1. Second order elliptic pde's (Introduction)

Let Ω be a domain (i.e. a connected open set) in $\mathbb{R}^n$ and define

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{ij} + \sum_{i=1}^{n} b_i u_i + c(x)\, u$$

where a_{ij}, b_i, $c(x)$ $\quad i,j = 1,\cdots,n$

are real valued functions on Ω,

$u(x) \in C^2(\Omega)$, $\left(u_i = \dfrac{\partial u}{\partial x_i},\ u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j} \right)$.

Then L is a linear operator, i.e

$L(u_1+u_2) = Lu_1 + Lu_2$, $\quad L(\alpha u) = \alpha Lu$, α a real constant.

The ~~linear~~ pde $Lu = f$, with f a given real function on Ω is called a (second order) linear p.d.e. If $f \equiv 0$, the equation is said to be homogeneous.

L is <u>elliptic</u> at a point $x \in \Omega$ if the quadratic form $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ is positive definite.

<u>L is elliptic on Ω</u> if it is elliptic at all $x \in \Omega$

L is <u>uniformly elliptic</u> on Ω if there exists real constants $\lambda, \Lambda > 0$ s. that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

$$\forall x \in \Omega,\ \xi \in \mathbb{R}^n$$

The classical prototype for elliptic equations is Poisson's equation

$$\Delta u := \sum_{i=1}^{n} u_{ii} = f$$

Here $a_{ij} = \delta_{ij}$ (Kronecker delta) $b_i = c_i = 0$.
Poisson's equation is uniformly elliptic
with $\lambda = \Lambda = 1$. The homogeneous Poisson
equation, i.e. $\Delta u = 0$
is called <u>Laplace's equation</u>.

More generally, a general (fully nonlinear)
second order equation on Ω,

$$F[u] = F(x, u, Du, D^2u) = 0$$

is elliptic (at a solution u) if the
(principal part) linearized operator

$$L = F^{ij} \partial_i \partial_j \quad , \quad F^{ij} = \frac{\partial F}{\partial u_{ij}}$$

is elliptic, for arguments (x, u, Du, D^2u) lying in an appropriate set.

<u>Remark</u> The <u>full</u> linearized operator

is $\mathcal{L}\phi = \frac{d}{dt} F[u+t\phi]\Big|_{t=0}$ (Gateaux derivative)

$= F^{ij}\phi_{ij} + F_{p_i}\phi_i + F_u\phi$

When $F[u]$ is linear in D^2u but nonlinear in Du, u, the equation is called quasilinear. The most famous example of a quasilinear elliptic (but not uniformly elliptic!)

equation is the minimal surface equation:

$$\sum_{i=1}^{n} \partial_i \left(\frac{u_i}{\sqrt{1+|Du|^2}} \right) = 0 \qquad \text{(divergence form)}$$

or

$$\sum_{i,j=1}^{n} \left((1+|Du|^2)\,\delta_{ij} - u_i u_j \right) u_{ij} = 0$$

(non-divergence form)

Closely related is the equation of prescribed mean curvature $H(x)$:

$$\sum_{i=1}^{n} \partial_i \left(\frac{u_i}{\sqrt{1+|Du|^2}} \right) = n\,H(x)$$

for the <u>graph</u> $S = \{ (x,u) : x \in \Omega \} \subset R^{n+1}$ oriented by the upward unit normal

$$\nu = \left(\frac{-\nabla u}{\sqrt{1+|Du|^2}}, \frac{1}{\sqrt{1+|Du|^2}} \right)$$

The Dirichlet problem for L uniformly elliptic on Ω is the problem

$$Lu = f \quad \text{on } \Omega$$
$$u = \phi \quad \text{on } \partial\Omega$$

where f, ϕ are given on Ω (of some ~~appropriate~~ appropriate smoothness) and u is the unknown to be found.

2. Laplace's equation & Harmonic functions.

Let Ω be a bounded domain in R^n with C^1 boundary. The Laplace operator, for $u \in C^2(\bar{\Omega})$ is then

$$\Delta u = \sum u_{ii} = \operatorname{div}(\nabla u)$$

The function u is called harmonic on Ω (respectively subharmonic, superharmonic) if $\Delta u = 0$ (respectively $\Delta u \geq 0$, $\Delta u \leq 0$) in Ω.

A basic tool in the development will be the divergence theorem.

For a vector field $X \in C^1(\bar{\Omega})$,

$$(1) \quad \int_\Omega \operatorname{div} X \, dx = \int_{\partial\Omega} X \cdot \nu \, ds$$

where ν is the exterior unit normal and ds denotes the $(n-1)$ dimensional surface area of $\partial\Omega$.

Let $u, v \in C^2(\bar{\Omega})$. Choosing $X = v Du$,

$$\operatorname{div} X = v \Delta u + Du \cdot Dv.$$

Inserting this into (1) gives <u>Green's first identity</u>:

$$(2) \quad \int_\Omega (v \Delta u + Du \cdot Dv) \, dx = \int_{\partial\Omega} v \, u_\nu \, ds$$

If we reverse the roles of u, v and subtract, we obtain <u>Green's</u> <u>second</u> <u>identity</u>,

$$(3) \quad \int_{\Omega} (v \Delta u - u \Delta v)\, dx = \int_{\partial\Omega} (v u_\nu - u v_\nu)\, ds$$

If $v \equiv 1$ in (2) we obtain

$$(4) \quad \int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} u_\nu \, ds$$

We now derive an important consequence of (4) known as the <u>mean</u> <u>value</u> <u>inequalities</u>.

