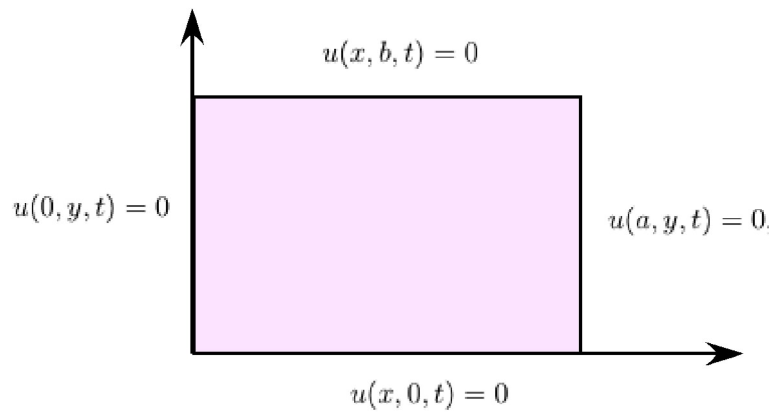
Wave Equation on a Two Dimensional Rectangle

In these notes we are concerned with application of the method of separation of variables applied to the wave equation in a two dimensional rectangle. Thus we consider

$$\begin{aligned} u_{tt} &= c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), \quad t>0, \quad (x,y)\in[0,a]\times[0,b], \qquad (1)\\ u(0,y,t)&=0, \quad u(a,y,t)=0, \quad u(x,0,t)=0, \quad u(x,b,t)=0\\ u(x,y,0)&=f(x,y), \quad u_t(x,y,0)=g(x,y)\end{aligned}$$



$$u(x,y)=X(x)Y(y)T(t).$$

Substituting into (1) and dividing both sides by $X(x)Y(y)$ gives

$$\frac{T''(t)}{c^2T(t)}=\frac{Y''(y)}{Y(y)}+\frac{X''(x)}{X(x)}$$

Since the left side is independent of x, y and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{T''(t)}{c^2T(t)}=\frac{Y''(y)}{Y(y)}+\frac{X''(x)}{X(x)}=\lambda.$$

We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions T, X and Y. We obtain

$$\frac{X''(x)}{X(x)}=\lambda-\frac{Y''(y)}{Y(y)} \qquad T''(t)-c^2\lambda T(t)=0.$$

Thus we conclude that there is a constant α

$$X''-\alpha X=0.$$

On the other hand we could also write

$$\frac{Y''(y)}{Y(y)}=\lambda-\frac{X''(x)}{X(x)}$$

so there exists a constant β so that

$$Y'' - \beta Y = 0.$$

Furthermore, the boundary conditions give

$$X(0)Y(y) = 0, \quad X(a)Y(y) = 0 \quad \text{for all } y.$$

Since $Y(y)$ is not identically zero we obtain the desired eigenvalue problem

$$X''(x) - \alpha X(x) = 0, \quad X(0) = 0, \quad X(a) = 0. \tag{2}$$

We have solved this problem many times and we have $\alpha = -\mu^2$ so that

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Applying the boundary conditions we have

$$0 = X(0) = c_1 \Rightarrow c_1 = 0 \quad 0 = X(a) = c_2 \sin(\mu a).$$

From this we conclude $\sin(\mu a) = 0$ which implies

$$\mu = \frac{n\pi}{a}$$

and therefore

$$\alpha_n = -\mu_n^2 = -\left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots .. \tag{3}$$

Now from the boundary condition

$$X(x)Y(0) = 0, \quad X(x)Y(b) = 0 \quad \text{for all } x.$$

This gives the problem

$$Y''(y) - \beta Y(y) = 0, \quad Y(0) = 0, \quad Y(b) = 0. \tag{4}$$

This is the same as the problem (2) so we obtain eigenvalues and eigenfunctions

$$\beta_m = -\nu_m^2 = -\left(\frac{m\pi}{b}\right)^2, \quad Y_m(y) = \sin(\nu_m y), \quad n = 1, 2, \cdots .. \tag{5}$$

