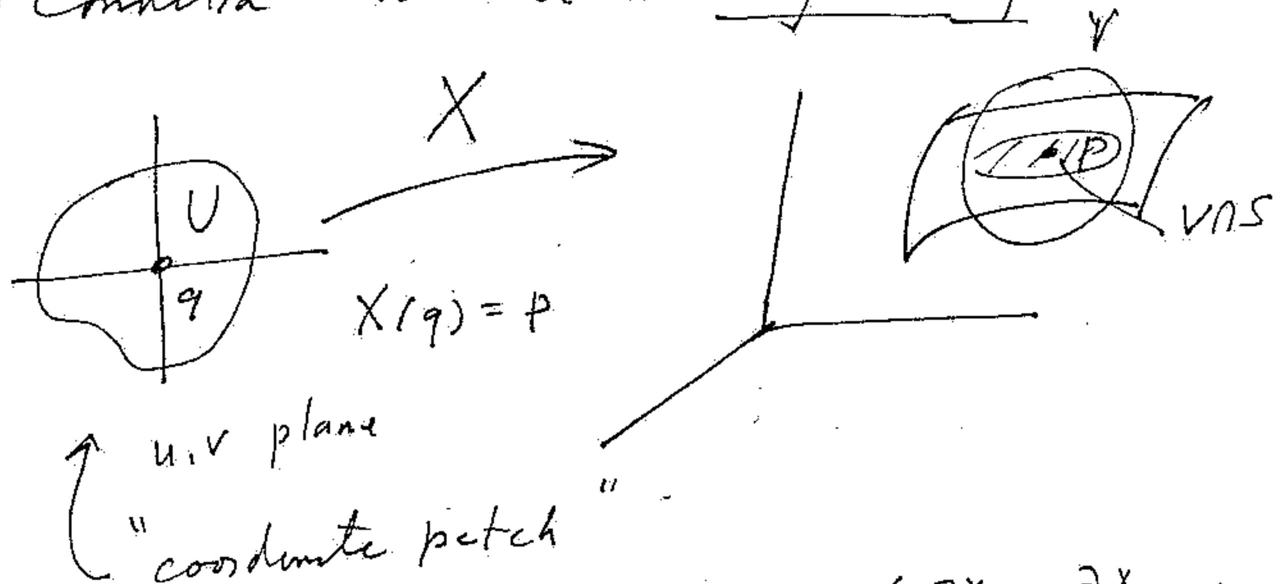
Defn A subset $S \subset R^3$ is a regular surface if for each $p \in S$, $\exists$ nhbd V in R^3 of p and a mapping $X : U \to V \cap S$ of an open $U \subset \mathbb{R}^2$ onto $V \cap S$ s.that:

1. X is C^1 (continuously differentiable, later we will assume smoother), i.e
$X(u,v) = (x(u,v), y(u,v), z(u,v))$
with $x, y, z \in C^1$

2. X is a homeomorphism (note by 1. X is continuous) $\Longleftrightarrow$ $\exists$ inverse $X^{-1} : V \cap S \to U$ which is continuous.

3. The differential $dX_q : \mathbb{R}^2 \to \mathbb{R}^3$ is 1-1 for each $q \in U$. This

condition is called <u>regularity</u>



X regular iff $dX_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$

has rank 2, i.e.

X_u, X_v linearly independent $\longleftrightarrow$

$X_u \times X_v \neq 0$ $\longleftrightarrow$

one of the Jacobian determinants

$\frac{\partial(x,y)}{\partial(u,v)}$, $\frac{\partial(y,z)}{\partial(u,v)}$ $\frac{\partial(z,x)}{\partial(u,v)}$

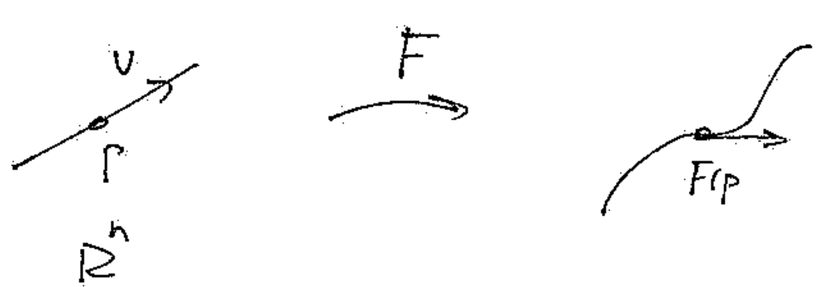
is non-zero at q

geometrically one defines the differential of a mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ as follows:

Let $F(p) = (f_1(p), f_2(p), \ldots, f_m(p))$
The f_i are the Euclidean coord. functions of $\mathbb{R}^m$

<u>Defn</u> 1. A tangent vector v to R^n at p, (notation $v \in T_p R^n$) is the velocity vector of a C^1 curve passing through p, i.e. if $\gamma: (-\varepsilon, \varepsilon) \to R^n$ $\gamma(0) = p$ then $v = \gamma'(0)$. For example $\gamma(t) = p + tv$

2. Now define $dF_p(v) = \frac{d}{dt}\Big|_{t=0} F(\gamma(t))$



so dF_p transforms a tangent vector to $\mathbb{R}^n$ at p to a tangent vector to $\mathbb{R}^m$ at $F(p)$.

<u>Exercise</u> Show

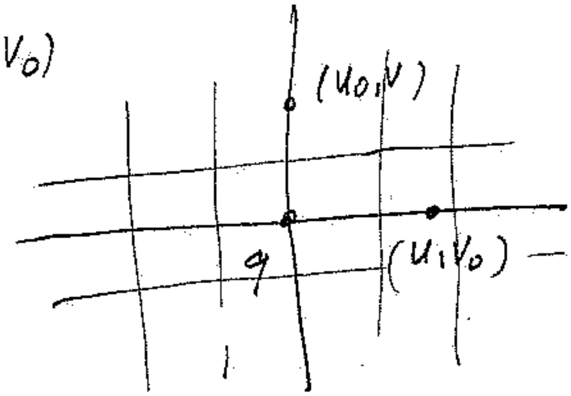
$$dF_p(v) = (v(f_1), \dots, v(f_m)) \text{ at } F(p)$$

where $v(f) = \nabla f \cdot v$ is the directional derivative of f in the direction v at p (tangent vectors "act on functions").

<u>Example</u> If $X : U \to R^3$ and $q \in U$

then
$$X_u(q) = dX_q(e_1)$$
$$X_v(q) = dX_q(e_2)$$

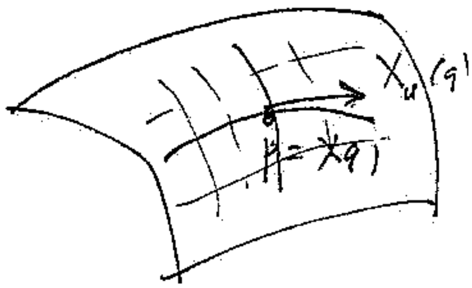
$q = (u_0, v_0)$



"horizontal and vertical curves in $\mathbb{R}^2$"

$X(u, v_0) = (x(u, v_0), y(u, v_0), z(u, v_0))$ image curve

$p = X(q)$

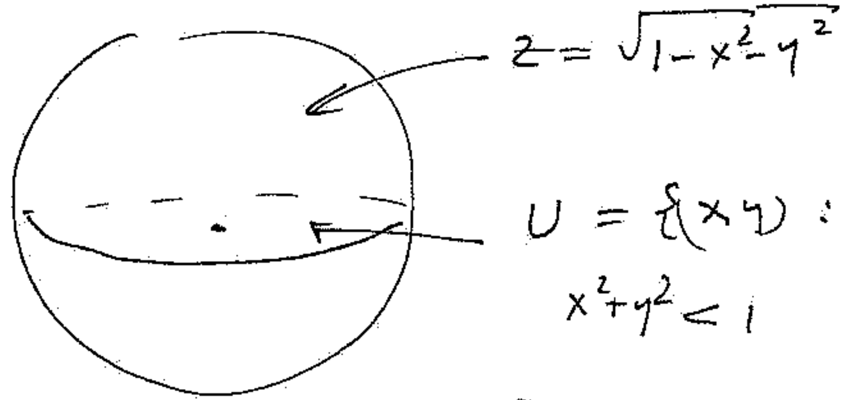


Thus representing dX_q with respect to the standard basis e_1, e_2 of $\mathbb{R}^2$ (it is indep't of basis) gives the matrix

$$dX_q = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

<u>Example 1</u> $S^2 = \{ (x, y, z) \in \mathbb{R}^3 ; x^2+y^2+z^2 = 1 \}$

is a regular surface. We can show this in many ways. Perhaps the simplest is to cover S^2 by graphical coordinate neighborhoods.



