Math 417 Practice Midterm 2,

1. a. Write the general form of the Sturm-Liouville differential equation $Lu = 0$ in one space dimension.

$$(p(x)u')' + (q(x) + \lambda\sigma(x))u = 0,\ a < x < b\ .$$

b. Write the corresponding Green's formula (Lagrange identity) for any two solutions $Lu = Lv = 0$, $a < x < b$.

$$\int_a^b (uLv - vLu)\ dx = p(x)(uv' - vu')|_a^b\ .$$

c. L is called self-adjoint for the boundary conditions if $\int_a^b (uLv - vLu)\ dx = 0$ for any two solutions $Lu = Lv = 0$ satisfying the boundary conditions
Show that the boundary conditions $u'(a) = 0$, $u'(b) = -hu(b)$ lead to a self-adjoint problem.

From part b, we need to show

$$p(b)(u(b)v'(b) - v(b)u'(b)) - p(a)(u(a)v'(a) - v(a)u'(a)) = 0\ .$$

But $p(b)(u(b)v'(b) - v(b)u'(b)) = p(b)(u(b)hv(b) - v(b)hu(b)) = 0$ and $p(a)(u(a)v'(a) - v(a)u'(a)) = p(a)(u(a)\cdot 0 - v(a)\cdot 0) = 0$.

2. Consider the second order differential equation

$$x^2u''(x) + 4xu'(x) + (\lambda - x^2)u(x) = 0,\ 1 < x < 2,\ u(1) = u(2) = 0\ .$$

a. Put the equation in Sturm-Liouville form.

We need an integrating factor $\mu(x)$ such that $x^2\mu(x) = p(x)$ and $4x\mu(x) = p'(x)$ so $x^2\mu'(x) + 2x\mu(x) = 4x\mu(x)$ or

$$\frac{\mu'(x)}{\mu(x)} = \frac{2}{x},\ \mu(x) = x^2,\ p(x) = x^4,\ q(x) = -x^4,\ \sigma(x) = x^2\ ,$$

$$(x^4u'(x))' + (\lambda - x^2)x^2u(x) = 0\ .$$

b. Write out the orthogonality condition for the eigenvalues.

$$\int_1^2 \phi_i(x)\phi_j(x)x^2\ dx = 0 \text{ for } i \neq j\ .$$

c. Show that all eigenvalues $\lambda > 0$.

We use the Rayleigh quotient formula for any eigenvalue λ obtained by multiplying the equation by ϕ and integrating by parts, i.e

$$0 = \int_a^b \phi\{(p(x)\phi')' + (q(x)+\lambda\sigma)\phi\}\, dx = \int_a^b \{-p(x)\phi'^2 + (q(x)+\lambda\sigma)\phi^2\}\, dx\ ,$$

where the boundary term is zero because $\phi(a) = \phi(b) = 0$. Hence,

$$\lambda = \frac{\int_a^b \{p(x)\phi'^2 - q(x)\phi^2\} dx}{\int_a^b \phi^2\sigma\ dx} > 0\ ,$$

since in this case $q(x) = -x^4 < 0$ on (1,2) and ϕ is not identically zero.

3. Consider the boundary value problem
$y''(x) + y(x) = f(x)$, $y(0) = y(\pi) = 0$.
a. Show that a necessary condition for a solution is that $< f, \sin x >=$.

Note that 1 (or 0 if you think of $y''_y + \lambda y = 0$)is an eigenvalue with eigenfunction $\sin x$ (with the Dirichlet BC). Hence using the Lagrange identity $0 = \int_)^\pi \sin x L y dx = \int_0^\pi f(x) \sin x\ dx$ since $L(\sin x) = 0$.

b. Assuming the orthogonality condition of part a., find the solution by the method of eigenfunction expansion.

The eigenvalues of $y'' + \lambda y = 0$, $y(0) = y(\pi) = 0$ are $(n\pi)^2$ with eigenfunctions $\sin nx$. Write $f(x) = \sum_{n=1}^\infty a_n \sin nx$ and note that $a_1 = 0$. The method of eigenfunction expansion looks for a solution in the form $y(x) = \sum_{n=2}^\infty b_n \sin nx$. Plugging into the equation gives

$$\sum_{n=2}^\infty b_n \cdot (1 - n^2) \sin nx = \sum_{n=2}^\infty a_n \sin nx\ ,$$

so $b_n = -\frac{a_n}{n^2-1}$.

4a. Show that the eigenvalue problem

$$e^{x^2}\phi'' + x\phi' + \lambda x^2\phi = 0,\ 1 < x < 2,\ \phi(1) = \phi(2) = 0$$

is a regular Sturm-Liouville eigenvalue problem and write down the orthogonality condition on the eigenfunctions.

