1.2 Parametrized Curves

Def'n A parametrized curve is a mapping $\vec{\alpha}: I \to R^n$, where $I \subset \mathbb{R}$ is an interval

$$I \ni t \longmapsto \alpha(t) \in R^n$$

We say $\alpha \in C^k$ $k = 0, 1, 2, \ldots$ if α is k times differentiable.

If we introduce $x_1, \ldots, x_n$ as coordinates in R^n then we may write $\vec{\alpha}(t) = (x_1(t), \ldots, x_n(t))$ and the $x_1(t), \ldots, x_n(t)$ are the coordinate functions of the curve $\vec{\alpha}$

If $\alpha \in C^1$

The velocity vector of the curve

is $\alpha'(t) = (x_1'(t), \dots, x_n'(t))$

and its speed is $|\vec{\alpha}'(t)| = \sqrt{x_1'(t)^2 + \cdots + x_n'(t)^2}$

For the most part we will take

$n = 2, 3$.

Ex. 1 the line through $p \in \mathbb{R}^n$
parallel to the vector $\vec{v} \in \mathbb{R}^n$

$$\vec{\alpha}(t) = p + t\bar{v}$$

has velocity vector $\vec{v}$ and speed $|\bar{v}|$

(constant)

Ex 2. a circle of radius R in $\mathbb{R}^2$, oriented counterclockwise :

$$\alpha(t) = (R\cos t, R\sin t) \in \mathbb{R}^2$$

$$0 \leq t \leq 2\pi$$

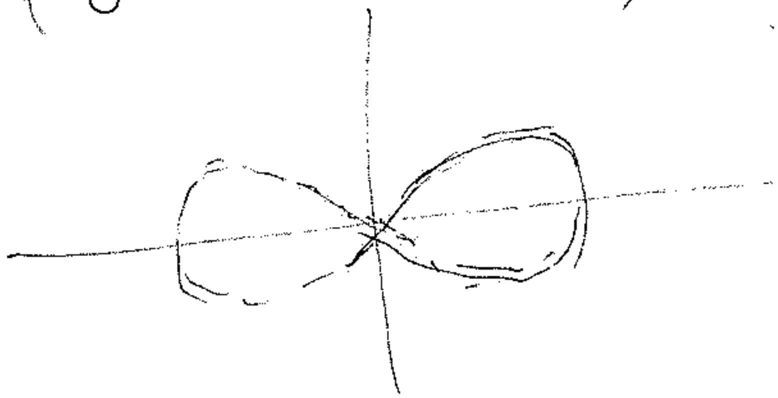
(note this curve is closed)

Ex 3 The "figure eight" curve

$$\alpha(t) = (\sin t, \tfrac{1}{2}\sin 2t) \in \mathbb{R}$$

$$0 \leq t \leq 2\pi$$

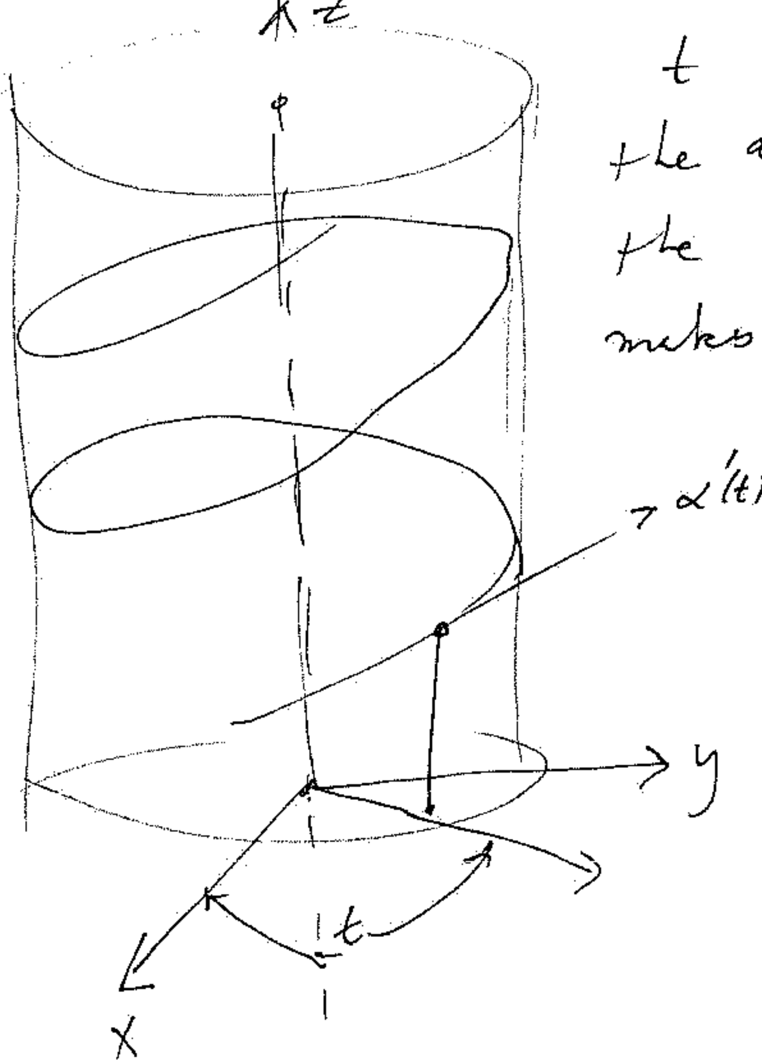
$$(y^2 = x^2(1-x^2))$$



Ex 4 The helix)

$$\alpha(t) = (a\cos t,\ a\sin t,\ bt)$$
$$t \in \mathbb{R}$$

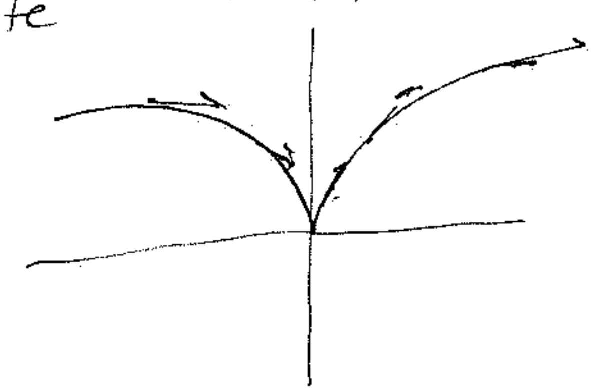
has its trace (image) in $\mathbb{R}^3$ a helix of "pitch $2\pi b$" on the cylinder $x^2+y^2=a^2$



t measures the angle the x axis makes with projection of $\vec{\alpha}(t)$ onto x-y plane

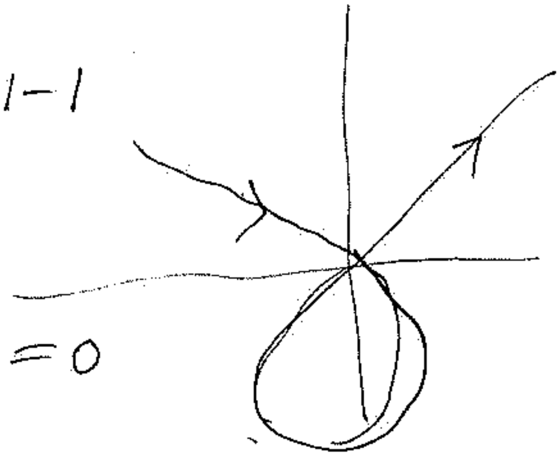
Ex. 5 $\alpha(t) = (t^3, t^2)$ $t \in \mathbb{R}$

Note $\alpha'(0) = 0$



Ex. 6 $\vec{\alpha}(t) = (t^3 - 4t, t^2 - 4)$
$t \in \mathbb{R}$

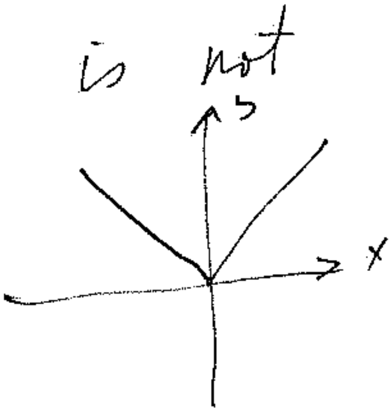
is not 1-1



$\alpha(2) = \alpha(-2) = 0$

Ex 7 $\vec{\alpha}(t) = (t, |t|)$ is not differentiable at $t = 0$

$y = |t|$



Ex 7 The two distinct parametrized curves

$$\vec{\alpha}(t) = (\cos t, \sin t)$$
$$\vec{\beta}(t) = (\cos 2t, \sin 2t)$$
$$t \in (0-\varepsilon, 2\pi+\varepsilon) \quad \varepsilon > 0$$

have the same trace (set of image pts) but β covers the trace twice.