<u>Theorem 1</u> Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ (respectively $\Delta u \geq 0$, $\Delta u \leq 0$) in Ω. Then for any ball $B = B_R(y) \subset\subset \Omega$,

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u \, ds \tag{5}$$

(respectively $u \leq$ rhs, $u \geq$ rhs)

$$u(y) = \frac{1}{\omega_n R^n} \int_B u \, dx \tag{6}$$

(respectively $u \leq$ rhs, $u \geq$ rhs)

(Here ω_n is the volume of the unit ball in $\mathbb{R}^n$)

<u>Proof</u> Let $\rho \in (0, R)$ and apply (4) to the ball $B_\rho = B_\rho(y)$:

$$\int_{\partial B_\rho} u_\nu \, ds = \int_{B_\rho} \Delta u \, dx = 0$$

(respectively ≥ 0, ≤ 0)

Introduce polar coordinates $r = |x-y|$, $\omega = \frac{x-y}{r}$ centered at y, and write $u(x) = u(y + r\omega)$. Then

$$\int_{\partial B_\rho} u_\nu \, ds = \int_{\partial B_\rho} \frac{\partial u}{\partial r}(y+\rho\omega)\, \rho^{n-1} d\omega = \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(y+\rho\omega)\, d\omega$$

$$= \rho^{n-1} \frac{\partial}{\partial \rho} \left(\rho^{1-n} \int_{\partial B_\rho} u \, ds \right) = 0$$

(respectively ≥ 0, ≤ 0)

So for $\rho \in (0, R)$,

$$\rho^{1-n} \int_{\partial B_\rho} u \, ds = R^{1-n} \int_{\partial B_R} u \, ds$$

(respectively $\leq$ rhs , $\geq$ rhs).

As $\rho \to 0$ $\rho^{1-n} \int_{\partial B_\rho} u \, ds$ tends to $n \omega_n u(y)$

and (5) follows. To obtain (6)

we write

$$n \omega_n \rho^{n-1} u(y) = \int_{\partial B_\rho} u \, ds$$

(respectively $\leq$ rhs , $\geq$ rhs)

and integrate in ρ from 0 to R. //

Here are some nice consequences of Theorem 1.

<u>Theorem 2</u> (Strong Maximum principle)

Let $\Delta u \geq 0$ (respectively ≤ 0) in Ω and suppose there exists a point $y \in \Omega$ at which u assumes its maximum (respectively minimum) i.e $u(y) = \sup_{\Omega} u$ (respectively $\inf_{\Omega} u$)

Then $u \equiv$ constant.

Proof Let $\Omega_M = \{x \in \Omega \mid u(x) = M = \sup_\Omega u\}$

By hypothesis, $\Omega_M \neq \phi$. Choose $z \in \Omega_M$ and apply the mean value inequality of Theorem 1 to the subharmonic function $u - M$ in a ball $B = B_R(z) \subset\subset \Omega$.

Then

$$0 = u(z) - M \leq \frac{1}{\omega_n R^n} \int_B (u - M)\, dx \leq 0.$$

so $u \equiv M$ in B. Since Ω is connected ~~[illegible]~~ $\Omega_M = \Omega$. //

Corollary 3 (weak maximum principle)

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$\Delta u \geq 0$ (respectively $\Delta u \leq 0$) in Ω.

Then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad \left(\text{respectly } \inf_{\Omega} u = \inf_{\partial\Omega} u\right)$$

Consequently if u harmonic in Ω

$$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u .$$

<u>Corollary 4</u> (uniqueness for classical Dirichlet problem for Poisson's equation). Let u, v $\in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $\Delta u = \Delta v$ in Ω, $u = v$ on $\partial\Omega$. Then $u \equiv v$ in Ω

<u>Theorem 3</u> Let u be a non-negative harmonic function in Ω. Then for any $\Omega' \subset\subset \Omega$ $\exists\ c = c(n, \Omega', \Omega)$ such that

$$\sup_{\Omega'} u \leq c \inf_{\Omega'} u$$

<u>Proof</u> Let $y \in \Omega$ and choose $B \overset{=}{} B_R(y)$ such that $B_{4R}(y) \subset \Omega$. Then for any $x_1, x_2 \in B_R(y)$,

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u\,dx \leq \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} u\,dx$$

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} u\,dx \geq \frac{1}{\omega_n (3R)^n} \int_{B_{2R}(y)} u\,dx$$

Hence $$\sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u \qquad (*)$$

Now for $\Omega' \subset\subset \Omega$, choose

$u(x_1) = \sup_{\Omega'} u$, $u(x_2) = \inf_{\Omega'} u$ and

let Γ be a closed arc joining x_1, x_2.

Choose R so that $4R < \mathrm{dist}(\Gamma, \partial\Omega)$.

By the Heine - Borel theorem we can

cover Γ by N balls of radius R.

Applying $*$ repeatedly gives

$$u(x_1) \leq 3^{nN} u(x_2) \quad \text{i.e. } C = 3^{nN}. \quad //$$

<u>Remark</u> C is invariant under similarity

and orthogonal transformations. In particular if $\Omega = B_R(x_0)$, $\Omega' = B_{\tau R}(x_0)$ $\quad \tau \in (0,1)$,

then $C = C(\tau)$.