Let $X_1(x,y) = (x, y, \sqrt{1-x^2-y^2})$

Note X_1 is regular since

$$\frac{\partial(x,y)}{\partial(x,y)} = 1$$

and X_1 is 1-1 with

X_1^{-1} = the restriction

of the (continuous) projection

$\pi(x,y,z) = (x,y)$ to

the set $X_1(U)$

Similarly $X_2(x,y) = (x, y, -\sqrt{1-x^2-y^2})$

$X_3(x,z) = (x, \sqrt{1-x^2-z^2}, z)$

$X_4(x,z) = (x, -\sqrt{1-x^2-z^2}, z)$

$X_5(y,z) = (\sqrt{1-y^2-z^2}, y, z)$

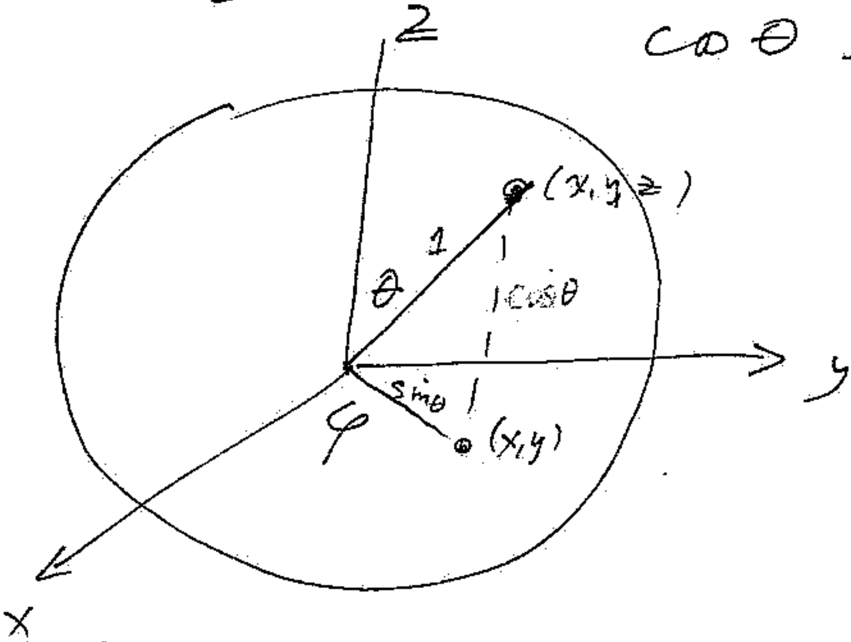
$X_6(y,z) = (-\sqrt{1-y^2-z^2}, y, z)$

Now every point in S^2 is covered.

Another useful and more geographical way is to introduce colatitude θ, and longitude φ:

Let $V = \{(\theta, \varphi) : 0 < \theta < \pi, \ 0 < \varphi < 2\pi\}$

and $X(\theta, \varphi) = \{\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta\}$



Then

$$\frac{\partial(x,y)}{\partial(\theta,\varphi)} = \cos\theta \sin\theta$$

$$\frac{\partial(y,z)}{\partial(\theta,\varphi)} = \sin^2\theta \cos\varphi$$

$$\frac{\partial(x,z)}{\partial(\theta,\varphi)} = \sin^2\theta \sin\varphi$$

$$\cos^2\theta \sin^2\theta + \sin^4\theta \cos^2\varphi + \sin^4\theta \sin^2\varphi$$

$$= \cos^2\theta \sin^2\theta + \sin^4\theta = \sin^2\theta$$

$$\neq 0 \quad \text{in } V$$

Another way :

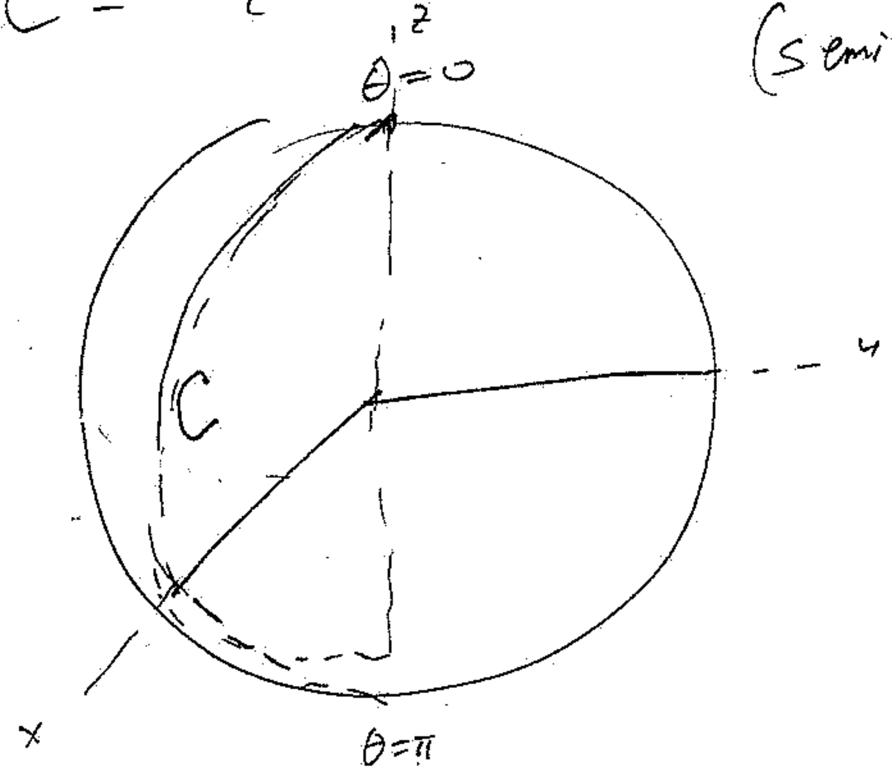
$X_\theta = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, -\sin\theta)$

$X_\varphi = (-\sin\theta \sin\varphi, \sin\theta \cos\varphi, 0)$

$X_\theta \times X_\varphi = (\sin^2\theta \cos\varphi, -\sin^2\theta \sin\varphi, \sin\theta \cos\theta$

$\neq 0$ in V

Let $C = \{(x, y, z) \in S^2 : y = 0, x \geq 0\}$ (semicircle in x, z plane)



If $(x, y, z) \in S^2 \setminus C$,

$\theta = \cos^{-1} z$ is uniquely

determined since $0 < \theta < \pi$.

Then we can find $\sin\varphi, \cos\varphi$

from $x = \sin\theta \cos\varphi$

$y = \sin\theta \sin\varphi$

so φ is uniquely determined in the interval $0 < \varphi < 2\pi$

Therefore X^{-1} exists. We will soon ~~see~~ (Prop 4 p 66 Do Carmo) that X^{-1} is automatically continuous provided S is regular (which we have shown already).

So $X(V)$ omits only a semicircle of S^2 and one other parametrization of this type suffices to cover S^2. //

We next present two useful cases where showing S is regular is easy.

<u>Proposition 1</u> Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Then graph(f) $= \{ (x, y, f(x,y)) : (x,y) \in U \}$ is a regular surface

<u>Pf</u> exercise.

In order to proceed we first review some important ideas from advanced calculus.

<u>Def'n</u>. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. We say $p \in U$ is a critical point of F if $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ (linear map) is not onto (surjective).

The image $F(p)$ of a critical point is called a critical value. Otherwise we say p is a regular pt (dF_p onto) and $F(p)$ is a regular value.

__Example__

Let $f: U \to \mathbb{R}^3 \to \mathbb{R}$. Then

$df_p = (f_x, f_y, f_z)$ and

$df_p : \mathbb{R}^3 \to \mathbb{R}$ is not surjective

iff $f_x = f_y = f_z = 0$ at p

(i.e $\nabla f(p) = 0$) ; i.e we

cannot solve the linear

equation $f_x(p) v_1 + f_y(p) v_2 + f_z(p) v_3 = \lambda \neq 0$

We next review the Inverse

Function theorem and Implicit

function theorem, which are

incredibly important analysis tools.

Inverse function theorem.

Suppose U open $\subset \mathbb{R}^n$ and $F: U \to \mathbb{R}^n$ is a C^1 mapping with $p \in U$ such that dF_p is nonsingular. (i.e. an isomorphism.). Then there exist nhbds V of p in U and W of $F(p)$ in $\mathbb{R}^n$ so that $F: V \to W$ has a C^1 inverse $F^{-1}: W \to V$ with $d(F^{-1})_y = [dF_{F^{-1}(y)}]^{-1}$ for $y \in W$ (Necessarily $W = F(V)$)

I will not delve into the proof which involves a lot of analysis.

