DAlembert's solution of the wave equation using the method of characteristics

We first recall the solution of the initial value problem for the inhomogeneous transport equation

$$w_t + bw_x = q(x,t) \ , \ w(x,0) = h(x) \ ,$$

by the method of characteristics. Set $z(s) = w(x+bs, t+s)$ (for (x,t) fixed); that is we restrict w to the charactistic line passing through (x,t). Then

$$\dot{z}(s) = w_x \cdot b + w_t = q(x+bs, t+s) \ .$$

We integrate this from $s = -t$ to $s = 0$:

$$w(x,t) - h(x-bt) = z(0) - z(-t) = \int_{-t}^{0} \dot{z}(s) \ ds =$$

$$\int_{-t}^{0} q(x+bs, t+s) \ ds = \int_{0}^{t} q(x+b(s-t), s) \ ds \ .$$

This gives that

$$w(x,t) = h(x-bt) + \int_{0}^{t} q(x+b(s-t), s) \ ds \ .$$

Now consider the Cauchy problem for the wave equation equation

$$u_{tt} = c^2 u_{xx} \text{ on } -\infty < x < \infty \ , \ t > 0 \ , \ u(x,0) = f(x) \ , \ u_t(x,0) = g(x) \ .$$

Observe that $u_{tt} - c^2 u_{xx} = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u$.

So if we set $v(x,t) = u_t - cu_x$, then v satisfies the homogeneous transport equation

$$v_t + cv_x = 0 \ , \ \text{with initial data } v(x,0) = g(x) - cf'(x \ .$$

In particular (with b=c and $q \equiv 0$ above), $v(x,t) = a(x-ct)$ where $a(x) = v(x,0) = g(x) - cf'(x)$. We recover $u(x,t)$ using that $u_t - cu_x = v(x,t)$ with b=-c , $h(x) = f(x)$ and $q(x,t) = v(x,t) = a(x-ct) = g(x-ct) - cf'(x-ct)$.

We obtain (noting $q(x+b(s-t),s) = f(x-c(t-s),s) = a(x+c(t-s),s)$ and changing variables $y = x - ct - 2s$)

$$u(x,t) = f(x+ct) + \int_0^t a(x+c(t-s)-cs)\ ds = f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} a(y)\ dy\ .$$

Inserting $a(x) = v(x,0) = g(x) - cf'(x)$ we obtain

$$u(x,t) = f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} (g(y) - cf'(y))\ dy\ ,$$

which gives after integrating the exact derivative

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ dy\ .$$

In particular, the solution can be written as the sum of two traveling waves:

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \text{ where} \\ F(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\ ds\ , \\ G(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\ ds\ , \end{aligned}$$