inner product $\langle \alpha, \beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_i$

$$= |\alpha|\,|\beta| \cos\theta$$

$\Rightarrow |\langle \alpha, \beta \rangle| \le |\alpha|\,|\beta|$

Let $\vec{\alpha}(t) : I \to R^n$

$\vec{\beta} \ : I \to R^n$

Then $\frac{d}{dt}\left(\langle \alpha(t), \beta(t) \rangle\right) =$

$\langle \alpha'(t), \beta(t) \rangle + \langle \alpha(t), \beta'(t) \rangle$

and

$$\frac{d}{dt}\left(\| \alpha(t) \|\right) = \frac{d}{dt}\sqrt{\langle \alpha(t), \alpha(t) \rangle}$$

$$= \frac{\langle \alpha(t), \alpha'(t) \rangle}{\| \alpha(t) \|}$$

If $\vec{\alpha}$ has unit speed

i.e. $\| \alpha'(t) \| = 1$ then

$\langle \alpha'(t), \alpha''(t) \rangle = 0$

i.e its velocity and acceleration are orthogonal

7.1

Cauchy-Schwarz inequality

$$|\langle v,w\rangle| \le |v||w|$$

Pf We may assume $v \neq 0,\ w \neq 0$; then

$$0 \le |v+tw|^2 = \langle v+tw, v+tw\rangle$$

$$= |v|^2 + 2t\langle v,w\rangle + t^2|w|^2$$

choose $t = \pm\frac{|v|}{|w|}$ $\quad \mp\frac{2|v|}{|w|}\langle v,w\rangle \le 2|v|^2$

$$\Rightarrow \boxed{|\langle v,w\rangle| \le |v||w|}$$

Cor (triangle inequality)

$$|v+w| \le |v| + |w|$$

Pf: $|v+w|^2 = |v|^2 + 2\langle v,w\rangle + |w|^2$

$$\le |v|^2 + 2|v||w| + |w|^2$$

$$= (|v|+|w|)^2$$

7-2

$$v \perp w \Leftrightarrow \langle v, w \rangle = 0$$

$$\cos\theta = \frac{\langle v, w \rangle}{|v|\,|w|}$$

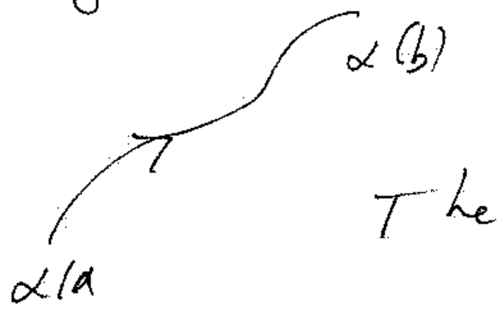
1.3 Arc Length

Def'n The length function

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|\,du$$

defines the arc length (starting at $\alpha(t_0)$) up to $\alpha(t)$

Remark Note that $\alpha(t)$ traverses the image with an order (direction) $a \le t \le b$



The "opposite order"

is ~~$\alpha(b+(t-a))$~~ $\alpha(a+b-t)$

8.1

Reparametrization by arc length

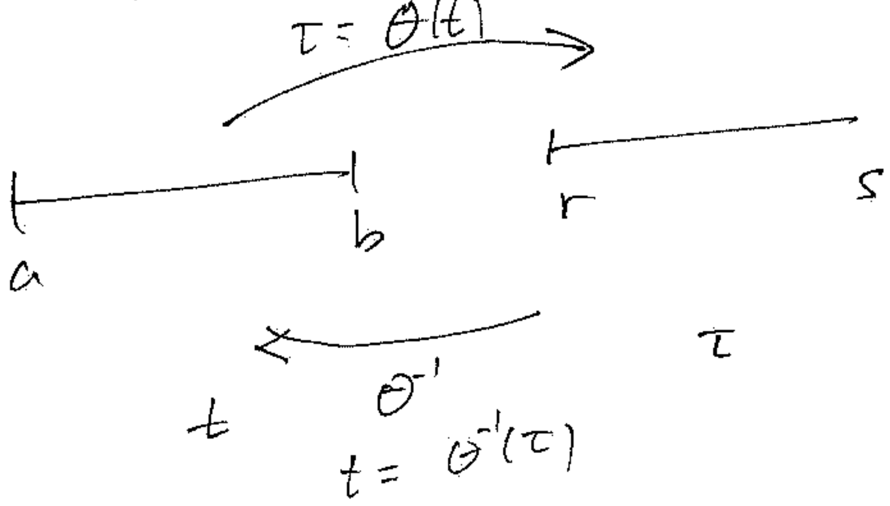
We say two curves

$\alpha(t)$ $\quad a \le t \le b$ $\quad$ and

$\beta(\tau)$ $\quad r \le \tau \le s$

are reparametrizations of each other

if $\beta(\tau) = \alpha(\theta(t))$ for a bijection $\theta : [a,b] \to [r,s]$



<u>Thm</u> $L_{[a,b]} = \int_a^b |\alpha'(t)| dt$

$= L_{[r,s]} = \int_r^s |\beta'(\tau)| d\tau$

length is invariant under repar.

$\frac{dt}{d\tau} = \frac{1}{\frac{d\tau}{dt}}$

$\beta'(\tau) = \alpha'(t) \frac{dt}{d\tau}$

<u>Def'n</u> α regular if $\alpha'(t) \neq 0$.

<u>Thm</u> Every <u>regular</u> curves possesses an arc length reparametr.

i.e $|\alpha'(s)| = 1 \quad 0 \leq s \leq L$

(unique up translation of parameter

<u>Pf</u> $\frac{ds}{dt} = |\alpha'(t)| > 0$ so

s is strictly increasing so

is 1-1 onto $[0, L]$ and

the inverse $t = t(s)$ is diff. and monotone.

Example The curves

$$\alpha:[0,1] \to R^2 \qquad \alpha(t) = (3, t)$$

$$\tilde{\alpha}\ [2,3] \to \mathbb{R} \qquad \tilde{\alpha}(t) = (3, t-2)$$

are both parametrized by arclength

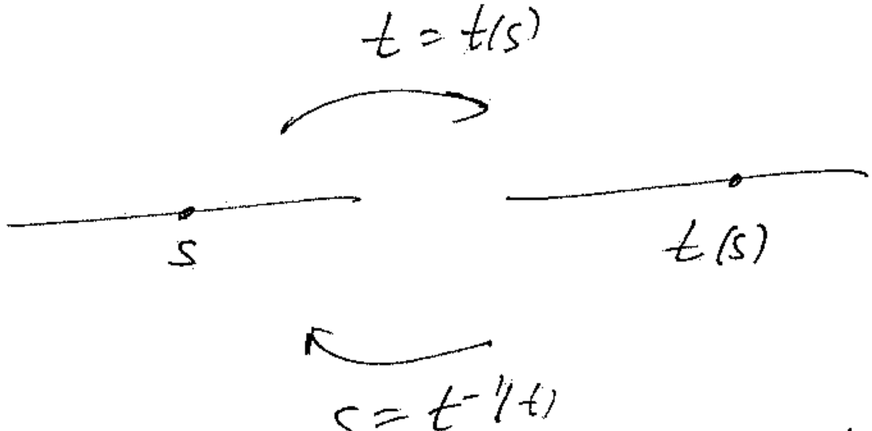
$$\tilde{\alpha}(\underset{\sigma(t)}{\underset{\|}{t+2}}) = (3, t) = \alpha(t)$$