If we look for a radial symmetric solution $u(x) = \phi(r)$ $\quad r = |x-y|$

$$u_i = \phi'(r)\,\frac{x_i - y_i}{r}$$

$$u_{ii} = \frac{\phi'(r)}{r} + \frac{(x_i-y_i)^2}{r^2}\left(\phi'' - \frac{\phi'}{r}\right)$$

$$\Delta u = \sum_{i=1}^{n} u_{ii} = \phi'' + \frac{n-1}{r}\,\phi = \frac{1}{r^{n-1}}\left(r^{n-1}\phi'\right)' = 0$$

$$\Rightarrow \quad r^{n-1}\phi' = c \qquad \phi = \frac{c_1}{r^{n-2}} + c_2 \quad n>2$$

$$\phi = c_1 \log r + c_2 \qquad n = 2$$

We introduce the <u>normalized fundamental solution</u>

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \dfrac{1}{n(n-2)\omega_n}\,|x-y|^{2-n} & \text{if } n>2 \\[2mm] -\dfrac{1}{2\pi}\log|x-y| & \text{if } n=2 \end{cases}$$

A straightforward computation gives

$$D_i \Gamma = -\frac{1}{n\omega_n}(x_i - y_i)|x-y|^{-n}$$

$$D_j D_i \Gamma = -\frac{1}{n\omega_n}\left(|x-y|^{-n}\delta_{ij} + n(x_i-y_i)(x_j-y_j)\,|x-y|^{-(n+2)}\right)$$

(Again we see that for $x \neq y$ $\sum_{i=1}^{n} \Gamma_{ii} = 0$)

It follows that

$$|D_i\Gamma| \le \frac{1}{n\omega_n}|x-y|^{1-n}$$

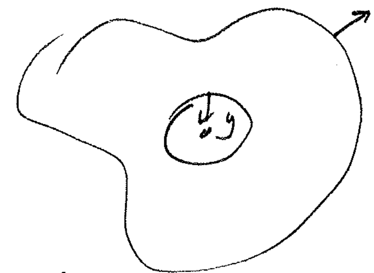
$$|D_j D_i \Gamma| \le \frac{n+1}{n\omega_n}|x-y|^{-n}$$

$$|D^\beta \Gamma| \le C\,|x-y|^{2-n-|\beta|}$$

for any multi-index $\beta = (\beta_1, \ldots, \beta_n)$
$\beta_j \ge 0$

Because Γ is singular at $x=y$, we cannot use Γ directly in Green's second identity.

We overcome this (in the usual way) by replacing Ω by $\Omega - \overline{B}_\varepsilon$, $B_\varepsilon = B_\varepsilon(y)$ for $\varepsilon > 0$ small.



We obtain

$$\int_{\Omega - \overline{B}_\varepsilon} \Gamma \Delta u \, dx = \int_{\partial\Omega} (\Gamma u_\nu - u \Gamma_\nu) \, ds + \int_{\partial B_\varepsilon} (\Gamma u_\nu - u \Gamma_\nu) \, ds$$

Now $\displaystyle\int_{\partial B_\varepsilon} \Gamma u_\nu \, ds = \Gamma(\varepsilon) \int_{\partial} u_\nu \, ds \leq n \omega_n \varepsilon^{n-1} \Gamma(\varepsilon) \sup_{B_\varepsilon} |D$

which tends to zero ($n=2$ also) as $\varepsilon \to 0$.

Also, $\displaystyle\int_{\partial B_\varepsilon} u \frac{\partial \Gamma}{\partial \nu} \, ds = -\Gamma'(\varepsilon) \int_{\partial B_\varepsilon} u \, ds = \frac{1}{n \omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon} u \, ds$

which tends to $u(y)$ as $\varepsilon \to 0$

Hence as $\varepsilon \to 0$ we obtain

$$(**) \qquad u(y) = -\int_{\partial\Omega} (u\Gamma_\nu - \Gamma u_\nu)\, ds_x - \int_{\Omega} \Gamma(x-y)\,\Delta u\, dx.$$

In particular if u has compact support in $\mathbb{R}^n$,
$$u(y) = -\int_{\mathbb{R}^n} \Gamma(x-y)\,\Delta u\, dx.$$

If u is harmonic in $(**)$,

$$u(y) = -\int_{\partial\Omega} (u\Gamma_\nu - \Gamma u_\nu)\, ds_x \qquad y \in \Omega.$$

Note that the integrand is an analytic function of y for $x \in \partial\Omega$. It follows that a $C^2(\Omega)$ harmonic function is in fact real analytic in Ω.

More generally, let $G = \Gamma + h$ where $h \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and $\Delta h = 0$ in Ω. Then the same argument gives

$$u(y) = -\int_{\partial\Omega} (u G_\nu - G u_\nu)\, dS_x - \int_{\Omega} G \Delta u \, dx$$

If $G = 0$ on $\partial\Omega$, G is called the (Dirichlet) Green's function for Ω. In this case,

$$u(y) = -\int_{\partial\Omega} u G_\nu \, ds - \int_{\Omega} G \Delta u \, dx$$

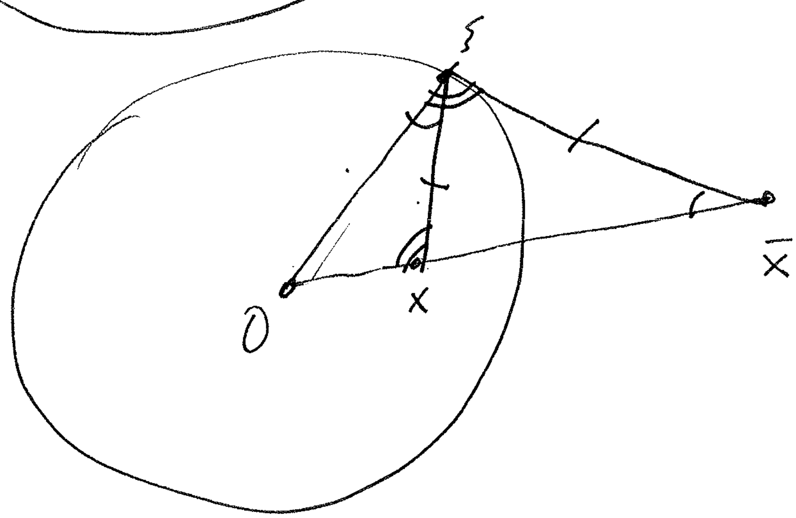
The existence of G depends upon the existence of a harmonic h in Ω with boundary values $h = -\Gamma$ on $\partial\Omega$

In the special case $\Omega = B_R(0)$ we can find the Green's function explicitly using the so called method of images.