Instead we will apply it to obtain the

<u>Implicit function theorem</u>

Let W be an open set in $\mathbb{R}^p \times \mathbb{R}^q$ with $(a,b) \in W$ and $F: W \to \mathbb{R}^q$ a C^1 mapping with $F(a,b) = 0$ and $D_y F(a,b)$ non-singular (note $D_y F(a,b)$ is a $q \times q$ matrix with respect to the standard basis so non-singular $\Leftrightarrow$ invertible)

Then there exists an open $U \subset \mathbb{R}^p$ with $a \in U$ and an open $V \subset \mathbb{R}^q$ containing b s. that :

i) For each $x \in U$, there is a unique $y = g(x)$ in V s.that

$$F(x, g(x)) = 0$$

and $$g(a) = b$$

ii) Moreover $g \in C^1$ and

$$(d_x g)_x = -(d_y F)^{-1}\big|_{(x,g(x))} (d_x F)\big|_{(x,g(x))}$$

<u>Proof</u> (sketch) Define $G(x,y) = (x, F(x,y))$ and compute that

$$\det dG_{(a,b)} = \det (d_y F)_{(a,b)} \neq 0$$

The Inverse function theorem thus

gives a C^1 inverse

$G^{-1}: W_1 \to V_1$ for some

open nhbds W_1 of $(a,0)$ in $\mathbb{R}^p \times \mathbb{R}^q$

and V_1 of (a,b) in $\mathbb{R}^p \times \mathbb{R}^q$

The set $U = \{x \in \mathbb{R}^p : (x,0) \in W_1\}$

is open in $\mathbb{R}^p$ and for each

point $x \in U$, $G^{-1}(x,0) = (x, g(x))$

for some $g(x) \in \mathbb{R}^q$. Moreover

$$\{(x,y) \in W_1 : F(x,y) = 0\} =$$

$$\{ G^{-1} \circ G)(W_1 \cap F^{-1}(0))$$

$$= \{(x, g(x)) : x \in U\}$$

One easily checks that

$$\frac{\partial g_i}{\partial x_j}(x) = \frac{\partial (G^{-1})_{p+i}}{\partial x_j}(x, 0)$$

for $i = 1, \cdots, q$, $j = 1, \cdots, p$ and $x \in W_1$

The formula for $(d_x g)_x$ follows

from differentiation of

$$F(x, g(x)) = 0 \text{ on } V. \quad //$$

Using a simple case of

the implicit function theorem,

we prove

Proposition 2 Let $f : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{R}$ be C^1 and $a \in f(\mathcal{O})$ be a regular value. Then $S = f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Pf Let $p = (x_0, y_0, z_0) \in S$. Then $f(p) = a$ and $(df)_p \neq 0$. Relabeling the axes if necessary, we may assume $f_z(p) \neq 0$. By the implicit function theorem, there is an open nhbd U of (x_0, y_0) in $\mathbb{R}^2$, an open nhbd V of z_0 in $\mathbb{R}$ and a C^1 function $g : U \to V$ such that $U \times V \subset \mathcal{O}$, $g(x_0, y_0) = z_0$ and

$$\{p \in U \times V \mid f(p) = a\} = \{(x, y, g(x,y)) \in \mathbb{R}^3 \mid (x,y) \in U\}$$

That is, $S \cap (U \times V)$ is a nhbd of p in S which is the graph of a C^1 function. By Prop 1, S is a regular surface. //.

<u>Example</u> (ellipsoid)

$$S = \{(x, y, z) \in R^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$$

$$= f^{-1}(0), \quad f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

is a regular surface since f is smooth and

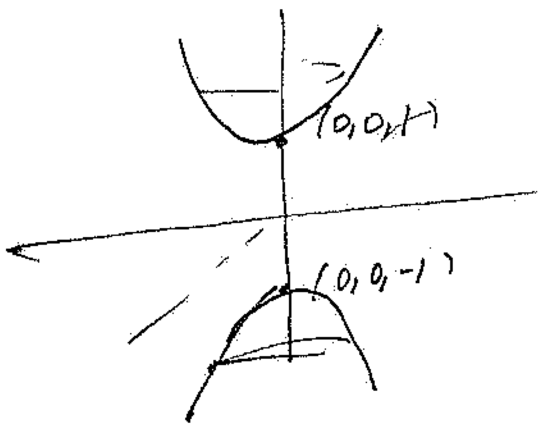
$$|\nabla f|^2 = \frac{4x^2}{a^4} + \frac{4y^2}{b^4} + \frac{4z^2}{c^4} \neq 0$$

on $f^{-1}(0)$. When $a = b = c = R > 0$, S is the sphere $B_R(0)$

The following example shows $f^{-1}(0)$ is not necessarily connected.

<u>Example</u> (hyperboloid of two sheets)

Let $f(x,y,z) = -x^2-y^2+z^2-1$



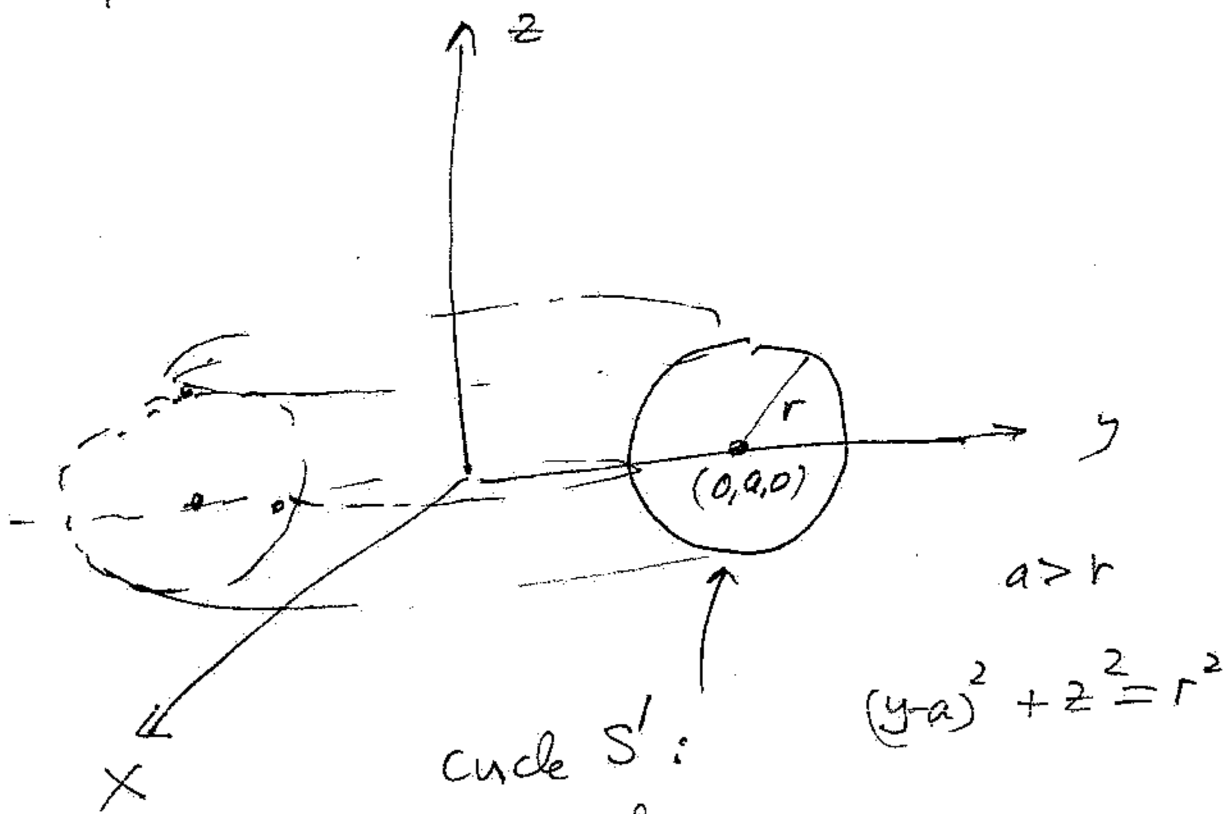
$|\nabla f|^2 = 4(x^2+y^2+z^2)$

$\neq 0$ on $f^{-1}(0)$

So $S = f^{-1}(0)$ is not connected and given and there does not exist any curve $\alpha(t) = (x(t), y(t), z(t))$ joining two points on different "sheets" of S, that is contained in S.