Pf of uniqueness

Let $\alpha = \tilde{\alpha} \circ \sigma \Rightarrow$

$$\alpha'(t) = \tilde{\alpha}'(\sigma(t))\, \sigma'(t)$$

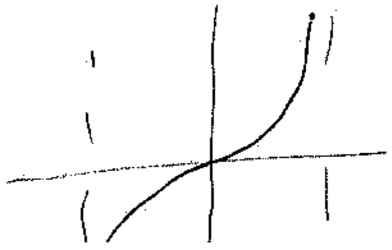
$$1 = |\alpha'(t) = |\tilde{\alpha}'(\sigma(t))| \underbrace{\sigma'(t)}_{+} \Rightarrow \begin{array}{l} \sigma'(t) = 1 \\ \sigma(t) = t + C \end{array}$$



$$\frac{ds}{dt}\,(t(s)) = \frac{1}{\frac{dt}{ds}(s)}$$

$$\frac{dt}{ds}(s) = \frac{1}{\frac{ds}{dt}(t(s))}$$

Ex $\quad t(s) = \tan s \qquad s(t) = \arctan t$



$$\frac{ds}{dt}(t(s)) = \frac{1}{1+t(s)^2} = \frac{1}{1+\tan^2 s}$$

$$= \frac{1}{\sec^2 s} = \text{[illegible]} = \frac{1}{\frac{dt}{ds}}$$

$\alpha : [a, b] \to \mathbb{R}^3$

$$s(t) = \int_a^t |\alpha'(\tau)| d\tau$$

$$\frac{ds}{dt} = |\alpha'(t)| > 0$$

s strictly $\nearrow$

$\Rightarrow$ inverse fcn $t = t(s)$ exists

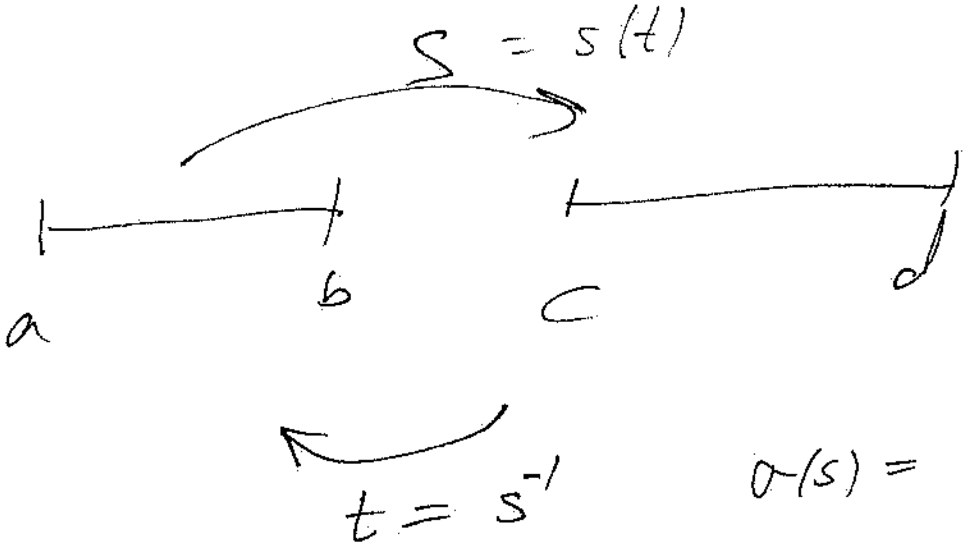
and $$\frac{dt}{ds}(s) = \frac{1}{\frac{ds}{dt}(t(s))}$$

$$= \frac{1}{\alpha'(t(s))}$$

Define $\tilde{\alpha}(s) = \alpha(t(s))$

$$|\tilde{\alpha}'(s)| = \left|\alpha'(t(s))\frac{dt}{ds}\right|$$

$$= |\alpha'(t(s))| \frac{1}{|\alpha'(t(s))|} = 1$$



$\sigma(s) = t(s)$

$\tilde{\alpha}(s) = \alpha(\sigma(s))$

8.6

Ex $\alpha(t) = (e^t\cos t, e^t\sin t, e^t)$

$-\infty < t < \infty$

$\alpha'(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t)$

$|\alpha'(t)| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1}$

$= \sqrt{3}\,e^t$

$s(t) = \int_0^t |\alpha'(\tau)|d\tau = \int_0^t \sqrt{3}\,e^t\,dt$

$= \sqrt{3}\,(e^t - 1)$

$0 < \tau = e^t < \infty$

$s(t) \in (-\sqrt{3}, \infty)$

$e^t = 1 + \frac{s}{\sqrt{3}}$

8-7

$$t = \log\left(\frac{s}{\sqrt{3}} + 1\right)$$

$$\tilde{\alpha}(s) = \alpha(t(s)) = \alpha\left(\log\left(\frac{s}{\sqrt{3}} + 1\right)\right)$$

$$= \left(\left(\frac{s}{\sqrt{3}} + 1\right) \cos \log\left(\frac{s}{\sqrt{3}} + 1\right), \left(\frac{s}{\sqrt{3}} + 1\right) \sin \log\left(\frac{s}{\sqrt{3}} + 1\right), \frac{s}{\sqrt{3}} + 1\right)$$

$$-\sqrt{3} < s < \infty$$

Let $g : [a,b] \to R^n$ be ~~continuous~~ C'

Then $\displaystyle L = \sup_{\|P\| \to 0} \sum_{j=0}^{N} |g(t_{j+1}) - g(t_j)|$ exists ⑨

and $\displaystyle L = \int_a^b |g'(t)|\,dt$.

(distance traveled obtained by integrating the speed)

$$\left| \int_{t_j}^{t_{j+1}} g'(t)\,dt - g'(t_j)\,(t_{j+1} - t_j) \right|$$

$$= \left| \int_{t_j}^{t_{j+1}} (g'(t) - g'(t_j))\,dt \right|$$

$$\leq \int_{t_j}^{t_{j+1}} | g'(t) - g'(t_j) |\,dt$$

$$g(t_{j+1}) - g(t_j) = \int_{t_j}^{t_{j+1}} g'(t)\,dt$$

so $$\left|\left(g(t_{j+1})-g(t_j)\right) - g'(t_j)\,(t_{j+1}-t_j)\right|$$

$$\leq \Big(\underbrace{\sup_{[t_j,t_{j+1}]} \left|g'(t)-g(t_j)\right|}_{\substack{\parallel\\ \delta}}\Big)\,(t_{j+1}-t_j)$$

By unif. continuity of g' we can make δ as small as desired ($\delta < \varepsilon$) by taking $\|P\|$ small enough

With $u = g(t_{j+1}) - g(t_j)$
$v = g'(t_j)(t_{j+1}-t_j))$

since $\left||u|-|v|\right| \leq |u-v| \Longrightarrow$

$$\left|\,|g'(t_j)|\,(t_{j+1}-t_j) - |g(t_{j+1})-g(t_j)|\,\right| \leq \varepsilon\,(t_{j+1}-t_j)$$

adding: $$\sum_{j=0}^{N}\left|\,|g'(t_j)|\,(t_{j+1}-t_j) - |g(t_{j+1})-g(t_j)|\,\right|$$
$$\leq \varepsilon \sum_{j=0}^{N}(t_{j+1}-t_j) = \varepsilon(b-a)$$

Example graph of $y = f(x)$

$a \leq x \leq b$

use x as a parameter

$$\vec{\alpha}(x) = (x, f(x))$$

$$\vec{\alpha}'(x) = (1, f'(x))$$

$$|\vec{\alpha}'(x)| = \sqrt{1 + f'^2}$$

$$L_{[a,b]} = \int_a^b \sqrt{1 + f'^2(x)}\, dx$$

1.4 The vector product

Def'n Vectors $u, v, w \in \mathbb{R}^3$ form a basis if every $v \in \mathbb{R}^3$ can be (uniquely) written as a linear combination

$$v = au + bv + cw \qquad a, b, c \in \mathbb{R}$$

"standard basis"

$$e_1 = (1,0,0)$$
$$e_2 = (0,1,0)$$
$$e_3 = (0,0,1)$$

$$v = (v_1, v_2, v_3) = v_1 e_1 + v_2 e_2 + v_3 e_3$$

The vectors u, v, w form a basis of $\mathbb{R}^3$ iff

(write

$$u = (u_1, u_2, u_3)$$
$$v = (v_1, v_2, v_3)$$
$$w = (w_1, w_2, w_3)$$

$$\det(u, v, w) \ :=$$

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \neq 0$$

It is not difficult to see that the vectors $u, v, w \in \mathbb{R}^3$ form a iff they are not contained in the same plane

This follows from the corresponding statement for $\mathbb{R}^2$: the vectors $u, v \in \mathbb{R}^2$ form a basis iff they are not parallel

(14)

The vector product

Fix $u, v \in \mathbb{R}^3$. Then

$w \longmapsto \det(u, v, w)$ is a linear function on $\mathbb{R}^3$ by properties of the determinant (the unique alternating multilinear form)

This implies $\exists!$ vector in $\mathbb{R}^3$, denoted $u \times v$ s.t.