For $x \in B_R(0)$, $x \neq 0$ the point $\bar{x}$ inverse to x with respect to B_R is given by
$$\bar{x} = \frac{R^2}{|x|^2} x$$

Let $|\xi| = R$. Then since

$$\frac{|\bar{x}|}{R} = \frac{R}{|x|} = \frac{|\xi|}{|x|}$$



triangle $O\xi x$ is similar to triangle $O\xi\bar{x}$

(common angle at O , SAS)

Hence $\quad |\xi - x| = \frac{|x|}{R} \, |\xi - \bar{x}|$

Now for fixed x (and $\bar{x}$) , $\Gamma\left(\frac{|x|}{R} |\xi - \bar{x}|\right)$

is a harmonic function which equals

$\Gamma(|\xi - x|)$ when $|\xi| = R$.

Therefore,

$$G(x,\xi) = \begin{cases} \Gamma(|x-\xi|) - \Gamma\left(\frac{|x|}{R}|\xi-\bar{x}|\right) & \text{if } x \neq 0 \\ \Gamma(|\xi|) - \Gamma(R) & \text{if } x = 0 \end{cases}$$

is the Green's function for $\Omega = B_R(0)$.

It is easily verified that G satisfies,

i) $G(x,\xi) = G(\xi,x)$

ii) $G(x,\xi) \geq 0$ for $x,\xi \in \overline{B_R}$

We proceed to compute $\frac{\partial G}{\partial \nu}\Big|_{|\xi|=R}$

(with x fixed, variable ξ).

$$\frac{\partial}{\partial \nu} \Gamma(|x-\xi|)\Big|_{|\xi|=R} = \frac{\xi_i}{R}\frac{\partial \Gamma}{\partial \xi_i} = \Gamma'(|x-\xi|)\,\frac{\xi_i - x_i}{|\xi - x|}$$

$$= -\frac{1}{n\omega_n}|x-\xi|^{-n}(\xi_i - x_i)\cdot\frac{\xi_i}{R} = -\frac{|x-\xi|^{-n}}{n\omega_n}\left(\frac{R^2 - x\cdot\xi}{R}\right)$$

and

$$\frac{\partial}{\partial \nu}\Gamma\left(\frac{|x|}{R}|\xi-\bar{x}|\right)\Big|_{|\xi|=R} = \frac{\xi_i}{R}\frac{\partial}{\partial \xi_i}\left(\Gamma\left(\frac{|x|}{R}|\xi-\bar{x}|\right)\right)\Big|_{|\xi|=R}$$

$$= \frac{\xi_i}{R}\,\frac{\Gamma'(|x-\xi|)}{|x-\xi|}\,{}^{|x-\xi|}\,\frac{|x|}{R}\,\frac{\xi_i - \bar{x}_i}{|\xi-\bar{x}|}\Big|_{|\xi|=R} = \frac{|x|^2}{R^2}\frac{\Gamma'(|x-\xi|)}{|x-\xi|}\,\frac{\xi}{R}\cdot(\xi - \bar{x})\Big|_{|\xi|=R}$$

$$\xi\cdot\bar{x} = \frac{R^2}{|x|^2}\,x\cdot\xi$$

$$= -\frac{|x-\xi|^{-n}}{n\omega_n}\left(\frac{|x|^2}{R} - \frac{x\cdot\xi}{R}\right)$$

Therefore,

$$\frac{\partial G}{\partial \nu}\Big|_{|\xi|=R} = -\frac{|x-\xi|^{-n}}{n\omega_n}\,\frac{R^2 - |x|^2}{R}$$

This gives the <u>Poisson integral formula</u>

$$u(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B_R} \frac{u(\xi)}{|x-\xi|^n}\, dS_\xi \qquad x \in B_R$$

for u harmonic in B_R. The Kernel

$$K(x,\xi) = \frac{R^2 - |x|^2}{n\omega_n R\,|x-\xi|^n} \qquad x \in B_R,\ \xi \in \partial B_R$$

is <u>called the Poisson Kernel</u>

In order to establish the existence of solutions to the Dirichlet problem, we need the converse of the representation.

<u>Theorem 4</u> Let $B = B_R(0)$ and suppose $\phi \in C^0(\partial B)$. Then

$$u(x) = \begin{cases} \dfrac{R^2 - |x|^2}{n\omega_n R} \displaystyle\int_{\partial B} \dfrac{\phi(y)}{|x-y|^n}\, dS_y & x \in B \\ \phi(x) & x \in \partial B \end{cases}$$

belongs to $C^2(B) \cap C^0(\bar{B})$ and satisfies $\Delta u = 0$ in B. In particular, the Dirichlet problem in a ball is solvable for arbitrary continuous boundary values.