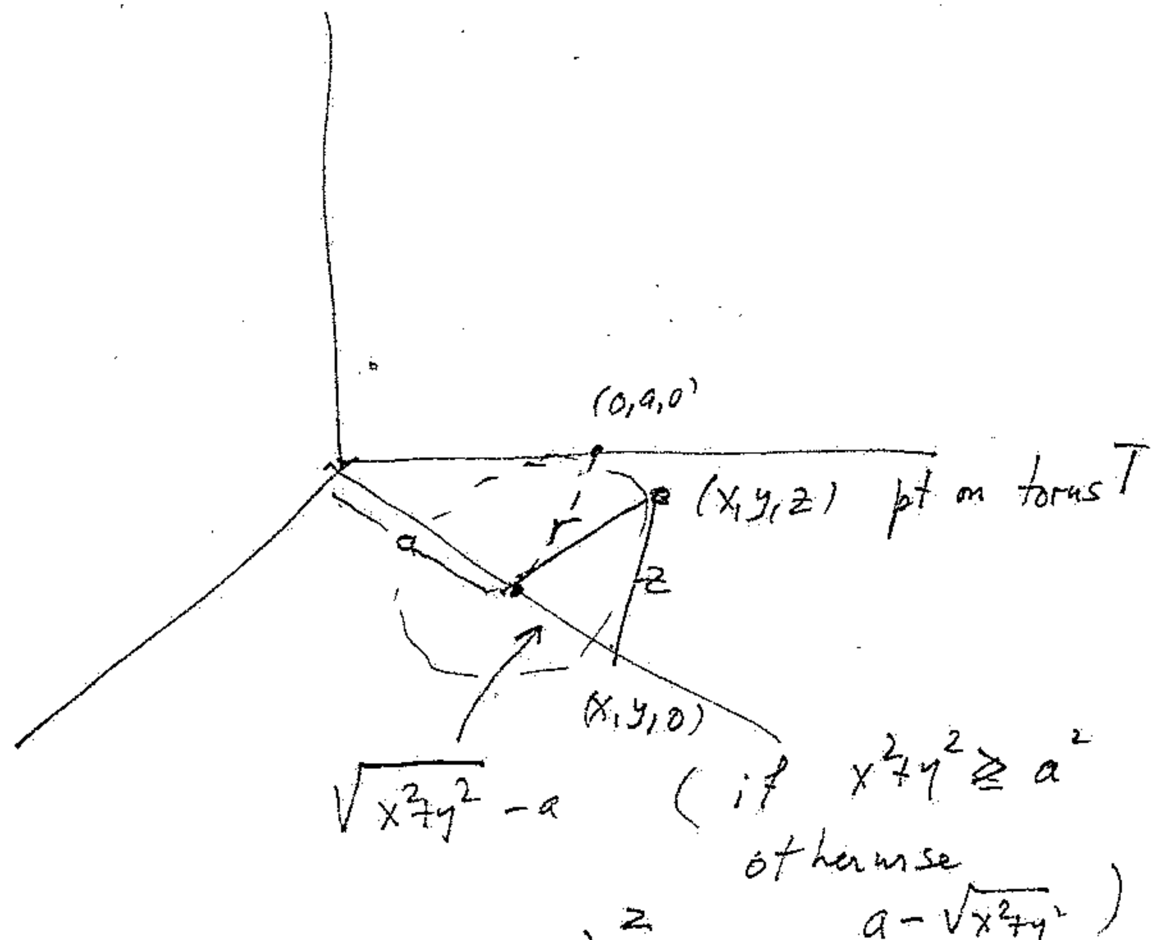
Conversely, if $f: S \to \mathbb{R}$ is continuous and non-zero and S is connected, then f does not change sign (exercise)

Example (torus of revolution)



in y,z plane

rotated about z axis



$$r^2 = z^2 + \left(\sqrt{x^2+y^2}-a\right)^2$$

Set $f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2}-a\right)^2 - r^2$

Then $T = f^{-1}(0)$ and

$$f_x = 2\left(\sqrt{x^2+y^2}-a\right)\frac{x}{\sqrt{x^2+y^2}}, \quad f_y = 2\left(\sqrt{x^2+y^2}-a\right)\frac{y}{\sqrt{x^2+y^2}}$$

$$f_z = 2z$$

$$|\nabla f|^2 = 4\left\{ \left(\sqrt{x^2+y^2}-a\right)^2 + z^2 \right\} = 4r^2 > 0$$

on T so T is regular

by Prop. 2. //.

We next prove that locally near $p \in S$, a regular surface, S may be represented as the graph of a differentiable function f, a kind of converse of Proposition 1.

<u>Proposition 3</u> Let $S \subset R^3$ and $p \in S$. Then there exists a nbhd V of p in S such that V is the

graph of a differentiable function which has one of the three forms $z = f(x,y)$, $y = g(x,z)$, $x = h(y,z)$

<u>Remark</u> Proposition 3 is really a "change of parameter" result in the sense of Prop. 1 p 72 of section 2.3 of do Carmo.

<u>Pf</u>. Let $X : U \to S$ be a parametrization with $X(p_0) = p$, say $X(u,v) = (x(u,v), y(u,v), z(u,v))$ Then

$$(dX)_{p_0} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} (p_0) \quad \text{has rank } 2$$

so one of the three Jacobian

determinants $\frac{\partial(x,y)}{\partial(u,v)}$, $\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial(z,x)}{\partial(u,v)}$

is not zero at p_0. We may

suppose $\frac{\partial(x,y)}{\partial(u,v)}(p_0) \neq 0$

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal

projection onto the xy plane i.e.

$\pi(x,y,z) = (x,y)$. Then the

composition $\pi \circ X$ is a differentiable

map of U to $\mathbb{R}^2$ and

$$d(\pi \circ X)_{p_0} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}(p_0)$$

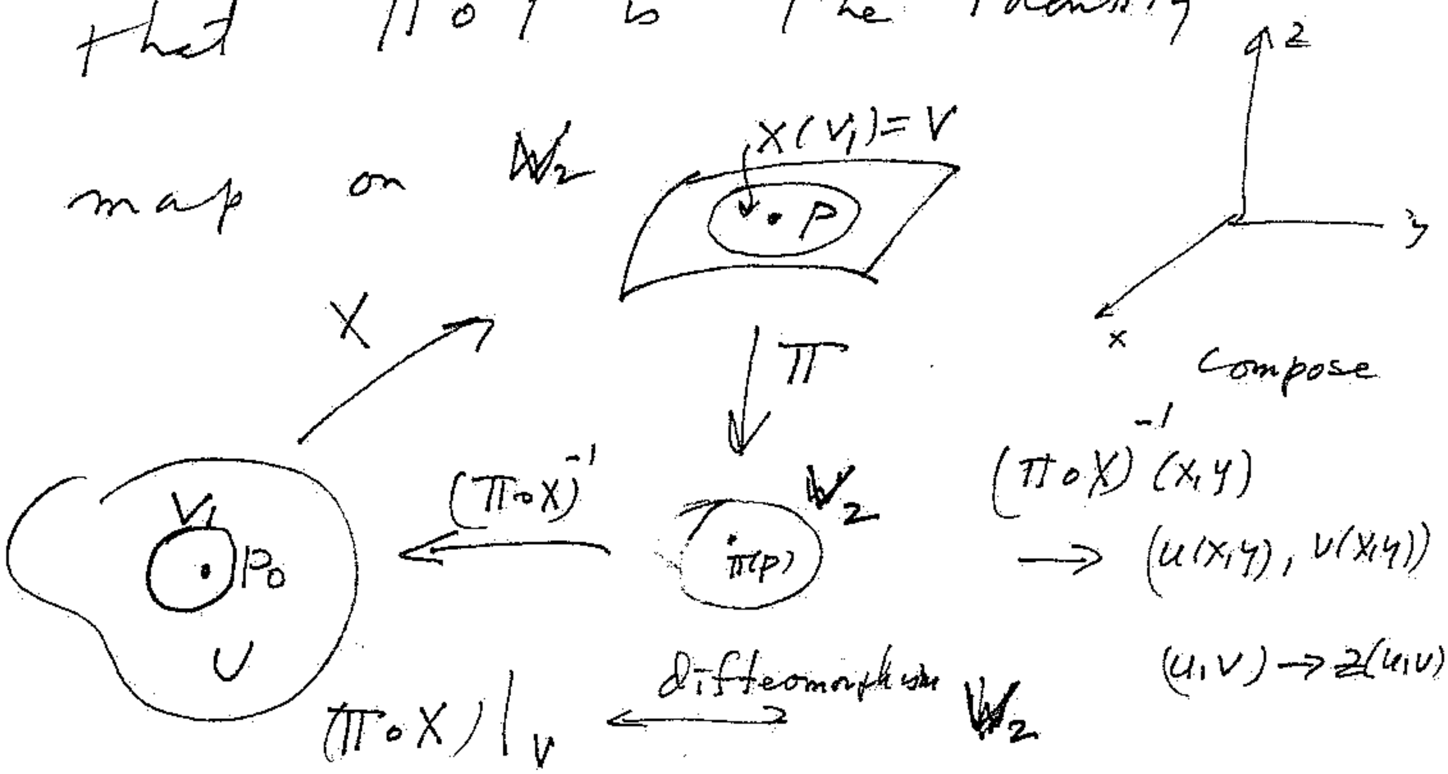
is regular. Applying the inverse

function theorem there is an open

nhbd V_1 of p_0 such that

$W_2 = (\pi \circ X)(V_1)$ is open in $\mathbb{R}^2$
and $(\pi \circ X)\, V_1 \to W_2$ is a diffeomorphism