$$\det(u, v, w) = (u \times v) \cdot w$$

called the vector product (cross product) of u and v

Explicitly :

$$u = (u_1, u_2, u_3)$$
$$v = (v_1, v_2, v_3)$$
$$w = (w_1, w_2, w_3)$$

$$(u \times v) \cdot w = \det(u, v, w)$$

$$= \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

cofactors

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} w_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} w_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} w_3$$

Putting $w = e_1, e_2, e_3$
(respectively)

gives that

$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3$$

Properties

1. $u \times v = -v \times u$

2. $u \times v$ linear in u, linear in v

3. $u \times v = 0$ iff u, v linear dependend i.e parallel

4. In general $u \times (v \times w) \neq (u \times v) \times w$ (not associative)

5. $u \times (v \times w) = (u \cdot w)\, v - (u \cdot v)\, w$

(Note that by definition,

$$u \times v \cdot u = u \times v \cdot v = 0$$

i.e $u \times v$ is orthogonal to the plane spanned by u and v

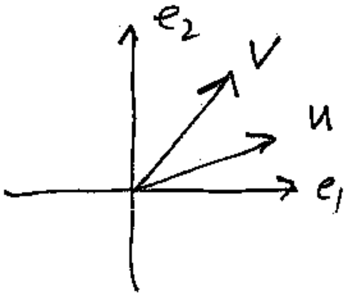
<u>Exercise</u> Verify property 5

6. $$\left(u(t) \times v(t)\right)' = u'(t) \times v(t) + u(t) \times v'(t)$$

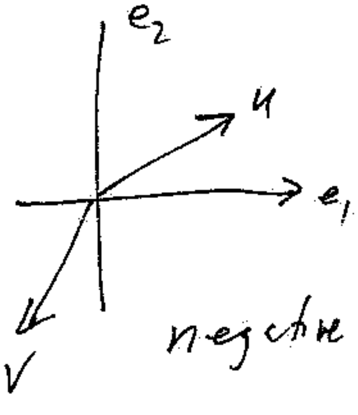
<u>Def'n</u> (orientation of a basis)

We say an ordered basis u, v, w is positively (respectively negatively) oriented if $\det(u, v, w) > 0$ (respectively < 0)

Example ($\mathbb{R}^2$)



positive



negative

1-5 Local theory of space curves ⑰

Curvature, torsion, Frenet formula

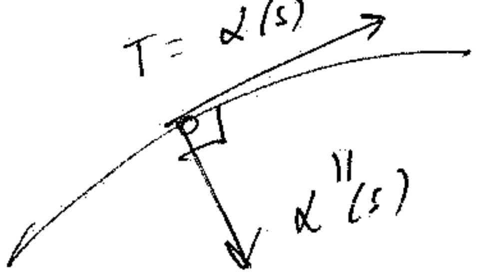
Let $\alpha : I \to R^3$ be a curve parametrized by arclength (unit speed)

Thus $T = \alpha'(s) \quad |T| = 1$

so $0 = \frac{d}{dt}(\alpha', \alpha')$

$= 2\, \alpha'(s) \cdot \alpha''(s)$

=



So $\alpha''(s)$ is $\perp$ to the velocity vector T and $|\alpha''(s)|$ measures how rapidly the curve pulls away from the tangent line at $\vec{\alpha}(s)$

Defn Let $\alpha : I \to R^3$ be parametrized by arclength s

Then $K(s) := |\alpha''(s)|$ is called the curvature of α at s. (or $\alpha(s)$)

If $\alpha(s)$ is a straight line

i.e $\alpha(s) = a + s\vec{N}$

then $|\alpha''(s)| = 0$

and conversely if $|\alpha''(s)| = 0$

then
$$\alpha''(s) = 0$$
$$\alpha'(s) = \vec{N}$$
$$\alpha(s) = a + s\vec{v}$$

Defn We say α is non-degenerate (or Frenet curve)

if $\alpha''(s) \neq 0$ $\forall s$

i.e $K(s) > 0$ $\forall s$.

Remark Let $\alpha = \alpha(t) : I \to \mathbb{R}^3$ be any regular space curve of class C^3 (not necessarily parametrized by arclength). Then α is a Frenet curve iff $\alpha'(t)$ and $\alpha''(t)$ are linearly indep't $\forall t$

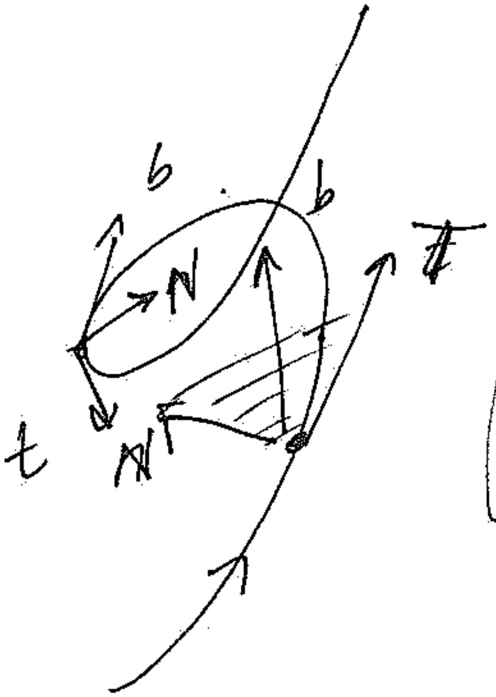
Def'n For a Frenet curve define

$T(s)$; $t(s) = \alpha'(s)$ unit tangent vector at s

$N(s)$ $n(s) := \dfrac{\alpha''(s)}{|\alpha''(s)|} =$ unit normal vector at s

Def'n

The plane determined by $\alpha'(s)$, $N(s)$ is called the osculating plane at s



$$\boxed{T' = k(s)\, N(s)}$$

The unit vector

$$\boxed{B(s) = T(s) \times N(s)}$$

is normal to the osculating plane is called the binormal

$|b'(s)|$ measures how rapidly the curve pulls away from the osculating plane

The vectors $\{T(s), N(s), B(s)\}$ form an o.n basis of R^3

$= \{T(s), N(s), T \times N(s)\}$

has positive orientation

called the Frenet frame at s

<u>Theorem</u> (Frenet formulas)

Let α be a Frenet curve.
Then $\exists$ function $\tau(s)$ s.t.

i) $\mathbf{T}'(s) = \kappa(s)\,\mathbf{N}(s)$

ii) $\mathbf{N}'(s) = -\kappa(s)\,\mathbf{T}(s) + \tau(s)\,\mathbf{B}(s)$

iii) $\mathbf{B}'(s) = -\tau(s)\,\mathbf{N}(s)$

Pf i) done already

$\mathbf{B}'(s) \perp \mathbf{B}(s)$

Since $\mathbf{B}(s)\cdot\mathbf{B}(s) = 1 \Rightarrow \boxed{\mathbf{B}'(s)\cdot\mathbf{B}(s) = 0}$

so

But $\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$

$$\mathbf{B}'(s) = \mathbf{T}'(s) \times \mathbf{N}(s) + \mathbf{T}(s) \times \mathbf{N}'(s)$$

$$= \kappa(s)\,\underset{\downarrow}{\mathbf{N}(s) \times \mathbf{N}(s)} + \mathbf{T}(s) \times \mathbf{N}'(s)$$

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Hence $B'(s)$ is also $\perp$ to $T(s)$

So $B'(s) \parallel N(s)$

Therefore we may write

$$B'(s) = -\tau(s)\, N(s)$$ proving iii)

for some smooth function $\tau(s)$.