So we obtain eigenvalues of the main problem given by

$$\lambda_{n,m} = -\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right) \tag{6}$$

and corresponding eigenfunctions

$$\varphi_{n,m}(x, y) = \sin(\mu_n x) \sin(\nu_m y).$$

We also find the solution to $T''(t) - c^2\lambda_{n,m}T(t) = 0$ is given by

$$T_{n,m}(t) = [a_{n,m}\cos(c\omega_{n,m}t) + b_{n,m}\sin(c\omega_{n,m}t)]$$

where we have defined

$$\omega_{n,m} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}.$$

So we look for u as an infinite sum

$$u(x,y,t) = \sum_{n,m=1}^{\infty} [a_{n,m}\cos(c\omega_{n,m}t) + b_{n,m}\sin(c\omega_{n,m}t)]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right). \tag{7}$$

The only problem remaining is to somehow pick the constants $a_{n,m}$ and $b_{n,m}$ so that the initial condition $u(x,y,0) = f(x,y)$ and $u_t(x,y,0) = g(x,y)$ are satisfied, i.e.,

$$f(x,y) = u(x,y,0) = \sum_{n,m=1}^{\infty} a_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right). \tag{8}$$

At this point we recall our orthogonality relations

$$\int_0^{\ell} \sin\left(\frac{j\pi\xi}{\ell}\right)\sin\left(\frac{c^2\pi\xi}{\ell}\right)\,d\xi = \begin{cases} \dfrac{2}{\ell} & j = k \\ 0 & j \neq k. \end{cases}$$

So we first multiply both sides of (8) by $\sin\left(\dfrac{n\pi x}{a}\right)$ and integrate in x from 0 to a to get

$$\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\,dx = \sum_{n=1}^{\infty}\left(\frac{2}{a}\right)a_{n,m}\sin\left(\frac{m\pi y}{b}\right).$$

Next we multiply this expression by $\sin\left(\dfrac{m\pi y}{b}\right)$ and integrate in x from 0 to b to get

$$\int_0^b\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\,dx = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)a_{n,m}.$$

Thus we conclude that

$$a_{n,m} = \left(\frac{4}{ab}\right)\int_0^b\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\,dx\,dy$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

$$g(x,y) = u_t(x,y,0) = \sum_{n,m=1}^{\infty} c\omega_{n,m}b_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right). \tag{9}$$

Just as above we get

$$\int_0^b \int_0^a g(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \, dx = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) b_{n,m} c\omega_{n,m}.$$

$$b_{n,m} = \left(\frac{4}{abc\omega_{n,m}}\right) \int_0^b \int_0^a g(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \, dx\, dy$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

General Methodology

The same methodology can be applied for more general boundary conditions. Consider $t > 0$, $(x, y) \in [0, a] \times [0, b]$, and

$$u_{tt} = c^2 \left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right), \tag{10}$$

$$\gamma_1 u(0,y,t) + \gamma_2 u_x(0,y,t) = 0,$$

$$\gamma_3 u(a,y,t) + \gamma_4 u_x(a,y,t) = 0,$$

$$\gamma_5 u(x,0,t) + \gamma_6 u_y(x,0,t) = 0,$$

$$\gamma_7 u(x,b,t) + \gamma_8 u_y(x,b,t) = 0$$

$$u(x,y,0) = f(x,y)$$
$$u_t(x,y,0) = g(x,y)$$

As usual we seek simple solution in the form

$$u(x,y) = X(x)Y(y)T(t).$$

Substituting into (10) and dividing both sides by $X(x)Y(y)$ gives

$$\frac{T''(t)}{c^2 T(t)} = \frac{Y''(y)}{Y(y)} + \frac{X''(x)}{X(x)}$$

As above we obtain three problems involving constants λ, α and β with $\lambda = \alpha + \beta$:

$$X''(x) - \alpha X(x) = 0, \quad \gamma_1 X(0) + \gamma_2 X'(0) = 0, \quad \gamma_3 X(a) + \gamma_4 X'(a) = 0. \tag{11}$$

After some work we obtain an infinite set of negative eigenvalues and eigenfunctions

$$\alpha_n = -\mu_n^2, \quad X_n(x), \quad n = 1, 2, \cdots. \tag{12}$$

Similarly,

$$Y''(y) - \beta Y(y) = 0, \; g_5 Y(0) + \gamma_6 Y'(0) = 0, \quad \gamma_7 Y(b) + \gamma_8 Y'(b) = 0. \tag{13}$$