Now observe that $Y = X \circ (\pi \circ X)^{-1}$
$: W_2 \to S$ is a parametrization
of S which covers p and
that $\pi \circ Y$ is the identity
map on W_2



Hence V is the graph of $z(u(x,y), v(x,y))$:
$x, y \in W_2$. //

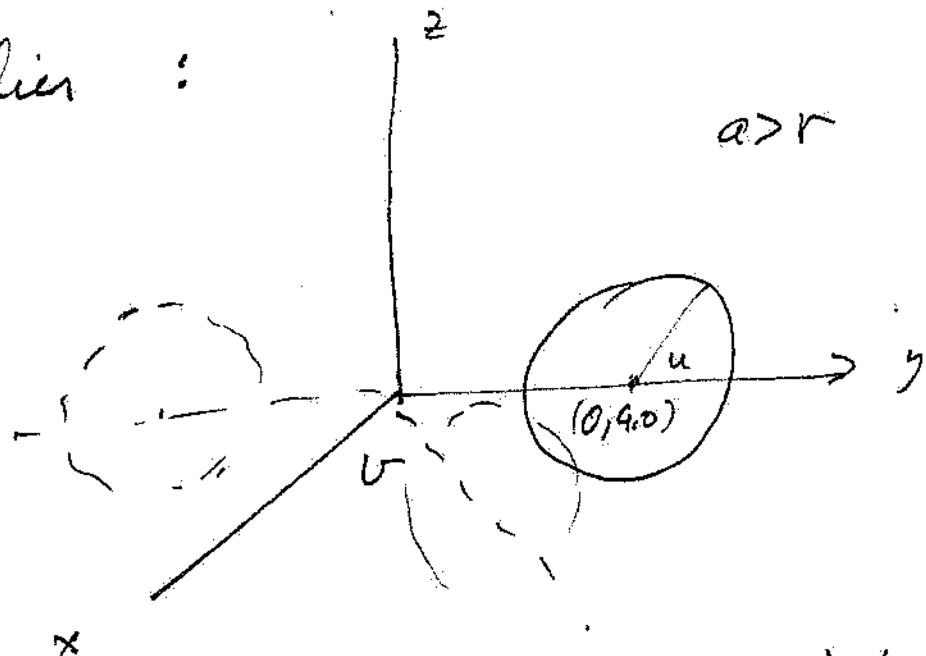
An immediate corollary of the proof of prop 3 is

<u>Prop 4</u> Let $p \in S$ be a point of a regular surface S and let $X : U \subset R^2 \to \mathbb{R}^3$ be a C^1 map with $p \in X(U) \subset S$ s.t. conditions 1, 3 of Def. 1 hold. Assume X is 1-1. Then X^{-1} is continuous.

<u>Pf</u> Restricted to $V = X(V_1)$,

$X^{-1} = (\pi \circ X)^{-1} \circ \pi$, the composition of continuous functions //

__Example__ The torus of revolution that we considered earlier:



may be parametrized by introducing the polar angle u in the circle of revolution

i.e
$$y = a + r\cos u$$
$$z = r\sin u$$

and the rotation angle v (measured from the x axis)

$$X(u,v) = \left((a+r\cos u)\cos v,\ (a+r\cos u)\sin v,\ r\sin u\right)$$

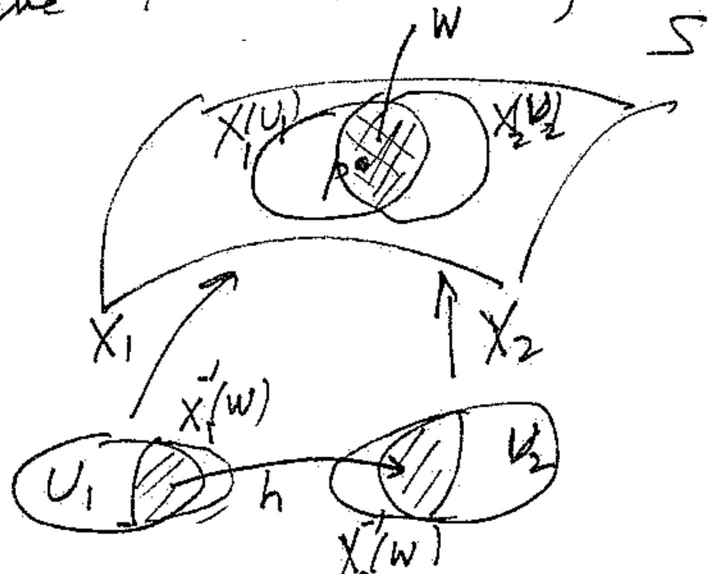
where $0 < u < 2\pi$, $0 < v < 2\pi$.

Since T is regular condition 2 is equivalent (by Prop 4) to showing X is 1-1. We have $\sin u = \frac{z}{r}$ and if $x^2+y^2 \le a^2$, then $\pi/2 \le u \le 3\pi/2$ while if $x^2+y^2 \ge a^2$ either $0 < u \le \pi/2$ or $\frac{3\pi}{2} \le u < 2\pi$

Thus given (x, y, z) u is determined uniquely (since $\sin$ is 1-1 on these intervals). Knowing u, x, y determines $\cos v$ and $\sin v$ which uniquely determines v in $0 < v < 2\pi$. //

2.3 change of parameters, differentiable functions on a surface

A point $p \in S$ may belong to several coordinate patches and we need to know that properties of S are independent of the (local) representation. Thus we have the following situation:



Assume $W = X_1(U_1) \cap X_2(V_2) \neq \phi$

Then the map $h = X_2^{-1} \circ X_1 : X_1^{-1}(W) \to X_2^{-1}(W)$ is a homeomorphism. We say that h is a change of parameters (or change of coordinates)

Prop. 1 h is a diffeomorphism,
(i.e h is differentiable with differentiable inverse)

Pf Since h is bijective and its inverse is also a change of parameters, we only need to show that h is differentiable at each point of $X_1^{-1}(W)$.

Let $q_1 = X_1^{-1}(p)$ $q_2 = X_2^{-1}(p)$

Applying the proof of Prop 3 of the last section, there is an open set $V_2 \subset X_2^{-1}(W)$ and a projection π, which we may assume to be the xy plane, such that $\pi \circ X_2 : V_2 \to (\pi \circ X_2)(V_2)$ is a diffeomorphism. Then $h^{-1}(V_2)$ is an open nbhd of q_1 and in this nbhd,

$$h = (\pi \circ X_2)^{-1} \circ (\pi \circ X_1)$$

Therefore h is differentiable on $h^{-1}(V_2)$ since it is the composition of the differentiable maps:

$X_1 : h^{-1}(V) \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ and
$(\pi \circ X_2)^{-1} : (\pi \circ X_2)(V_2) \subset \mathbb{R}^2 \to V_2 \subset \mathbb{R}^2$.
//

Now we can define what it means for a function on a regular S to be differentiable (independent of the parametrization of S)

<u>Defn</u> Let $f : V \subset S \to \mathbb{R}$ be a function defined on an open subset V of a regular surface S. We say f is <u>differentiable</u> at $p \in V$ if for some parametrization $X : U \subset \mathbb{R}^2 \to S$ with $p \in X(U) \subset V$, the composition $f \circ X : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $X^{-1}(p)$. We say f is differentiable on V if f is differentiable at p $\forall p \in V$

By Prop 1 the def'n of differentiability of f at p is independent of choice of X (since if $Y : V \subset \mathbb{R}^2 \to S$ is another parametrization with $p \in Y(V)$ and if $h = X^{-1} \circ Y$, then $f \circ Y = f \circ X \circ h$ is also differentiable, (all restricted to the proper domains)).