Proof. To show that $u(x)$ is harmonic in B, we write

$$K(x,y) = \frac{1}{n\omega_n R} \frac{R^2 - |x|^2}{|x-y|^n} = \frac{1}{n\omega_n R} \frac{1}{n-2} \cdot$$

$$\sum_{i=1}^{n} \left\{ y_i \frac{\partial}{\partial x_i} |x-y|^{2-n} + x_i \frac{\partial}{\partial x_i} |x-y|^{2-n} \right\}$$

(Check: $\sum y_i \frac{\partial}{\partial x_i} |x-y|^{2-n} = \frac{2-n}{|x-y|^n} \sum y_i \cdot (x_i - y_i) = \frac{2-n}{|x-y|^n} (x \cdot y - |y|^2)$

$\sum x_i \frac{\partial}{\partial x_i} |x-y|^{2-n} = \frac{2-n}{|x-y|^n} \sum x_i (x_i - y_i) = \frac{(2-n)}{|x-y|^n} (|x|^2 - x \cdot y)$

sum $:= \frac{2-n}{|x-y|^n} (|x|^2 - |y|^2) = \frac{n-2}{|x-y|^n} (R^2 - |x|^2)$ ✓)

But $\frac{\partial}{\partial x_i} |x-y|^{2-n}$ is harmonic since if

h is harmonic $\Delta h_k = \sum h_{kii} = (\sum h_{ii})_k = 0$,

(for fixed k)

and also $\Delta(\sum x_k h_k)$

$$= \sum_i \left(\sum_k x_k h_k\right)_{ii} = \sum_i \left(\sum_k (x_k h_{kii} + \delta_{ki} h_{ki})\right)$$

$$= \sum_k x_k \Delta h_k + 2\Delta h = 0$$

Hence $u(x)$ is harmonic. By the Poisson integral formula with $u \equiv 1$ we see that $\int_{\partial B} K(x,y)\, dS_y = 1 \quad \forall x \in B$

Let $x_0 \in \partial B$ and $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $|\phi(y) - \phi(x_0)| < \varepsilon$ if $|y - x_0| < \delta$, and let $M = \sup_{\partial B} |\phi|$.

Then for $|x - x_0| < \delta/2$,

$$|u(x)-u(x_0)| = \left| \int_{\partial B} K(x,y)\,(\phi(y)-\phi(x_0))\,dS_y \right|$$

$$\leq \int_{|y-x_0|<\delta} K(x,y)\,|\phi(y)-\phi(x_0)|\,dS_y + \int_{|y-x_0|\geq\delta} K(x,y)\,|\phi(y)-\phi(x_0)|\,dS_y$$

$$\leq \varepsilon + 2M \frac{(R^2-|x|^2)\,R^{n-2}}{(\delta/2)^n} \leq 2\varepsilon \quad \text{by}$$

Choosing $|x-x_0|$ small enough

(note that $R-|x| \leq |x-x_0|$)

so that $\dfrac{2M\cdot 2R^{n-1}}{(\delta/2)^n} \leq \varepsilon$.

Hence u is continuous at x_0. //

We state some consequences of the Poisson formula.

<u>Theorem 5</u> A $C^0(\Omega)$ function u is harmonic if and only if for every ball $B = B_R(y)$ $\subset\subset \Omega$, u satisfies the mean value property

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u\, ds$$

<u>Pf</u>. By Theorem 4, for any $B \subset\subset \Omega$ there exists a harmonic function h in B with $h = u$ on ∂B. The difference $w = u - h$ satisfies the mean value property and vanishes identically on ∂B. By the maximum principle

$w \equiv 0$ in B.

Corollary 4 The limit of a uniformly convergent sequence of harmonic functions is harmonic.

Theorem 7 (Harnack's convergence theorem) ~~The limit of a uniformly~~ Let $\{u_n\}$ be a monotone increasing sequence of harmonic functions in Ω and suppose that for some point $y \in \Omega$ $\{u_n(y)\}$ is bounded. Then the sequence converges uniformly on any compact subdomain $\Omega' \subset\subset \Omega$ to a harmonic function

<u>Proof</u>. Since $\{u_n(y)\}$ converges, given $\varepsilon > 0$ $\exists\, N$ s. that $0 \le u_m(y) - u_n(y) < \varepsilon$ for $m \geq n > N$. By the Harnack inequality Theorem 3,

$$\sup_{\Omega'} |u_m(x) - u_n(x)| \le C\varepsilon$$

with C independent of ε. Hence $\{u_n\}$ converges uniformly on Ω' to a harmonic function. (Cor. 6). //

<u>Theorem 8</u> Let u be harmonic in Ω and let $\Omega' \subset\subset \Omega$ with $d = \mathrm{dist}(\Omega', \partial\Omega)$. Then

$$\sup_{\Omega'} |D^\alpha u| \le \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup |u|$$

(exercise)

<u>Corollary 9</u> (Liouville) Let u be a bounded harmonic function on R^n. Then u is constant

<u>Pf</u>. Fix $x_0 \in \mathbb{R}^n$. Then for any $R>0$

$$|Du(x_0)| \leq \frac{n}{R} \sup_{B_R(x_0)} |u| \leq \frac{C}{R} \to 0 \ //$$

<u>Corollary 10</u> Any bounded sequence of harmonic functions on Ω contains a subsequence which converges uniformly on compact sets (in any topology) to a harmonic function.

The Perron process and the Dirichlet problems in general domains

Definition A $C^0(\Omega)$ function u is called subharmonic (superharmonic) in Ω if for every ball $B \subset\subset \Omega$ and every harmonic function h in B with $h \geq u$ on ∂B ($h \leq u$ on ∂B), we have $h \geq u$ ($h \leq u$) in B.

Note that u is subharmonic iff $-u$ is superharmonic.

Also if $u \in C^2$ and $\Delta u \geq 0$,
u is subharmonic by the maximum
principle.

<u>Lemma 1</u> (Strong maximum princi-
for subharmonic functions)

Let u be subharmonic in Ω and
suppose for some pt $x_0 \in \Omega$, $u(x_0) = \sup_\Omega u$.
Then $u \equiv u(x_0)$.