Using the method of problem 2, the integrating factor μ satisfies

$$\frac{\mu'}{\mu} = xe^{-x^2} - 2x,\ \log\mu = -\frac{1}{2}e^{-x^2} - x^2,\ \ \mu = e^{-(x^2 + \frac{1}{2}e^{-x^2})}\ ,$$

so the SL form is $(e^{-\frac{1}{2}e^{-x^2}}\phi')' + \lambda x^2 e^{-(x^2+\frac{1}{2}e^{-x^2})}\phi = 0$. $p(x) = e^{-\frac{1}{2}e^{-x^2}} > 0$ and since we are on the interval (1,2), the weight $\sigma(x) = x^2 e^{-(x^2+\frac{1}{2}e^{-x^2})} > 0$ so the problem is regular. The orthogonality condition is

$$\int_1^2 \phi_i(x)\phi_j(x)x^2 e^{-(x^2+\frac{1}{2}e^{-x^2})}\ dx = 0 \text{ for } i \neq j\ .$$

b. It is known (see Haberman section 5.9) that for n large the large eigenvalues are asymptotically given by the formula

$$\lambda_n \approx \left(\frac{n\pi}{\int_1^2 \sqrt{\frac{\sigma(x)}{p(x)}}\ dx}\right)^2 .$$

Find the asymptotic value for λ_n.

From part a, $\sqrt{\dfrac{\sigma(x)}{p(x)}} = xe^{-\frac{1}{2}x^2}$, $\displaystyle\int_1^2 xe^{-\frac{1}{2}x^2}\ dx = -e^{-\frac{1}{2}x^2}|_1^2 = e^{-\frac{1}{2}} - e^{-2}$ so $\lambda_n \approx (\frac{n\pi}{e^{-\frac{1}{2}} - e^{-2}})^2$.

5a. Find the Green's function for the problem

$$u''(x) - u(x) = f(x),\ 0 < x < 1,\ u(0) = 0,\ u'(1) = 0\ ,$$

by direct construction from $u_1(x) = \sinh x$, $u_2(x) = \cosh(x-1) = \cosh 1 \cosh x - \sinh 1 \sinh x$.

Set $G(x, x_0) = \begin{cases} A\sinh x & 0 < x < x_0 \\ B\cosh(x-1) & x_0 < x < 1 \end{cases}$

Then we need A, B to satisfy

$$A\sinh x_0 = B\cosh(x_0 - 1),\ B\sinh(x_0 - 1) - A\cosh x_0 = 1\ .$$

The solution is $B = -\frac{\sinh x_0}{\cosh 1}$, $A = -\frac{\cosh x_0 - 1}{\cosh 1}$.

Hence $G(x, x_0) = \begin{cases} -\frac{\cosh x_0 - 1}{\cosh 1}\sinh x & 0 < x < x_0 \\ -\frac{\sinh x_0}{\cosh 1}\cosh(x-1) & x_0 < x < 1 \end{cases}$

b. Use the Green's function to find the explicit solution for $f(x) = x$. Check directly that your solution is correct.

$$u(x) = \int_0^1 G(x, y)f(y)dy = -\frac{1}{\cosh 1}\{\int_0^x \cosh(x-1)\sinh y\ ydy + \int_x^1 \sinh x \cosh(y-1)\ ydy\}$$

$= \frac{\sinh x}{\cosh 1} - x$ after simplification .

6. Consider the problem $y'' + k^2 y = f(x)$, $0 < x < \pi$, $y(0) = y(\pi) = 0$.
a. Find the eigenfunctions and eigenvalues of $y'' + k^2 y + \lambda y = 0$, $0 < x < \pi$, $y(0) = y(\pi) = 0$.

Set $\tilde{\lambda}_n = k^2 + \lambda_n = n^2$, $\lambda_n = n^2 - k^2$ with eigenfunctions $\phi_n = \sin nx$.

b. Express $f(x)$ as a Fourier series and solve the original problem. What assumptions are needed?

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x_0) \sin x_0 dx_0$.
Set $y(x) = \sum_{n=1}^{\infty} a_n \sin nx$. Then $(k^2 - n^2) a_n = b_n$, $a_n = \frac{b_n}{k^2 - n^2}$.
This is valid if k^2 is not an eigenvalue or if it is an eigenvalue we must have $\int_0^{\pi} f(y) \sin |k| y \, dy = 0$, i.e the Fredholm alternative.

c. Use your answer from part b. to immediately write down the Green's function $G(x, x_0)$ for the original problem.

$G(x, x_0) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx_0 \sin nx}{k^2 - n^2}$.