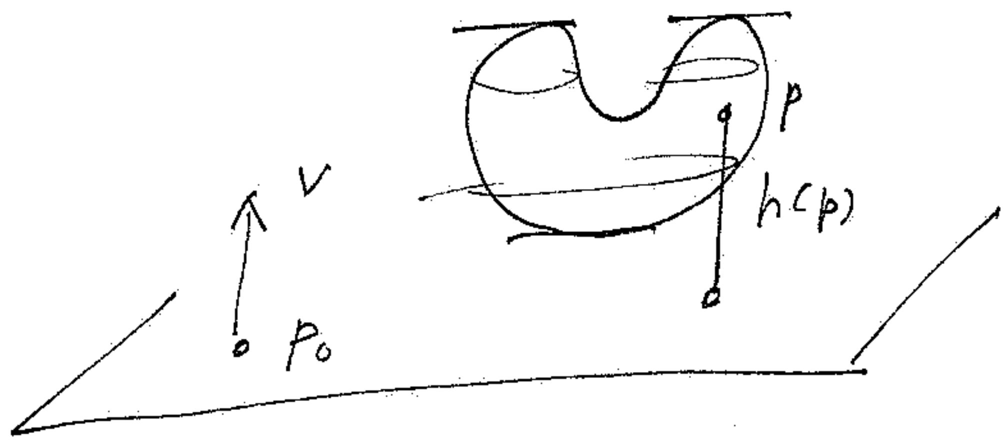
<u>Remark</u> It is common to abuse notation by writing f, $f \circ X$ just as $f(u,v)$ i.e. we identify $X(U)$ with U. This (hopefully) will not cause confusion.

Example (restrictions)

Let S be regular, $S \subset V$ (open) $\subset \mathbb{R}^3$ and let $f : V \to \mathbb{R}$ be differentiable. Then $f|_S$ (restriction of f to S) is differentiable. For if $p \in S$ and $X : U \subset \mathbb{R}^2 \to S$, $p \in X(U)$ is a parametrization, then $f \circ X : U \to \mathbb{R}$ is clearly differentiable

1. (The height function) Let v be the (unit) normal vector to the plane passing through a pt P_0. If S is a regular surface define

$$h(p) = (p - p_0) \cdot v \quad \forall\, p \in S$$



Then $h(p)$ is differentiable on S because it is the restriction of a linear function on $\mathbb{R}^3$.

2. (square of the distance) For any $p_0 \in \mathbb{R}^3$, S regular, define $f: S \to \mathbb{R}$ by $f(p) = |p - p_0|^2$ $\forall p \in S$, the square of the Euclidean distance to p_0. Again f is differentiable

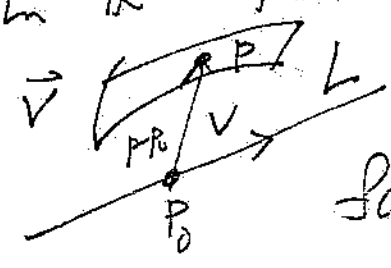
3. (distance function)

Let $p_0 \in \mathbb{R}^3 \setminus S$, S regular

Then $f(p) = |p - p_0|$ $\forall p \in S$

measuring the Euclidean distance to p_0 since it is the restriction of the diff. fcn $|p - p_0|$ on $\mathbb{R}^3 - \{p_0\}$

<u>Exercise</u> (square of the distance to a straight line) Let L be a line in $\mathbb{R}^3$ through p_0 with unit direction $\vec{v}$



show that for $p \in S$

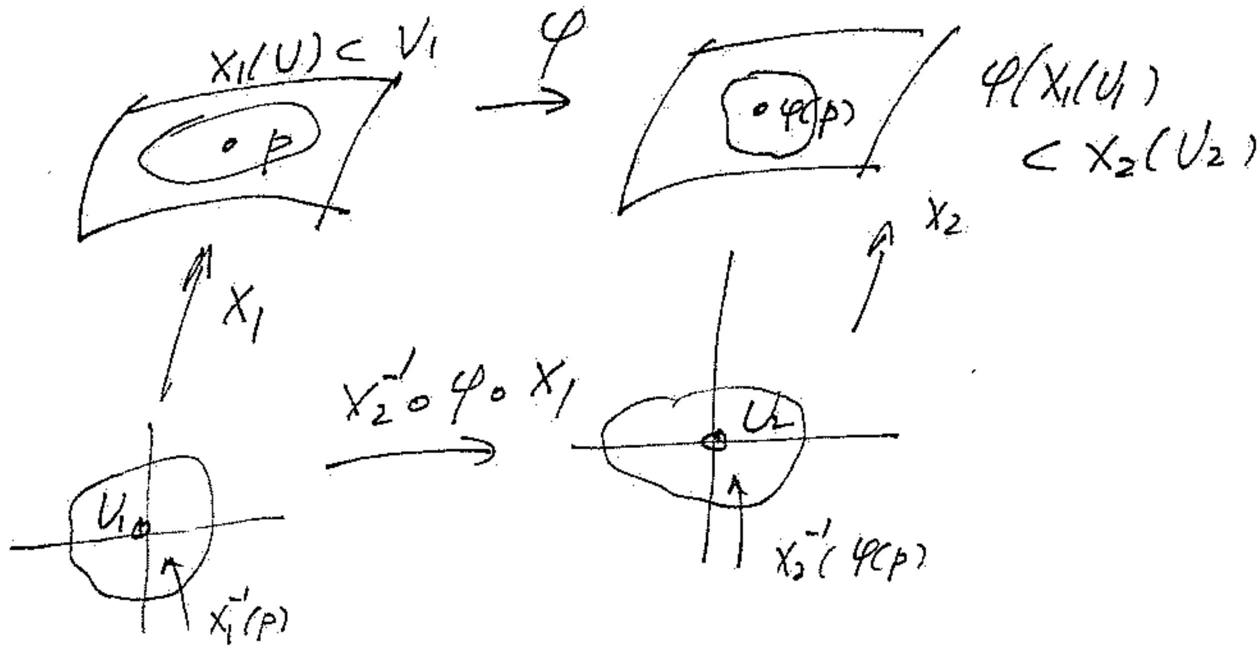
$$f(p) = |p - p_0|^2 - (p - p_0)\cdot \vec{v} = |(p - p_0) \times \vec{v}|^2$$

is the square of the distance from p to L is differentiable.

The defn of differentiability naturally extends to mappings:

$\varphi : V_1 \subset S_1 \rightarrow S_2$, S_1, S_2 regular surfaces.

V_1 open



φ is differentiable (assume φ continuous)

at $p \in$ if $X_2^{-1} \circ \varphi \circ X_1 : U_1 \rightarrow U_2$

is differentiable at $q = X_1^{-1}(p)$

Exercise Check all the details of Example 2 p 76 of Do Carmo.

2.4. The tangent plane, differential of a map

Given $p \in S$ a regular surface, we want to define the best linear approximation at p to S.

<u>Def'n</u> (tangent vectors and tangent plane)
Given $p \in S$ a regular surface, we say $v \in R^3$ is tangent to S at p if we can find a curve
$\alpha : (-\varepsilon, \varepsilon) \to S$, $\varepsilon > 0$ with
$\alpha(0) = p$ and $\alpha'(0) = v$. The set of all such tangent vectors is called the tangent plane T_pS to S at p.

<u>Prop 1</u> Let S be a regular surface and $X : U \subset R^2 \to S$ be a (local) parametrization of S with $p = X(q)$ $q \in U$. Then

$$T_pS = (dX)_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

<u>Pf</u> Let $w \in \mathbb{R}$ and let $\beta : (-\varepsilon, \varepsilon) \to U$ be the line segment $\beta(t) = q + tw$

Let $\alpha = X \circ \beta$; then $\alpha(0) = p$,

$\alpha'(0) = (X \circ \beta)'(0) = dX_q(w)$. Hence

$(dX_q)(\mathbb{R}^2) \subset T_pS$ since the image of α lies in S.

Conversely let $v \in T_pS$. By def'n $\exists$ curve $\alpha : (-\varepsilon, \varepsilon) \to S$ $\alpha(0) = p$ $\alpha'(0) = v$

For ε small enough we may

assume by continuity of α that its trace (image) is contained in $X(U)$. Define a curve in U by $\beta = X^{-1} \circ \alpha$.

Then $\beta(0) = q$ and $\alpha = X \circ \beta$

Thus $v = \alpha'(0) = (X \circ \beta)'(0)$
$= (dX)_q\,(\beta'(0))$, i.e. any vector $v \in T_pS$ lies in the image of $(dX)_q$. //

<u>Remarks</u> 1. T_pS is in fact a plane in $\mathbb{R}^3$ spanned by $X_{u_1}(q)$, $X_{u_2}(q)$ (for $v = \alpha'(0) =$ assume $\beta(t) = (u_1(t), u_2(t))$

$$\frac{d}{dt} X(u_1(t), u_2(t))\Big|_{t=0} = X_{u_1}(q)\, u_1'(0) + X_{u_2}(q)\, u_2'(0),$$

2. Convince yourself that T_pS does not depend on the parametrization X.

<u>Example</u> If $\mathcal{O} \subset \mathbb{R}^3$ is open and $f : \mathcal{O} \to \mathbb{R}$ is differentiable we have earlier shown that $S = f^{-1}(a)$ is a regular surface if a is a regular value of f.