(the $-$ sign is convention)

Since $B(s) = T(s) \times N(s) \Rightarrow$

$N(s) = B(s) \times T(s)$,

$$\Rightarrow \quad N'(s) = B'(s) \times T(s) + B(s) \times T'(s)$$

$$= -\tau(s)\, N(s) \times T(s) + B(s) \times \kappa N(s)$$

$$n'(s) \quad = -\kappa\, T(s) + \tau(s)\, B(s)$$

proving ii)

The function $\tau(s)$ is called the torsion

Thm (Fundamental theorem of local theory of curves)

Given C' differentiable functions

$k(s) > 0$, $\tau(s)$ $s \in I$

$\exists$ regular par. curve $\alpha : I \to R^3$

s. that s = arc length

$k(s)$ = curvature

$\tau(s)$ = torsion

Moreover any other sol'n $\tilde{\alpha}$ differs from α by a rigid motion (i.e $\exists$ orthogonal linear transf. $A : R^3 \to R^3$ $\det A = 1$ s. that $\tilde{\alpha} = A \circ \alpha + \vec{c}$

$$(*)\quad \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}' = \begin{bmatrix} 0 & K(s) & 0 \\ -K(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

Idea fr uniqueness after a regid motion $\tilde\alpha(s_0) = \alpha(s_0)$

and $(\tilde T_0, \tilde N_0, \tilde B_0) = (T_0, N_0, B_0)$

By Frenet eqn's

$$\frac{1}{2}\frac{d}{ds}\left\{ |T-\tilde T|^2 + (N-\tilde N)^2 + (B-\tilde B)^2 \right\}$$

$$= \langle T-\tilde T, T'-\tilde T'\rangle + \langle B-\tilde B, B'-\tilde B'\rangle + \langle N-\tilde N, N'-\tilde N'\rangle$$

$$= k\,\cancel{\langle T-\tilde T, N-\hat N\rangle} + \tau\,\cancel{\langle B-\tilde B, N-\tilde N\rangle}$$

$$-k\,\cancel{(N-\hat N, T-\tilde T\rangle} - \tau\,\cancel{(N-\tilde N, B-\tilde B\rangle}$$

$$= 0$$

Hence since at $s = s_0$ the expression in $\{\}$ is zero, it is identically 0.

Existence (Sketch)

$$\vec{X}'(t) = \vec{F}(X(t), t) = (f_1(\vec{X},t), f_2(\vec{X},t), f_3())$$

Thm (ode local existence) If F is Lipschitz (C') in all argument
$\exists$ soln defined in a nbhd of
$(x_1^0, x_2^0, x_3^0, t_0)$

linear case

$$X'(s) = A(s)\,X(s)$$
$$X(s_0) = \bar{X}_0$$

Remark

<u>Example</u> $y' = y^2$

singular when $y(x_0) = 0$

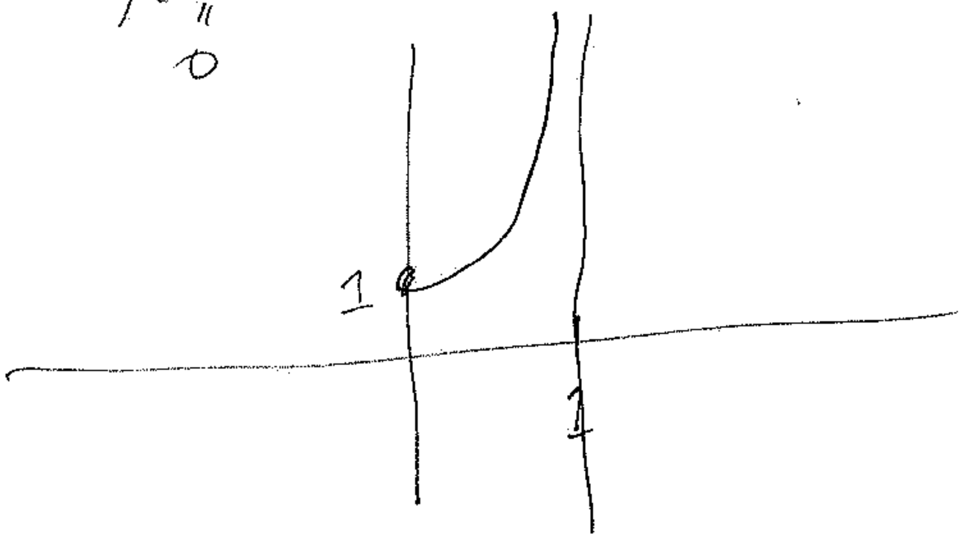
$$-\left(\frac{1}{y}\right)' = 1$$

$x_0 = 0$

$$\frac{1}{y(x_0)} - \frac{1}{y(x)} = x - x_0$$

$$-\frac{1}{y(x)} = x - x_0 - \frac{1}{y(x_0)}$$

$$y(\underset{\overset{\shortparallel}{0}}{x_0}) = 1 \qquad y(x) = -\frac{1}{x-1}$$



Thm (ode existence, linear case)

Given a continuous matrix

fcn $A : (a,b) \rightarrow M^{n\times n}$

$s_0 \in (a,b) \quad x_0 \in R^n$

$\exists !$ vector fcn $\vec{X} = \vec{X}(s) : (a,b) \rightarrow R^n$

s.that $X'(s) = A(s)\, X(s)$

$\vec{X}(s_0) = \vec{X}_0$

$X_1(s) = T(s) = \alpha'(s)$

Define $\alpha(s) = \alpha(s_0) + \int_{s_0}^{s} T(\tau)\,d\tau$

Claim $\{T(s), N(s), B(s)\}$ is a pos.tively oriented o.n basis of $\mathbb{R}^3$

$$X = \begin{bmatrix} T(s) \\ N \\ B \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} = -A^T$$

$$(X^T X)' = (X^T)' X + X^T X' =$$

$$(X')^T X + X^T X'$$

$$= (AX)^T X + X^T (AX)$$

$$= X^T A^T X + X^T A X = 0$$

So $X^T X = C$

$X^T X(S_0) = I$ so $C = I$

$\det(T, N, B) = \pm 1$

but $\det(T, N, B)(S_0) = 1$

and det is a continuous

fcn so $\det(T, N, B)(s) \equiv 1$

Claim $|\alpha'(s)| = 1$

Since $\alpha' = T$

<u>Claim</u> α is Frenet i.e.

$$\alpha'' \neq 0 \qquad T = X_1$$

$$\alpha''(s) = T' = \kappa N \qquad \kappa > 0$$

$$|\alpha''(s)| = \kappa(s) > 0$$

$$T'(s) = X_1'(s) = \kappa(s)\, X_2(s) \qquad \text{so } X_2 = N$$

Remains to show τ is the torsion. Since $\{X_1, X_2, X_3)$ pos. oriented on frame

$$X_3 = X_1 \times X_2 = T \times N = B$$

$$B'(s) = X_3' = -\tau(s)\, X_2(s)$$

$$= -\tau(s)\, N(s) \quad //$$

Thm Every Frenet curve α in R^3 with constant curv. $\kappa > 0$ and constant torsion $\tau \neq 0$ is a helix

Pf For $a = \frac{\kappa}{\kappa^2+\tau^2}$ $b = \frac{\tau}{\kappa^2+\tau^2}$

the helix $\alpha(t) = (a\cos t, a\sin t, bt)$

has curv. κ & torsion τ

By the uniqueness part of the fund. thm. $\exists$ rigid motion so that $R \circ \alpha$ = helix.

Thm a. a curve $\alpha: I \to R^3$ par. by arclength has image contained in a line iff $K \equiv 0$

b: A Frenet curve has image contained in a plane iff $\tau \equiv 0$

Pf of b: $\Rightarrow$ $\alpha \subset$ plane

$\Rightarrow$ $T = \alpha'$, $N = \frac{\alpha''}{K}$ contained in the plane so $B = T \times N$ is the unit vector $\perp$ plane i.e. B constant

$\Rightarrow$ $\tau = |B'| = 0$

$\Longleftarrow$ If $\tau \equiv 0$ then $B' \equiv 0$.

so $B \equiv$ constant

Fix $s_0 \in I$ and consider

$$f(s) = (\alpha(s) - \alpha(s_0)) \cdot B$$

$$f'(s) = \alpha'(s) \cdot B = T \cdot B = 0$$

so $f(s) \equiv f(s_0) = 0$ //

<u>How</u> <u>to</u> <u>compute</u> <u>curvature</u> <u>and</u> <u>torsion</u> <u>in</u> <u>arc</u> <u>length</u> <u>parametrization</u>

<u>Theorem</u> If $\alpha : I \to \mathbb{R}^3$ is a Frenet curve parametrized by arc length, then

$$\kappa(s) = |\alpha''(s)|, \quad \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{|\alpha''(s)|^2}$$