<u>Proof</u> Let B_ρ be a ball centered at x_0
of radius ρ and suppose $\overline{B_\rho} \subset \Omega$.
Let h be harmonic in B_ρ with

$h = u$ on ∂B_ρ. Then

$$h(X_0) \geq u(X_0) \geq \sup_{\partial B_\rho} u = \sup_{\partial B_\rho} h.$$

By the strong maximum principle for h,
$h \equiv h(X_0)$ on B_ρ. Thus
$u \equiv u(X_0) = h(X_0)$ on ∂B_ρ. But
the same is true for any smaller ρ
So $u \equiv u(X_0)$ in B_ρ. Since Ω is
connected, $u \equiv u(X_0)$ in Ω. //

Lemma 2 i) If u_1, u_2 are subharmonic on Ω
so is $u_1 + u_2$

ii) If $u_1, \dots, u_n$ are subharmonic on Ω,
so is $u = \max(u_1, \dots u_n)$

Pf i) obvious

ii) Let B be any ball with $\overline{B} \subset \Omega$ and let h_i, h be harmonic on B with $h_i = u_i$, $h = u$ on ∂B. Since $h \geq h_i$ on ∂B we have $h \geq h_i$ on B.

Since u_i subharmonic, $h_i \geq u_i$ on B.

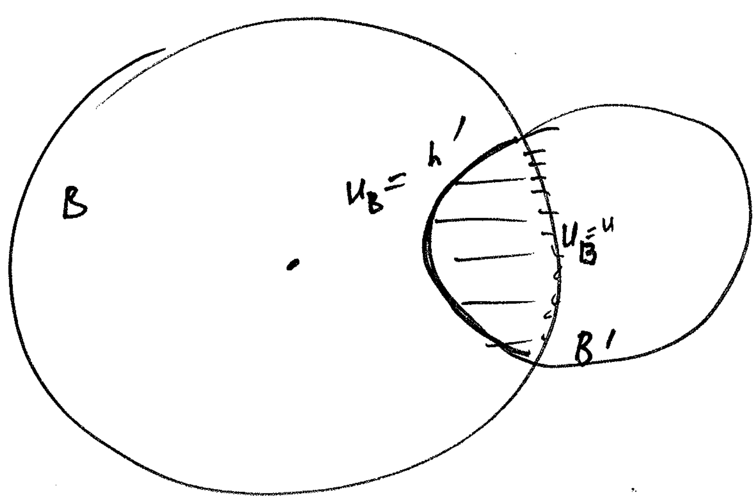
Thus $h \geq u_i$ on B $\forall i$ so $h \geq u$ on $\overline{B}$, i.e. u is subharmonic //

Def^n For u subharmonic in Ω and B a ball with $\overline{B} \subset \Omega$, define

$$u_B = \begin{cases} u & \text{in } \Omega - \overline{B} \\ h & \text{in } B \\ \Delta h = 0 & h = u \text{ on } \partial B \end{cases}$$

Lemma 3 u_B is subharmonic in Ω

Proof Let B' be any ball with $\overline{B'} \subset \Omega$ and let h' be harmonic in B' with $h' = u_B$ on $\partial B'$. We may assume $B \cap B' \neq \emptyset$ otherwise $u_B = u \leq h'$ on B'



Note that in $B' \cap B$, $u_B \geq u$ since u is subharmonic, u_B is harmonic and $u_B = u$ on ∂B
~~$u_B = u$ on $\partial B \cap B'$, $u_B = h'$~~ (so $u_B \geq u$ in B)

But $u_B = u$ in $\Omega' - B$,

Hence $u_B \geq u$ in B'

Therefore $h' \geq u_{B'}$ in B'

(since $h' = u_B \overset{\geq u}{}$ on $\partial B'$ and
$u_{B'} = u$ on $\partial B'$
$\Delta h' = \Delta u_{B'} = 0$ in B')

For points in $\overline{B'} - B$, $u_B = u \leq u_{B'} \leq h'$.

In particular, $u_B \leq h'$ on $\partial B \cap B'$.

Also, $u_B = h'$ on $\partial B' \cap B$

Thus $u_B \leq h'$ on $\partial(B \cap B')$

hence $u_B \leq h'$ in $B \cap B'$

So $\quad u_B = u \leq h' \quad$ in $B' \setminus \bar{B}$

$u_B \leq h' \quad$ in $B \cap B'$

$\Rightarrow \quad u_B \leq h'$ in B',

i.e. $\quad u_B$ subharmonic. //

Now let ϕ be continuous on $\partial\Omega$

and set

$$S(\phi) = \{ u \in C^0(\bar{\Omega}) : u \text{ subharmonic in } \Omega, \; u \leq \phi \text{ on } \partial\Omega \}$$

<u>Theorem</u> The function $u(x) = \sup\limits_{v \in S(\phi)} v(x)$

is harmonic in Ω.

<u>Proof</u>. By the maximum principle, any $v \in S(\phi)$ satisfies

$v \leq \sup\limits_{\partial\Omega} \phi$ so u is well-defined.

Then $\inf\limits_{\partial\Omega} u \leq u \leq \sup\limits_{\partial\Omega} \phi$

<u>Claim</u>: u is continuous on Ω.

To see this, let B_δ be a ball

s.that $\bar{B}_\delta \subset \Omega$ and let

$x' \in B_{\delta/2}$ and $\varepsilon > 0$ be given.

By definition of u $\exists\, v \in S(\phi)$

so that $v(x') > u(x') - \varepsilon/2$.

