Math 417 Midterm 2 Solutions

1. Consider the second order differential equation

$$xu''(x) + 2u'(x) + (\lambda - x)u(x) = 0,\ 1 < x < 2,\ u(1) = u(2) = 0\ .$$

a. (10pts) Put the equation in Sturm-Liouville form.

The integrating factor $\mu(x)$ must solve

$$x\mu' + \mu = 2\mu,\ \frac{\mu'}{\mu} = \frac{1}{x},\ \mu(x) = x\ ,$$

so SL form: $(x^2u')' + (-x^2 + x\lambda)u = 0$.

b. (10pts) Write out the orthogonality condition for the eigenfunctions.

$$\int_1^2 \phi_i(x)\phi_j(x)\ xdx = 0\ \text{ for } i \neq j\ .$$

c. (10pts) Show that all eigenvalues $\lambda > 0$.

Any eigenfunction pair $(\lambda, \phi(x))$ must satisfy

$$\lambda = \frac{\int_1^2 (x^2\phi'^2 + x^2\phi^2)\ dx}{\int_1^2 \phi^2\ xdx} > 0$$

since ϕ is not identically zero.

2. Consider the boundary value problem
$y''(x) + 4y(x) = f(x),\ 0 < x < \pi,\ y(0) = y(\pi) = 0$.

a. (15pts) Show that a necessary condition for a solution is that

$$\int_0^\pi f(x)\sin 2x\ dx = 0$$

.

Note that $\sin 2x$ is an eigenfunction (i.e, $L(\sin 2x) = 0$) so integrating the Lagrange identity yields

$$\int_0^\pi ((\sin 2x)Ly - yL(\sin 2x))\ dx = 0\ \text{ or }\ \int_0^\pi f(x)\sin 2x\ dx = 0\ .$$

b.(15pts) Assuming this condition, find the solution by the method of eigenfunction expansion.

Write

$$f(x) = b_1 \sin x + \sum_{n=3}^{\infty} b_n \sin nx,\ \ y(x) = a_1 \sin x + \sum_{n=3}^{\infty} a_n \sin nx\ .$$

Then the a_n must satisfy $3a_1 = b_1$, $(4-n^2)a_n = b_n$, $n \geq 3$ or $a_1 = \frac{1}{3}b_1$, $a_n = \frac{b_n}{4-n^2}$, $n \geq 3$.

3a (20pts) . Find the Green's function for the problem

$$(xu')' = f(x),\ 1 < x < e,\ u(1) = 0,\ u(e) = 0\ ,$$

by direct construction from two linearly independent solutions. Hint: Find $u_1(x)$, $u_2(x)$ linearly solutions of $(xu')' = 0$ with $u_1(1) = u_2(e) = 0$

$(xu') = c$, $u = c\log x + d$ is the general solution to the homogeneous equation so take $u_1(x) = \log x$, $u_2(x) = \log x - 1$.

Now let $G(x, x_0) = \begin{cases} A\log x & 1 < x < x_0 \\ B(\log x - 1) & x_0 < x < e \end{cases}$

where $A\log x_0 = B(\log x_0 - 1)$, $\frac{B}{x_0} - \frac{A}{x_0} = \frac{1}{x_0}$.

The coefficients A, B are then $A = \log x_0 - 1$, $B = \log x_0$ so

$G(x, x_0) = \begin{cases} (\log x_0 - 1)\log x & 1 < x < x_0 \\ \log x_0(\log x - 1) & x_0 < x < e \end{cases}$

3b. (20 pts) Use the Green's function to find the explicit solution for $f(x) = x$. (You do not need to derive the formula for the representation of the solution in terms of the Green's function). Check directly that your solution is correct.

$$u(x) = \int_1^e G(y, x) f(y) dy\ .$$

$$u(x) = \{\int_1^x (\log x - 1)\log y\ ydy + \int_x^e \log x(\log y - 1)\ ydy$$

$$= (\log x - 1)(\frac{1}{2}y^2\log y - \frac{1}{4}y^2)|_1^x + \log x(\frac{1}{2}y^2\log y - \frac{3}{4}y^2)|_x^e$$

$$= (\log x - 1)(\frac{1}{2}x^2\log x - \frac{1}{4}x^2 + \frac{1}{4}) + \log x(-\frac{1}{4}e^2 - \frac{1}{2}x^2\log x + \frac{3}{4}x^2)$$

$$= \log x(\frac{1}{2}x^2 + \frac{1}{4}(1 - e^2)) - (\frac{1}{2}x^2\log x - \frac{1}{4}x^2 + \frac{1}{4})$$

$$= \frac{1}{4}\{(1 - e^2)\log x + (x^2 - 1)\}\ .$$

Check:

$$u(1) = u(e) = 0,\ u' = \frac{1}{4}((1-e^2)\frac{1}{x} + 2x),\ xu' = \frac{1}{4}((1-e^2) + 2x^2),\ (xu')' = \frac{1}{4}\cdot 4x = x\ .$$