<u>Pf</u> We need to compute $\tau(s)$. From the third Frenet formula, we have since $B' = -\tau N$,

$$\tau = -(T \times N)' \cdot N = -(T' \times N + T \times N') \cdot N =$$

$$-(\kappa \underbrace{N \times N}_{0} + T \times N') \cdot N = (N' \times T) \cdot N =$$

$$\det(N', T, N) = \det(T, N, N').$$ Since

$$T = \alpha', \ N = \frac{\alpha''}{\kappa}, \ N' = \frac{\kappa \alpha''' - \alpha'' \kappa'}{\kappa^2} = \frac{\alpha'''}{\kappa} - \frac{\kappa'}{\kappa^2}\alpha'',$$

$$\tau = \det(T, N, N') = \det\left(\alpha', \frac{\alpha''}{\kappa}, \frac{\alpha'''}{\kappa} - \frac{\kappa'}{\kappa^2}\alpha''\right)$$

$$= \det\left(\alpha', \frac{\alpha''}{\kappa}, \frac{\alpha'''}{\kappa}\right) - \underbrace{\det\left(\alpha', \frac{\alpha''}{\kappa}, -\frac{\kappa'}{\kappa^2}\alpha''\right)}_{0}$$

$$= \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}. \ //$$

In most situations, the curve is given in a parametrisation *which is not* arc length

$$\alpha = \alpha(t) : I \to \mathbb{R}^3$$

Then the arc length reparametization

$$\tilde{\alpha}(s) = \alpha(t(s))$$

of α, is a Frenet curve iff $\alpha'(t), \alpha''(t)$ are linearly indep't (exercise). Then define (since $\alpha(t) = \tilde{\alpha}(s(t))$)

$$\kappa(t) = \tilde{\kappa}(s(t)), \quad \tau(t) = \tilde{\tau}(s(t)$$

and

$$T(t) = \tilde{T}(s(t)), \quad N(t) = \tilde{N}(s(t)$$

$$B(t) = \tilde{B}(s(t)).$$

Theorem If $\alpha : I \to \mathbb{R}^3$ is Frenet, then

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \quad B(t) = \frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t) \times \alpha''(t)|}, \quad N(t) = B(t) \times T$$

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}, \quad \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{|\alpha'(t) \times \alpha''(t)|^2}$$

Pf Clearly $T(t) = \alpha'(t)/|\alpha'(t)|$

By the Chain Rule,

$$\alpha'(t) = \tilde{\alpha}'(s(t))\, s'(t) = \tilde{T}(s(t))\, s'(t) = T(t)\, s'(t)$$

$$\alpha''(t) = \tilde{\alpha}''(s(t))\, s'^2(t) + \tilde{\alpha}'(s(t))\, s''(t)$$

$$= \tilde{N}(s(t))\tilde{\kappa}(s(t))\, s'^2(t) + \tilde{T}(s(t))\, s''(t)$$

$$(*) \quad = N(t)\,\kappa(t)\, s'^2(t) + T(t)\, s''(t)$$

Then $\alpha' \times \alpha'' = T(t)\, s'(t) \times \left(N(t)\kappa(t) s'^2(t) + T(t)\, s''(t) \right)$

$$= \kappa\, s'^3\, T \times N\,(t) = \kappa(t)\, s'^3(t)\, B(t)$$

So $(**)$ $B = \dfrac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$, $|\alpha' \times \alpha''| = \kappa\, s'^3$

and $\kappa(t) = \dfrac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}$

It remains to compute $\tau(t)$. By $(*)$

$$\alpha''' = \tilde{N}' s' \kappa s'^2 + N(t)\left(\kappa(t) s'^2(t)\right)'$$

$$+ \left(\underset{''\ \tilde{\kappa}(s(t))\,\tilde{N}(s(t))}{\tilde{T}'(s(t))}\, s'(t)\right) s''(t) + T(t)\, s'''(t)$$

$$= \tilde{N}' \kappa (s')^3 + a(t)\, N(t) + b(t)\, T(t)$$

We use $\det(\vec{u}, \vec{v}+q\vec{u}, w+a_2\vec{u}+b\vec{v})$
$= \det(\vec{u}, \vec{v}, \vec{w})$ to obtain

$$\det(\alpha', \alpha'', \alpha''') =$$

$$\det(Ts', N\kappa s'^2 + \cancel{Ts''}, \tilde{N}'\kappa s'^3 + \cancel{aN} + \cancel{bT}) =$$

$$\det(Ts', N\kappa s'^2, \tilde{N}'\kappa s'^3) =$$

$$\kappa^2 s'^6 \det(T, N, \tilde{N}') = \text{(by Frenet for } \tilde{N}'(s(t)),$$

$$\kappa^2 s'^6 \det(T, N, \underset{-\kappa T}{\cancel{-\tilde{\kappa}(s(t))\tilde{T}(s(t))}} + \underset{\tau}{\overset{}{\tilde{\tau}(s(t))}}\,\underset{B}{\tilde{B}(s(t))}))$$

$$= \tau\kappa^2 s'^6 \underbrace{\det(T, N, B)}_{1}$$

Hence $$\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2 s'^6} = \frac{\det(\alpha', \alpha'', \alpha''')}{|\alpha' \times \alpha''|^2}$$

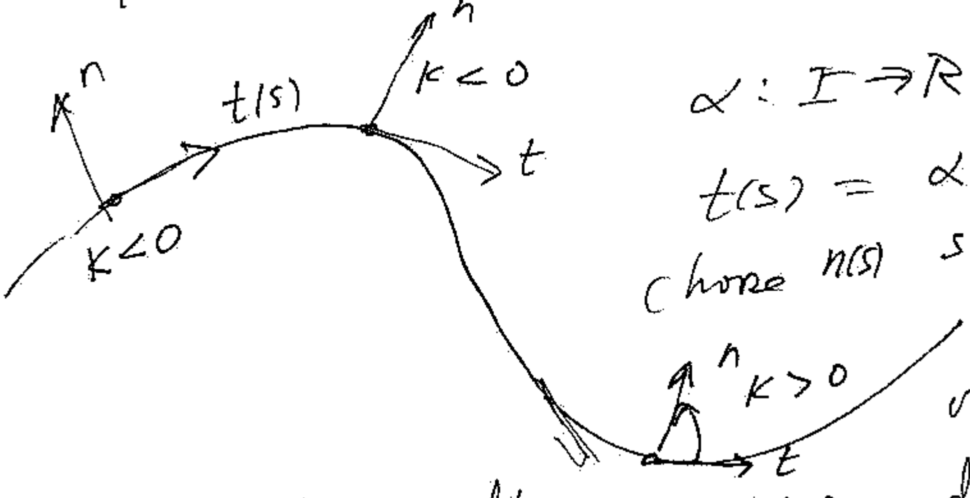
by $(**)$

Exercise Show (being careful) that the Frenet equation for $\alpha(t)$ become

$$\begin{cases} T'_{(t)} = |\alpha'(t)| \; \kappa(t) \, N(t) \\ N'(t) = |\alpha'(t)| \left(-\kappa(t) \, T(t) + \tau(t) \, B(t) \right) \\ B'(t) = |\alpha'(t)| \left(-\tau(t) \, N(t) \right) \end{cases}$$

Curves in the plane

Curvature is a measure of how fast a curve is turning. When that curve lies in a plane we may assign the curvature a sign which indicates whether it is rotating clockwise (+) or counter clockwise (-1)



$\alpha : I \to R^2$ regular

$t(s) = \alpha'(s)$

Choose $n(s)$ so that (t, n) pos. oriented.

Then $\frac{dt}{ds} = K(s)\, n(s)$ defines signed curvature $K(s)$

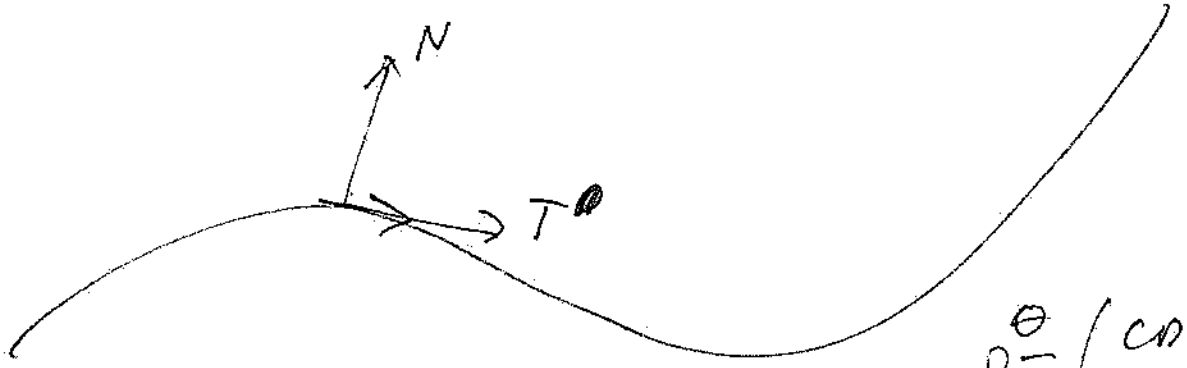
JT = counter clockwise rotation of $t(s)$

$K = \langle t'(s), Jt \rangle$

curves in the plane

$$T'(s) = \kappa N$$

$$N = JT \qquad (T, N) \text{ pos. oriented.}$$



$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad R^\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$Je_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_2$$

$$N' = (JT)' = J\,T' = J(\kappa N)$$

$$= \kappa JN = -\kappa T$$

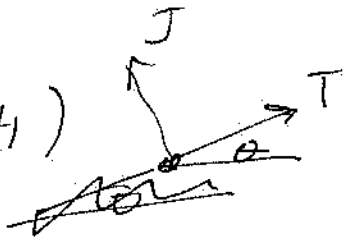
$$\begin{cases} T'(s) = \kappa(s)\, N(s) \\ N'(s) = -\kappa(s)\, T(s) \end{cases}$$

Note also that $\tau \equiv 0$ for a plane curve

Another simple but important way to define signed curvature is in terms of the turning angle θ defined as follows.