Let $w = \max(\inf_{\partial\Omega} \phi, v) \in S(\phi)$

(Lemma 2). Clearly,

$$w(x') > u(x') - \varepsilon/2$$

Hence

$$W_{B_\delta}(x') > u(x') - \varepsilon/2$$

By Theorem , $|DW_{B_\delta}| \le \frac{C}{\delta}$ in $B_{\delta/2}$

so $\forall x \in B_{\delta/2}$,

$$|W_{B_\delta}(x) - W_{B_\delta}(x')| \leq \frac{C}{\delta}|x-x'|$$

So if $x \in B_{\delta/2}$ and $|x-x'| < \frac{\delta}{2C}\varepsilon$,

we have

$$u(x') - \varepsilon/2 < W_{B_\delta}(x') \leq W_{B_\delta}(x) + \varepsilon/2$$

$$\leq u(x) + \varepsilon/2 \quad \text{(by defn' of } u$$

$$\{\text{since } w \in S(\phi) \Rightarrow W_{B_\delta} \in S(\phi)\,)$$

Combining gives

$$u(x') \leq u(x) + \varepsilon$$

for $x, x' \in B_{\delta/2}$, $|x-x'| < \frac{\varepsilon\delta}{2C}$

Reversing x, x' $\Rightarrow$

$|u(x) - u(x')| < \varepsilon$ so

u is continuous on $B_{\delta/2}$.

Since $B_{\delta/2}$ arbitrary, u is continuous on Ω.

To prove u is harmonic on Ω, let B be any ball with $\bar{B} \subset \Omega$, and let $\varepsilon > 0$ be given

For each $x_0 \in \bar{B}$ $\exists$ $v \in S(\phi)$ with

$$u(x_0) - \varepsilon/2 < v(x_0)$$

By continuity of u, v,

$$u(x) - \varepsilon < v(x) \quad \text{on } B(x_0)$$

By compactness of $\bar{B}$ we can cover $\bar{B}$ by a finite number of such (open) balls with centers $x_1, \ldots, x_k$. Let $v_1, \ldots, v_k$ be the corresponding subharmonic functions in $S(\phi)$ s.t.

$$u(x) - \varepsilon < v_i(x) \quad \text{in } B(x_i), \quad i = 1, \ldots, k$$

Then $v = \max(v_1, \ldots, v_k) \in S(\phi)$ and

$$u(x) - \varepsilon \leq v(x) \quad \forall x \in \bar{B}$$

Then $v_{\bar{B}} \in S(\phi)$ and $v_{\bar{B}} \geq v$ so

$$u(x) - \varepsilon \leq v_B(x) \leq u(x) \quad \forall x \in \bar{B}$$

Thus u can be uniformly approximated by harmonic functions on $\bar{B}$. By Corollary ( ) u is harmonic on B. Since B arbitrary, u is harmonic on Ω. //

So ~~far~~ Ω was any bounded open connected set. To prove u is continuous on $\bar{\Omega}$ and agrees with ϕ on $\partial\Omega$, we need some assumptions on Ω.

Before we prove this we note

<u>Proposition</u> If the Dirichlet problem is solvable for ϕ the Perron solution u must be the solution.

<u>Pf</u> If $\exists$ solution u_1, then $u_1 \in S(\phi)$ so $u_1 \le u$. On the other hand, $\forall v \in S(\phi)$ $v \le u_1$ so $u \le u_1$

Thus $u \equiv u_1$. //

<u>Definition</u> Let $x_0 \in \partial\Omega$. We say w is a <u>barrier function for Ω at x_0</u> if w continuous on $\bar{\Omega}$, w superharmonic on Ω, $w > 0$ at and $\bar{\Omega} - \{x_0\}$ and $w(x_0) = 0$

<u>Theorem</u> The Dirichlet problem is solvable for arbitrary continuous ϕ iff $\exists$ barrier function for Ω at each $x_0 \in \partial\Omega$.

<u>Pf</u>. If the DP is always solvable, for $x_0 \in \partial\Omega$ consider $\phi(x) = |x - x_0|$. Then the resulting harmonic w is a barrier function for Ω at x_0. Conversely, let N be a nbhd of $x_1 \in \partial\Omega$ and let w be a barrier function for Ω at x_0. We choose N so small that $\phi(x) \leq \phi(x_0) + \varepsilon \quad \forall x \in \partial\Omega \cap N$.

On $\bar{\Omega} \sim N$, w has a strictly positive minimum. Thus for suitable $\lambda > 0$,

$$\phi(x_0) + \varepsilon + \lambda w \geq \phi \quad \text{on } \partial\Omega.$$

Then if $v \in S(\phi)$, $v \leq \phi$ on $\partial\Omega$ so

$$v(x) \leq \phi(x_0) + \varepsilon + \lambda w(x)$$

at each $x \in \bar{\Omega}$. Hence

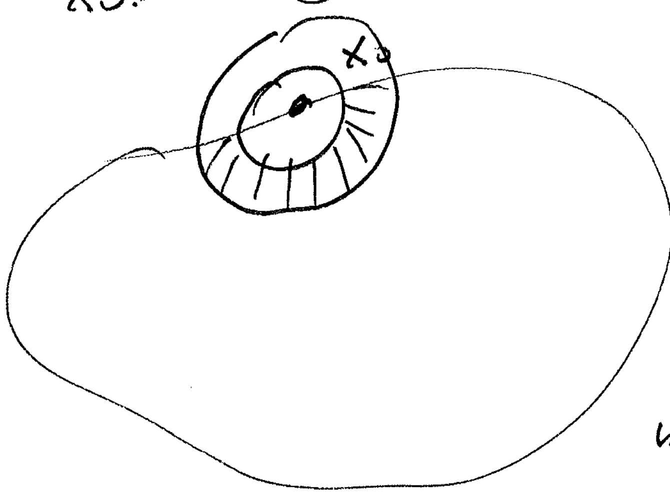
$$u(x) \leq \phi(x_0) + \varepsilon + \lambda w(x) \qquad x \in \bar{\Omega}$$

Observe that for $\lambda >> 1$,

$$\phi(x_0) - \varepsilon - \lambda w \in S(\phi)$$

Remark It suffices to construct "local barriers", that is we need only find a ball B about x_0 and w superharmonic in $B \cap \bar{\Omega}$ such that $w(x_0) = 0$, $w(x) > 0$ for $x \in B \cap \bar{\Omega}$, $x \neq x_0$.