Once again we obtain eigenvalues and eigenfunctions

$$\beta_m = -\nu_m^2, \quad Y_m(y), \quad n = 1, 2, \cdots .. \tag{14}$$

We also find the solution to $T''(t) - c^2 \lambda_{n,m} T(t) = 0$. To this end let us again define

$$\omega_{n,m} = \sqrt{-\lambda} = \sqrt{(\mu_n)^2 + (\nu_m)^2}.$$

Then the solution is given by

$$T_{n,m}(t) = \left[a_{n,m} \cos(c\omega_{n,m} t) + b_{n,m} \sin(c\omega_{n,m} t)\right].$$

So we look for u as an infinite sum

$$u(x, y, t) = \sum_{n,m=1}^{\infty} \left[a_{n,m} \cos(c\omega_{n,m} t) + b_{n,m} \sin(c\omega_{n,m} t)\right] X_n(x) Y_m(y). \tag{15}$$

Finally we need to pick the constants $a_{n,m}$ and $b_{n,m}$ so that the initial conditions $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y)$ are satisfied, i.e.,

$$f(x, y) = u(x, y, 0) = \sum_{n,m=1}^{\infty} a_{n,m} X_n(x) Y_m(y). \tag{16}$$

$$g(x, y) = u_t(x, y, 0) = \sum_{n,m=1}^{\infty} b_{n,m} c\omega_{n,m} X_n(x) Y_m(y). \tag{17}$$

The general Sturm-Liouville theory guarantees that the eigenfunctions corresponding to distinct eigenfunctions are distinct, i.e.

$$\int_0^a X_j(x) X_k(x)\, dx = \begin{cases} \kappa_j & j = k, \\ 0 & j \neq k. \end{cases}, \qquad \int_0^b Y_j(x) Y_k(x)\, dx = \begin{cases} \widetilde{\kappa}_j & j = k, \\ 0 & j \neq k. \end{cases} \tag{18}$$

for some positive constants κ_j and $\widetilde{\kappa}_j$. So we obtain

$$a_{n,m} = \left(\frac{1}{\kappa_n \widetilde{\kappa}_m}\right) \int_0^b \int_0^a f(x, y) X_n(x) Y_m(y)\, dx\, dy \tag{19}$$

and

$$b_{n,m} = \left(\frac{1}{c\omega_{n,m} \kappa_n \widetilde{\kappa}_m}\right) \int_0^b \int_0^a g(x, y) X_n(x) Y_m(y)\, dx\, dy \tag{20}$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

Example 1. To simplify the problem a bit we set $c = 1$, $a = \pi$ and $b = \pi$. Namely we consider

$$\begin{aligned} u_{tt} &= (u_{xx}(x,y,t) + u_{yy}(x,y,t)), \quad t > 0, \quad (x,y) \in [0,\pi] \times [0,\pi] \\ u(0,y,t) &= 0, \quad u(\pi,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,\pi,t) = 0 \\ u(x,y,0) &= x(\pi - x)y(\pi - y), \quad u_t(x,y,0) = 0. \end{aligned} \tag{21}$$

In this case we obtain eigenvalues

$$\lambda_{n,m} = -(n^2 + m^2), \quad \alpha_n = -n^2, \quad \beta_m = -m^2, \quad n, m = 1, 2, \cdots.$$

The corresponding eigenfunctions are given by

$$X_n(x) = \sin(nx), \qquad Y_m(y) = \sin(my).$$

Our solution is given by (15) which here has the form

$$u(x,y,t) = \sum_{n=1}^{\infty} [a_{n,m} \cos(\omega_{n,m} t) + b_{n,m} \sin(\omega_{n,m} t)] \sin(nx) \sin(my)$$

where we have defined

$$\omega_{n,m} = \sqrt{n^2 + m^2}.$$

The coefficients $a_{n,m}$ are obtained from (18) where in this case

$$\kappa_n = \widetilde{\kappa}_m = \frac{\pi}{2}, \quad n, n = 1, 2, \cdots.$$

We have

$$x(\pi - x)y(\pi - y) = \sum_{n,m=1}^{\infty} a_{n,m} \sin(nx) \sin(my).$$

From (19) we have

$$a_{n,m} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} x(\pi - x)y(\pi - y) \sin(nx) \sin(my)\, dx\, dy = \frac{16((-1)^n - 1)((-1)^m - 1)}{n^3 m^3 \pi^2}.$$