<u>Claim</u>: For any planar curve $\alpha : I \to \mathbb{R}^2$ $\exists$ $\theta : I \to \mathbb{R}^2$ s.that

$$T(t) = (\cos\theta(t), \sin\theta(t))$$



$JT =$

Then assuming $t = s$

$$K(s) = \theta'(s) .$$

$$T'(s) = (-\sin\theta \cdot \theta', \cos\theta\, \theta') = (-\sin\theta, \cos\theta)\,\theta' = \theta' JT(s)$$

Note that $t(s) : I \to S^1$
I cpct.

<u>Proposition</u> For any continuous function $T : I \to S^1$ $I = [a, b]$ cpct

$\exists$ a continuous $\theta : I \to S^1$

s. that $T = (\cos\theta, \sin\theta)$.

<u>Pf</u> T is unif. continuous, i.e given $\varepsilon > 0$ $\exists\, \delta > 0$ s that if $|t_1 - t_2| < \delta$
$|T(t_1) - T(t_2)| < \varepsilon$ and choose ε_0.

Set $0 < \delta_0 < 1$

Choose partition $a =: x_0 \leq x_1 \leq \dots \leq x_n =: b$
$|x_i - x_{i-1}| < \varepsilon_0$ $i = 1, \dots, n$

Then $T|_{[x_{i-1}, x_i]}$ is not onto.

and we may define $\theta_i : [x_{i-1}, x_i] \to \mathbb{R}$

by $\theta_i = \measuredangle \in [0, 2\pi)$ measured

counterclockwise between $T(x_{i-1})$ and $T(x_i)$

Define $$\Theta(x) := \theta_0 + \sum_{i=1}^{k-1} \theta_i(x_i) + \theta_k(x)$$

if $x \in [x_{k-1}, x_k]$

$K = \langle T', JT \rangle$ The circle

Ex $\alpha(t) = (R\cos t, R\sin t)$ $R > 0$

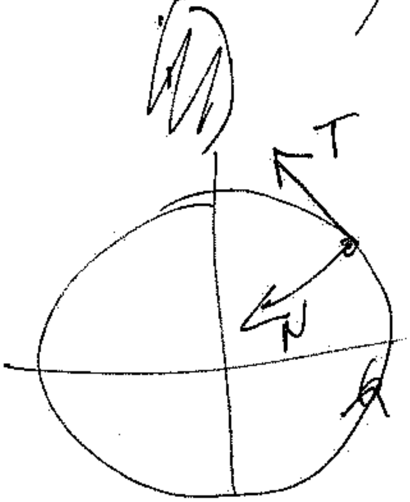
$0 \leq t \leq 2\pi$

i) has oriented curvature $\frac{1}{R}$

and the circle $\alpha(t) = (R\sin t, R\cos t)$

ii) has oriented curvature $-\frac{1}{R}$

i) $\alpha' = R(-\sin t, \cos t)$

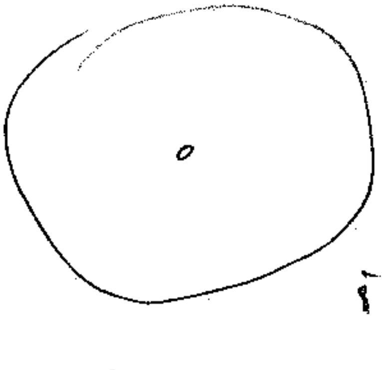


$T = (-\sin t, \cos t)$

$T' = (-\cos t, -\sin t)$

Theorem A regular curve
$\alpha: I \to R^2$ has
constant curvature κ iff
$\alpha \subset$ circle of radius $\frac{1}{|\kappa|}$ $\kappa \neq 0$
line if $\kappa = 0$

Pf Assume $\kappa > 0$ and
assume α par. by s
α contained in circle
$\oplus$ of radius $\frac{1}{|\kappa(s_0)|}$



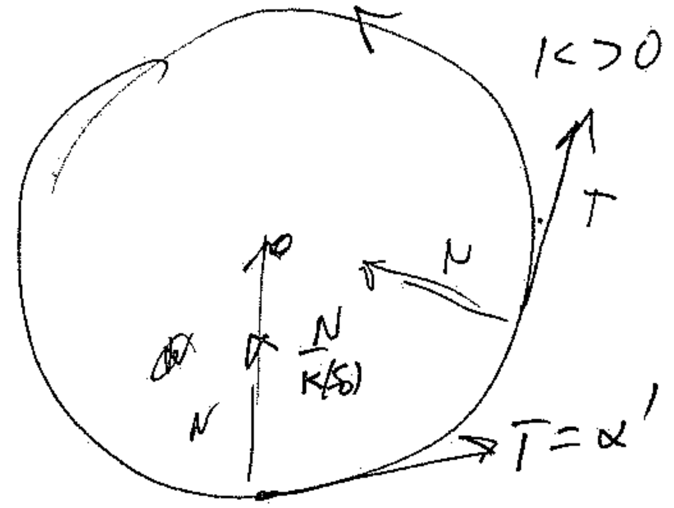
iff

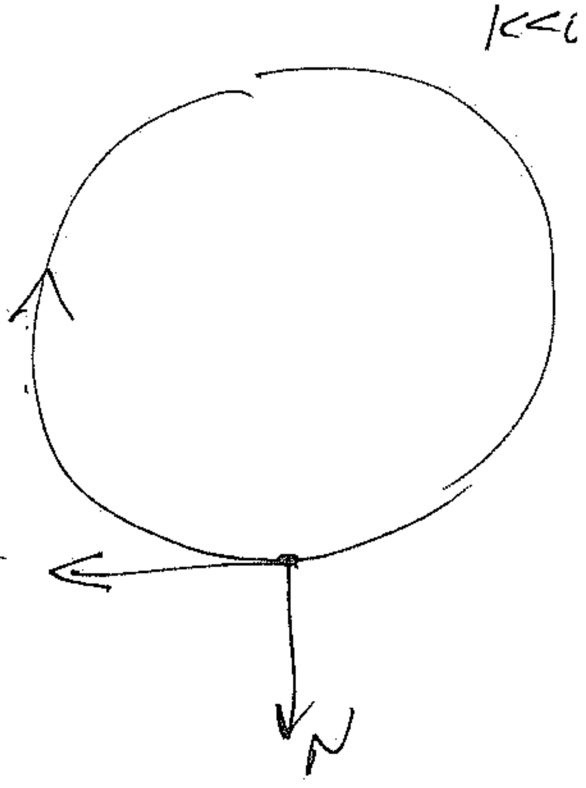
$$(*) \quad m(s) := \alpha(s) + \frac{1}{\kappa(s)} N(s) = \text{constant} \; m_0$$

Fn if $m(s) = m_0$ constant

the distance of $\alpha(s)$ to m_0

is $|\alpha(s) - m_0) = \left| \frac{N(s)}{\kappa(s_0)} \right| = \frac{1}{|\kappa(s_0)|}$





differentiate (A)

$$\left(\alpha(s) + \frac{1}{\kappa(s_0)} N(s)\right)' = 0$$

$N' = -\kappa T$

$$T(s) + \frac{1}{\kappa(s_0)}\left(-\kappa(s)\,T(s)\right) = 0$$

so $\kappa(s) = \kappa(s_0)$

Defn The circle with center $\alpha(s_0) + \frac{1}{\kappa(s_0)} N(s_0)$ & radius $\frac{1}{|\kappa(s_0)|}$ is called the osculating circle

(order 2 contact with curve α at $\alpha(s_0)$)