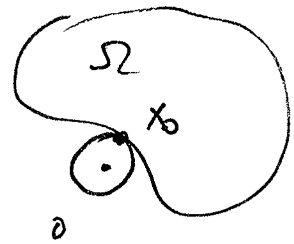
For let B' be a smaller ball about x_0. Set $m = \min\{ w(x) \mid x \in (B - B') \cap \bar{\Omega} \} > 0$.



Then
$$w' = \begin{cases} m & \Omega - B \\ \min(m, w) & B \end{cases}$$
is a barrier for Ω at x_0.

Existence of Barriers

1. (exterior ball condition) We say Ω satisfies exterior ball condition at $x_0 \in \partial\Omega$ if $\exists$ ball $B_R(0)$, $B_R(0) \cap \bar{\Omega} = \{x_0\}$



always true if $\partial\Omega \in C^2$ near x_0.

Take $$w = \begin{cases} R^{2-n} - r^{2-n} & n > 2 \\ -\log \frac{R}{r} \end{cases}$$

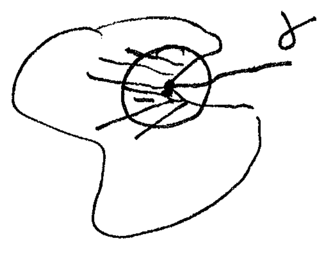
as barrier.

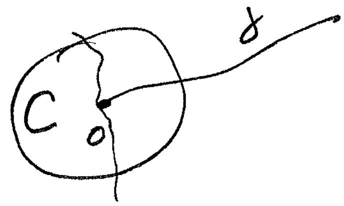
2. Theorem. If $n=2$ and if there is a connected set γ (containing more than one pt)

and $\gamma \cap \bar{\Omega} = \{x_0\}$, then $\exists$ barrier for Ω at x_0.

<u>Pf</u> The condition means x_0 is "accessible" from the outside (for example by a continuous path)

Take $x_0 = \{0\}$ and let C be a disk around 0 such that some point of γ is outside C.





In $C - \gamma$ one can define a single valued branch of $\arg z$

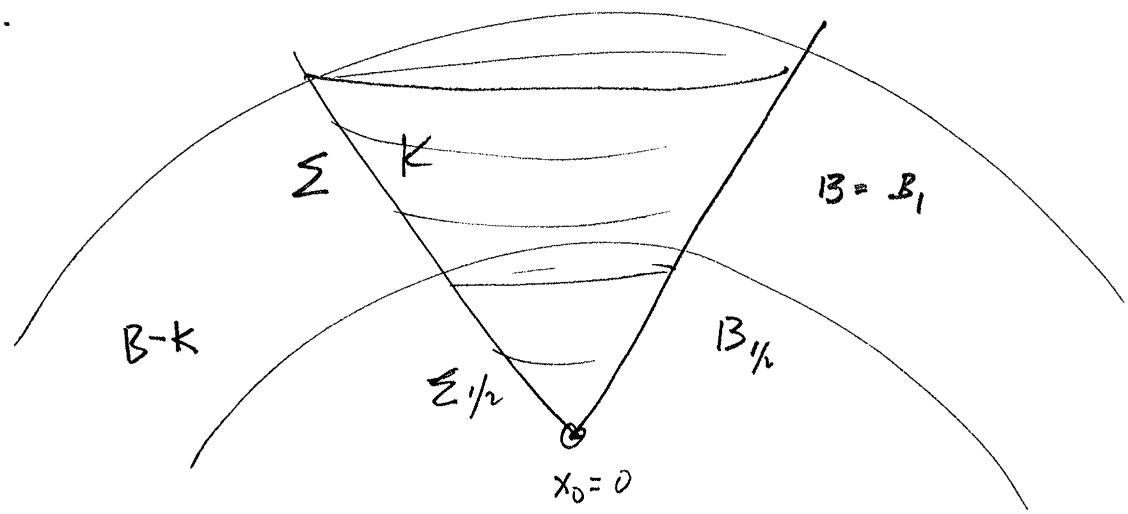
Set $w = -\mathrm{Re}\,\frac{1}{\log z}$ n

$$w = -\frac{\log r}{(\log r)^2 + \theta^2}.$$ Then $w > 0$ harmonic in $(\mathbb{C} - \gamma) \cap \overline{\Omega}$

is a local barrier, $w(0) = 0$

3. $n \geq 3$ <u>Theorem</u> If there is a cone with vertex $x_0 \in \partial\Omega$ lying outside $\overline{\Omega} \setminus \{x_0\}$, there is a barrier at x_0.

<u>Pf</u>



On $B-K$ consider boundary values $\phi = r = |X|$. and let $v(x)$ be the Perron solution. $\Delta v = 0$ in $B-K$

$v = |X|$ on $\partial(B \sim K)$ except possibly at $x=0$ since we have an exterior ball condition. Clearly $0 < v \le 1$

and $v < 1$ in the interior of $B-K$

Let $m = \sup_{\partial B_{1/2} \cap (B-K)} v < 1$

On $\Sigma_{1/2} - \{0\}$, $v(X) = \frac{1}{2} v(2X)$

$\Rightarrow$ $v(x) < c\, v(2X)$ $X \in \partial(B_{1/2} - K)$

where $c = \max(\frac{1}{2}, m)$

except possibly at $x=0$