<u>Claim</u> $T_pS = \ker((df)_p : \mathbb{R}^3 \to \mathbb{R})$.

For if $v \in T_pS$ $\exists$ curve $\alpha : (-\varepsilon, \varepsilon) \to S$ $\alpha(0) = p$, $\alpha'(0) = v$. Hence $f \circ \alpha(t) = a$ $\forall t$. Differentiate at $t = 0$ $\Rightarrow$ $(df)_p(v) = (f \circ \alpha)'(0) = 0$ i.e. $v \in \ker df_p$. Since T_pS and $\ker (df)_p$ are linear subspaces of dimension 2 and $T_pS \subseteq \ker (df)_p$ they must coincide. //

<u>Example</u> $S^2(p_0, r) :=$

$\{ p \in \mathbb{R}^3 \mid |p - p_0|^2 = r^2 \}$

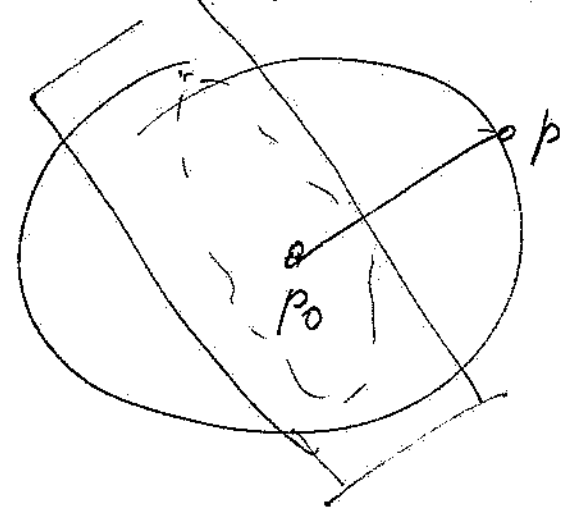
is the sphere of center p_0 and radius r. Then $S^2(p_0, r)$

$= f^{-1}(r^2)$ where $f(p) = |p - p_0|^2$

So $T_p S^2(p_0, r) = \ker(df)_p$

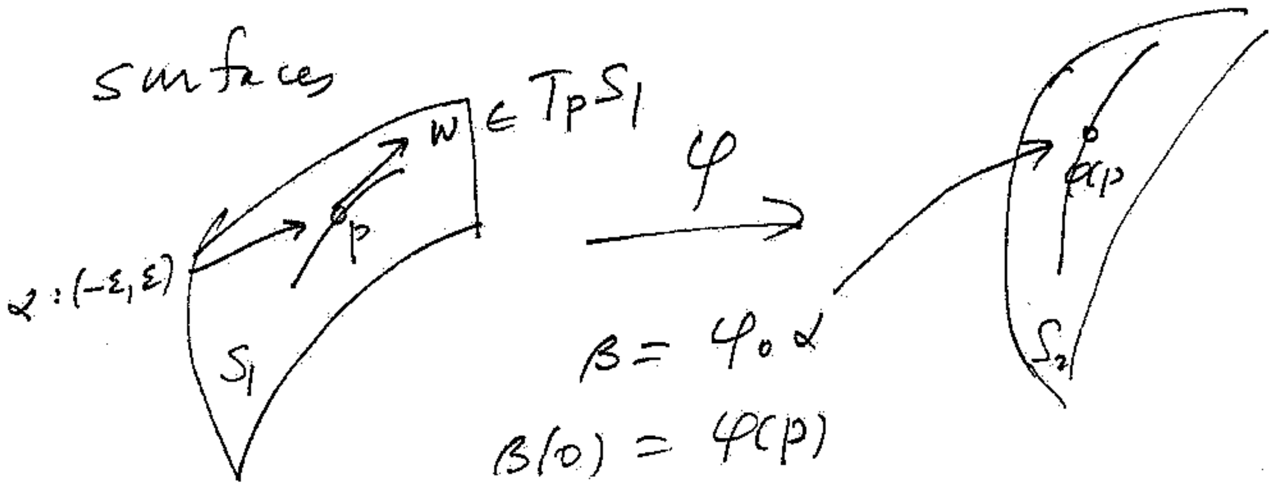
$= \{ v \in \mathbb{R}^3 : (p - p_0) \cdot v = 0 \}$

= linear plane orthogonal to the position vector $p - p_0$ relative to p_0



Extension to $\varphi : \underset{\text{open}}{V} \subset S_1 \longrightarrow S_2$

differentiable map, S_1, S_2 regular surfaces



$\beta = \varphi \circ \alpha$

$\beta(0) = \varphi(p)$

$\Rightarrow \beta'(0) \in T_{\varphi(p)} S_2$

Prop 2 $\beta'(0)$ is independent of choice of α and defines $d\varphi_p(w)$ (linear)

Read pf p 87 do Carmo.

Def'n

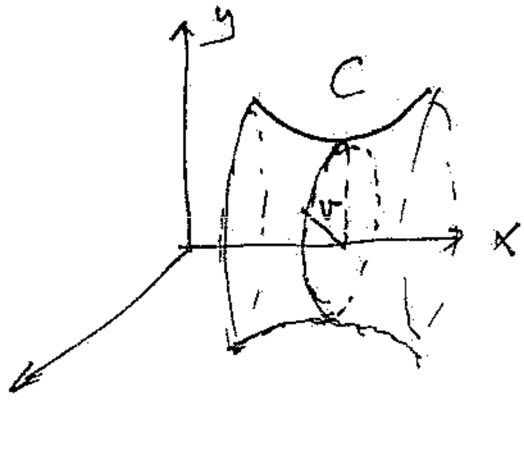
We have seen that locally in a parametrization ("coordinate patch"), the "tangent plane T_pS is spanned" by $X_u, X_v(u_0,v_0)$. In particular, the (unit) normal vector to S at p is given by

$$\frac{X_u \times X_v}{|X_u \times X_v|}(u_0, v_0)$$

Example (surface of revolution)

Let C be the parametrized curve in the x, y plane:

$$x = g(u)$$
$$y = h(u) > 0$$



that lies above the x axis. Now rotate C about the x axis to generate a "surface of revolution" S given by

$$X(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v)$$

(where v represents the rotation angle)

and $a < u < b$

$0 < v < 2\pi$

Then
$$X_u = (g'(u),\ h'(u)\cos v,\ h'(u)\sin v)$$
$$X_v = (0,\ -h(u)\sin v,\ h(u)\cos v)$$

$X_u \times X_v =$

$$(h(u)h'(u),\ -h(u)g'(u)\cos v,\ -h(u)g'(u)\sin v)$$

$$|X_u \times X_v| = h(u)\sqrt{h'^2(u) + g'^2(u)} > 0$$

(assume C is <u>regular</u>)

Then

$$N(u,v) := \frac{X_u \times X_v}{|X_u \times X_v|} =$$

$$\frac{(h'(u), -g'(u)\cos v, -g'(u)\sin v)}{\sqrt{h'^2(u) + g'^2(u)}}$$

2.5 The first fund. form and area

The inner (dot) product of $\mathbb{R}^3$ induces on each tangent plane T_pS of S an inner product $\langle\ ,\ \rangle_p$; i.e. if $w_1, w_2 \in T_pS$ define $\langle w_1, w_2\rangle_p = \langle w_1, w_2\rangle_{\mathbb{R}^3}$. Note that $\langle w_1, w_2\rangle_p$ is a symmetric bilinear form ($\langle w_1, w_2\rangle_p = \langle w_2, w_1\rangle$ and linear in both w_1, w_2)

<u>Defn</u>. The quadratic form $I_p : T_p(S) \to \mathbb{R}$ given by $I_p(w) = \langle w, w\rangle = |w|^2 \geq 0$ is called the <u>first fundamental form</u> of S at $p \in S$.