Since $u_t(x,y,0) = g(x,y) = 0$ we have

$$b_{n,m} = 0.$$

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c_{n,m} e^{k\lambda_{n,m} t} \sin(n\pi x) \sin(m\pi y). \tag{22}$$

Example 2. To simplify the problem a bit we set $c = 1$, $a = 1$ and $b = 1$. Namely we consider

$$\begin{aligned} u_{tt} &= (u_{xx}(x,y,t) + u_{yy}(x,y,t)), \quad t > 0, \quad (x,y) \in [0,\pi] \times [0,\pi] \\ u_x(0,y,t) &= 0, \quad u_x(\pi,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,\pi,t) = 0 \\ u(x,y,0) &= x(\pi - x)y, \qquad u_t(x,y,0) = 0. \end{aligned} \tag{23}$$

We get eigenvalue problem in x given by

$$X'' - \alpha X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0.$$

Therefore we have eigenvalues and eigenvectors

$$\alpha_0 = 0, \quad X_0(x) = 1, \qquad \alpha_n = -n^2, \quad X_n(x) = \cos(nx), \quad n = 1, 2, 3, \cdots .$$

The eigenvalue problem in y is given by

$$Y'' - \beta Y = 0, \quad Y(0) = 0, \quad Y(\pi) = 0.$$

The corresponding eigenvalues are

$$\beta_m = -m^2, \quad X_m(x) = \sin(mx), \quad m = 1, 2, 3, \cdots .$$

In this case we obtain eigenvalues

$$\lambda_{n,m} = -(n^2 + m^2), \quad \alpha_n = -n^2, \quad \beta_m = -m^2, \quad n, m = 1, 2, \cdots .$$

The corresponding eigenfunctions are given by

$$X_n(x) = \sin(n\pi x), \qquad Y_m(y) = \sin(m\pi y).$$

For this example we also have eigenvalues

$$\lambda_{0,m} = -m^2, \quad X_0(x) = 1.$$

Our solution is given by (15) which here has the form

$$\begin{aligned} u(x,y,t) = &\sum_{m=1}^{\infty} [a_{0,m} \cos(\omega_{0,m} t) + b_{0,m} \sin(\omega_{0,m} t)] \sin(my) \\ &+ \sum_{n,m=1}^{\infty} [a_{n,m} \cos(\omega_{n,m} t) + b_{0,m} \sin(\omega_{0,m} t)] \cos(nx) \sin(my). \end{aligned}$$

Setting $t = 0$ we obtain

$$x(\pi - x)y = \sum_{n=1}^{\infty} a_{n,0} \cos(nx) + \sum_{n,m=1}^{\infty} a_{n,m} \sin(nx) \sin(my).$$

We have

$$a_{n,m} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi x(\pi - x)y\cos(nx)\sin(my)\,dx\,dy = \frac{-8((-1)^n - 1)((-1)^m - 1)}{n^2 m^3 \pi^2}.$$

Finally we obtain the coefficients $a_{n,0}$ from

$$a_{0,m} = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi x(\pi - x)y\sin(my)\,dx\,dy = \frac{-2((-1)^m - 1)}{m^3}.$$

Finally we arrive at the solution

$$\begin{aligned} u(x,y,t) = &\sum_{m=1}^{\infty}\frac{-2((-1)^m - 1)}{m^3}\cos(\omega_{0,m}t)\sin(my) \\ &+ \sum_{n,m=1}^{\infty}\frac{-8((-1)^n - 1)((-1)^m - 1)}{n^2 m^3 \pi^2}\cos(\omega_{n,m}t)\cos(nx)\sin(my) \end{aligned}$$

with $\omega_{n,m} = \sqrt{n^2 + m^2}$.