Def'n The total signed curvature of closed $\alpha : \underset{[0,L]}{\underset{\shortparallel}{I}} \to R^2$

is defined as

$\alpha(a) = \alpha(b)$
$T(a) = T(b)$

$$\text{total } \bar{\kappa}(\alpha) = \int_I \bar{\kappa}(s)\, ds$$

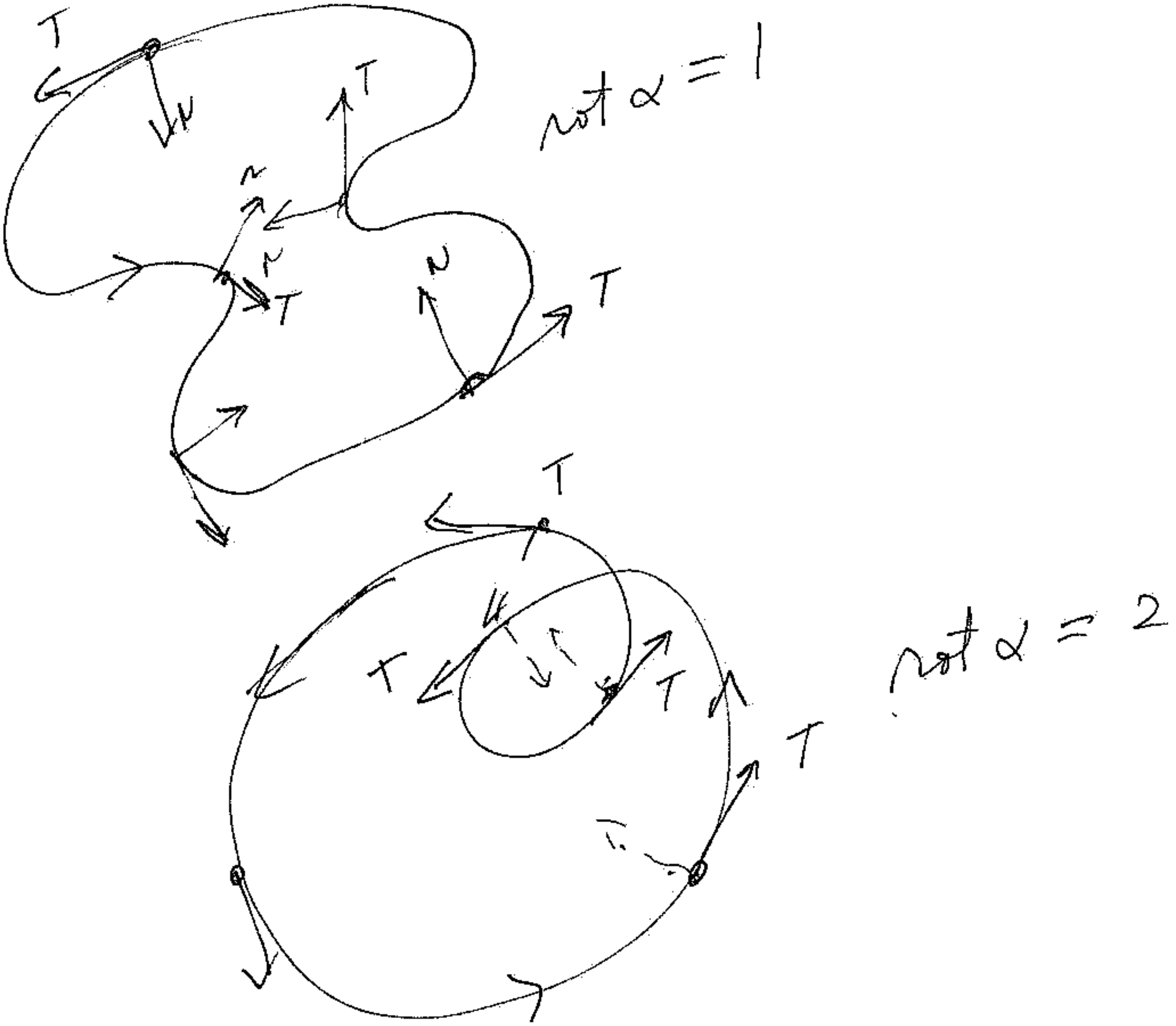
Note $\bar{\kappa}(s) = \theta'(s)$

$$= \theta(L) - \theta(0)$$

$$= 2\pi \cdot \boxed{\text{integer}}$$

rotation index rot (α)

Thm (Hopf) Any simple closed curve has rot. $(\alpha) = \pm 1$


T
N
T
N
N
T
N
T
not α = 1
T
N
T
T
T
not α = 2
T

Theo. Let C be a simple closed curve in R^2 bounding a domain D. Let $L =$ length(C), $A =$ area(D). Then

$$L^2 \geq 4\pi A$$

with equality only for a circle.

2.1. **Hurwitz's proof of the the isoperimetric inequality.** A fairly simple modern analytic proof of the isoperimetric inequality was given by A. Hurwirtz in 1902. It uses the formula for the area A of a domain D in R^2 in terms a line integral around its (positively oriented) boundary C:

$$A = \int_C xdy = -\int_C ydx = \frac{1}{2}\int_C xdy - ydx$$

where the line integrals are defined by the usual limit of Riemann sums. For our purposes, if $C : x = x(t)$, $y = y(t)$ $a \leq t \leq b$ is a parametrization of a simple curve C, then

$$\int_C xdy = \int_a^b x(t)y'(t)dt \ , \ \int_C ydx = \int_a^b y(t)x'(t)dt \ .$$

The formulas for A follow from the divergence theorem:

$$\int_D \mathrm{div} V(\vec{x}, y)dxdy = \int_C \vec{V} \cdot \vec{n} ds \ ,$$

where if $V = (V^1, V^2)$, div $V(\vec{x}, y) = V^1_x + V^2_y$, $\vec{n}$ is the outer unit normal to C and s is arc length. Choosing $V = (x, y)$, div $\vec{V} = 2$. Noting that in terms of the parametrization of C,

$$\vec{n} = \frac{(y'(t), -x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \text{ and } ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt \ ,$$

the formulas for A follow easily.

Hurwitz's proof also uses a famous inequality known as Wirtinger's inequality.

Lemma 2.5. *Let $f(t)$ be a C^1 periodic function of period 2π and average $\int_0^{2\pi} f(t)\, dt = 0$. Then*

$$\int_0^{2\pi} f'(t)^2\, dt \geq \int_0^{2\pi} f(t)^2\, dt$$

with equality only if $f(t) = a\cos t + b\sin t$.

Proof. Expand $f(t)$ in a Fourier series

$$f(t) = \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

(note that the constant term $\frac{1}{2}a_0$ is zero by the average zero condition). Then

$$f'(t) = \sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt)$$

By Parseval's formula,

$$\int_0^{2\pi} f(t)^2\,dt = \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\ ,$$

$$\int_0^{2\pi} f'(t)^2\,dt = \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2)\ .$$

Hence,

$$\int_0^{2\pi} (f'(t)^2 f(t)^2)\,dt = \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2)\ ,$$

which is strictly positive unless $a_n = b_n = 0$ for all $n > 1$, proving the lemma.

For simplicity assume that the length L of C is 2π (convince yourself that we can do this without loss of generality). We may also assume by translating C that

$$\int_0^{2\pi} x(t)\,ds = 0\ .$$

Then

$$2\pi = \int_0^{2\pi} ((x')^2 + (y')^2)\,ds\ ,$$

and $A = \int_0^{2\pi} x(s)y'(s)\,ds$. Hence,

$$2(\pi - A) = \int_0^{2\pi} ((x')^2 - 2xy' + (y')^2)\,ds = \int_0^{2\pi} ((x')^2 - x^2)\,ds + \int_0^{2\pi} (x - y')^2\,ds\ .$$

The first integral is nonegative by the lemma and the second integral is also clearly nonnegative. Hence $4\pi A \leq (2\pi)^2 = L^2$ which is the isoperimetric inequality. The equality sign holds only if

$$x = a\cos s + b\sin s\ ,\ y' = x$$

which gives

$$y = a\sin s - b\cos s + c\ .$$

Thus $x^2 + (y - c)^2 = a^2 + b^2$ and C is a circle.

3. Calculus of Variations formalism: necessary conditions

~~The simplest and most common problems in the Calculus of Variations are of the type where we seek to minimize (or maximize) a functional~~

$$\xcancel{E(u) = \int_a^b f(x, u(x), u'(x))\,dx}$$

~~where $f = f(x, z, p)$ is a given (say C^2) function of its arguments and the function $u(x)$ belongs to a certain admissible class, for example C^1 functions~~