We next calculate I_p in a local parametrization $X(u,v) : U \to S$ $p \in X(U)$

given $w \in T_pS$ $\quad w = \alpha'(0)$,
$\alpha(0) = p$ where
$$\alpha(t) = X(u(t), v(t)) \quad t \in (-\varepsilon, \varepsilon)$$
$$p = X(u_0, v_0) = X(u(0), v(0))$$. Hence

$$I_p(w) = \langle \alpha'(0), \alpha'(0) \rangle_p =$$
$$\langle X_u u' + X_v v', X_u u' + X_v v' \rangle_p$$
$$= \langle X_u, X_u \rangle_p u'^2 + 2\langle X_u, X_v \rangle_p u'v' + \langle X_v, X_v \rangle_p v'^2$$
$$!= E u'^2 + 2F u'v' + G v'$$ where we have set
$$E(u_0, v_0) = \langle X_u, X_u \rangle_p$$
$$F(u_0, v_0) = \langle X_u, X_v \rangle_p$$
$$G(u_0, v_0) = \langle X_v, X_v \rangle_p \,,$$

the "coefficients of the first fundamental form" in the basis $\{X_u, X_v\}$ of T_pS

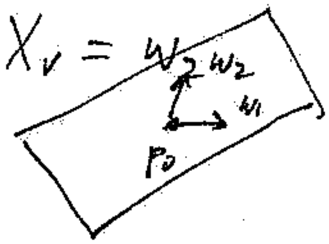
Thus as p varies over U we may think of $E(u,v)$, $F(u,v)$, $G(u,v)$ which are smooth if X smooth

<u>Example 1</u> (plane). Consider a plane P in $\mathbb{R}^3$ passing through $p_0 = (x_0, y_0, z_0)$ and containing the orthonormal (orthogonal of unit length) vectors

$w_1 = (a_1, a_2, a_3)$ $w_2 = (b_1, b_2, b_3)$

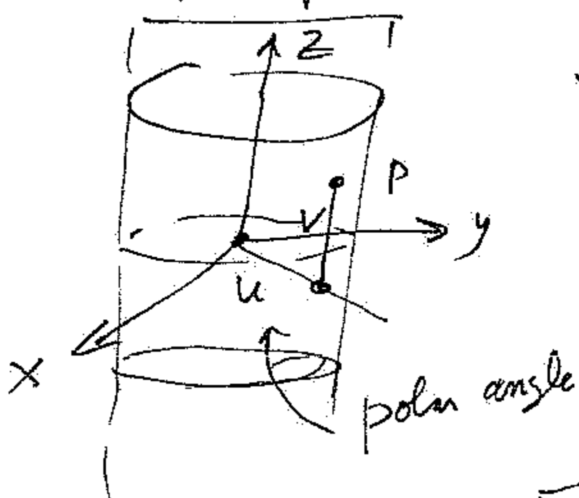
Then $X(u,v) = p_0 + u\,w_1 + v\,w_2$

$(u,v) \in \mathbb{R}^2$ and $X_u = w_1$, $X_v = w_2$



Hence $E=1$ $F=0$ $G=1$

<u>Example 2</u> (right cylinder)



$X(u,v) = (\cos u, \sin u, v)$

$U = \{(u,v) \in \mathbb{R}^2 \; 0 < u < 2\pi, \; -\infty < v < \infty\}$

$X_u = (-\sin u, \cos u, 0)$

$X_v = (0, 0, 1)$

$\Rightarrow E=1$ $F=0$ $G=1$

Example 3 (helix) Consider the helix $(\cos u, \sin u, a u)$, say $0 < u < 2\pi$, and draw a line parallel to the x, y, plane through each pt. We obtain a surface called the <u>helicoid</u> parametrized by

$$X(u,v) = (v \cos u, v \sin u, a u)$$

$$0 < u < 2\pi$$

$$-\infty < v < \infty$$

(we could extend u to $-\infty < u < \infty$)

where the strip of width 2π in the x, y plane is mapped by X onto the part of the surface corresponding to a rotation of 2π along the helix.

One easily computes

$$E(u,v) = v^2 + a^2, \quad F(u,v) = 0 \quad G(u,v) = 1$$

Thus the arc length of the parametrized curve $\alpha(t) = X(u(t), v(t))$ (say starting at $t = 0$) is

$$s(t) = \int_0^t \sqrt{I(\alpha'(t))}\, dt$$
$$= \int_0^t \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\, dt$$

and we can also measure the angle θ of intersection of curves

$\alpha : (a,b) \to S$ at $t = t_0$.
$\beta\ (a,b) \to S$

$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0) \rangle}{|\alpha'(t_0)|\ |\beta'(t_0)|}$$

In particular for the coordinate curves of $X(u,v)$ meeting at angle φ.

$$\cos\varphi = \frac{\langle X_u, X_v\rangle}{|X_u|\,|X_v|} = \frac{F}{\sqrt{EG}}$$

<u>Remark</u>:

Sometimes we write for I

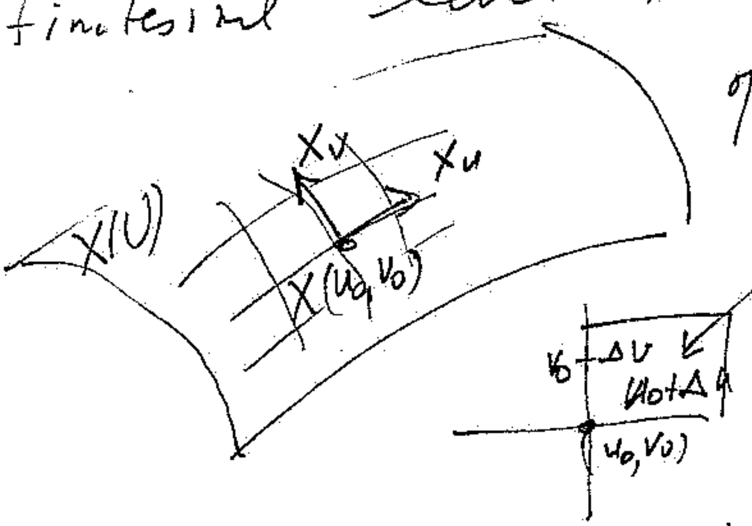
"$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$"

meaning that if $\alpha(t) = X(u(t), v(t))$

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

Review Example 4 p 98 do Carmo

We next give a (heuristic) derivation of how the first fund. form determines the area of a coordinate patch. On an infinitesimal level, the area of the image



X(R) of an "infinitesimal rectangle R (to first order)

of area $\Delta u \Delta v$ is the area

$$|X_u(u_0,v_0)\,\Delta u \times X_v(u_0,v_0)\,\Delta v| = |X_u \times X_v(u_0,v_0)|\,\Delta u\,\Delta v$$

of the parallelogram spanned by $X_u \Delta u$, $X_v \Delta v$

Taking Riemann sums,

we arrive at the

<u>Def'n</u> Let Q be a bounded region in a coordinate patch $X: U \to \mathbb{R}^3$ and let $R = X(Q)$. Then the area of S is given by

$$A(R) = \iint_{Q} |X_u \times X_v| \, du \, dv$$

<u>Remark</u> $A(S)$ does not depend on the parametrization. For if $\bar{X}: \bar{U} \to \mathbb{R}^3$ is another parametrization with $R \subset \bar{X}(\bar{U})$, set

$$\bar{Q} = \bar{X}^{-1}(R). \quad \text{Then}$$

$$\iint_{\bar{Q}} |\bar{X}_u \times \bar{X}_v| \, d\bar{u} \, d\bar{v} \quad = \text{(see exercise)}$$

$$\iint_{\bar{Q}} |X_u \times X_v| \left| \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} \right| d\bar{u} \, d\bar{v}$$

$$= \iint_{Q} |X_u \times X_v| \, du \, dv$$

from the <u>change</u> of <u>variables</u> <u>formula</u> <u>for</u> <u>double integrals</u> in Calculus 3

<u>Exercise</u> a. Show

$$|X_u \times X_v| = \sqrt{EG - F^2}$$

<u>Hint</u> $|X_u \times X_v| = |X_u|\,|X_v| \sin\varphi$

where φ is the angle between the coordinate lines.

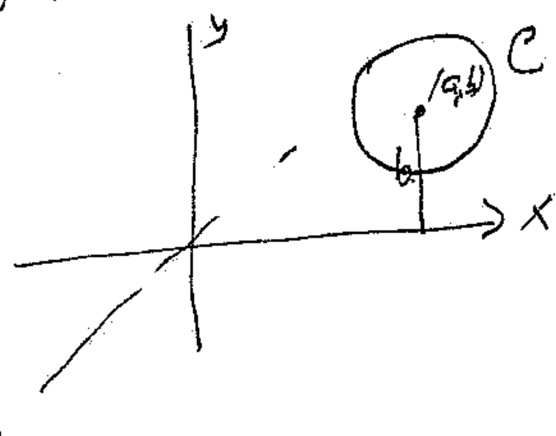
Thus we define

b. Show that in the Remark following the def'n of $A(S)$

$$|\bar{X}_{\bar{u}} \times \bar{X}_{\bar{v}}| = |X_u \times X_v| \left| \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} \right|$$

Example Recall from earlier example of a surface of revolution and take

$$C: \begin{aligned} x = g(u) &= a + r\cos u \\ y = h(u) &= b + r\sin u \end{aligned} \qquad b > r$$



to generate a torus T of revolution

Here $0 < u < 2\pi$
$0 < v < 2\pi$

Then

$$A(T) = r\int_0^{2\pi}\int_0^{2\pi}(b + r\cos u)\,du\,dv =$$

$4\pi^2 rb$ (this is really an "improper integral" since the parametrization omits a "meridian" and a "parallel")