MATH 417 PRACTICE MIDTERM 2,

1. a. Write the general form of the Sturm-Liouville differential equation $Lu = 0$ in one space dimension.
b. Write the corresponding Green's formula (Lagrange identity) for any two solutions $Lu = Lv = 0$, $a < x < b$.
c. L is called self-adjoint for the boundary conditions if $\int_a^b (uLv - vLu)\, dx = 0$ for any two solutions $Lu = Lv = 0$ satisfying the boundary conditions
Show that the boundary conditions $u'(a) = 0$, $u'(b) = -hu(b)$ lead to a self-adjoint problem.

2. Consider the second order differential equation

$$x^2u''(x) + 4xu'(x) + (\lambda - x^2)u(x) = 0,\ 1 < x < 2,\ u(1) = u(2) = 0\ .$$

a. Put the equation in Sturm-Liouville form.
b. Write out the orthogonality condition for the eigenvalues.
c. Show that all eigenvalues $\lambda > 0$.

3. Consider the boundary value problem
$y''(x) + y(x) = f(x)$, $y(0) = y(\pi) = 0$.
a. Show that a necessary condition for a solution is that $< f, \sin x >=$.
b. Assuming the orthogonality condition of part a., find the solution by the method of eigenfunction expansion.

4a. Show that the eigenvalue problem

$$e^{x^2}\phi'' + x\phi' + \lambda x^2\phi = 0,\ 1 < x < 2,\ \phi(1) = \phi(2) = 0$$

is a regular Sturm-Liouville eigenvalue problem and write down the orthogonality condition on the eigenfunctions.
b. It is known (see Haberman section 5.9) that for n large the large eigenvalues are asymptotically given by the formula

$$\lambda_n \approx \left(\frac{n\pi}{\int_1^2 \sqrt{\frac{\sigma(x)}{p(x)}}\, dx}\right)^2 .$$

Find the asymptotic value for λ_n.

5a. Find the Green's function for the problem

$$u''(x) - u(x) = f(x),\ 0 < x < 1,\ u(0) = 0,\ u'(1) = 0\ ,$$

by direct construction from $u_1(x) = \sinh x$, $u_2(x) = \cosh(x-1) = \cosh 1 \cosh x - \sinh 1 \sinh x$.

b. Use the Green's function to find the explicit solution for $f(x) = x$. Check directly that your solution is correct.

6. Consider the problem $y'' + k^2 y = f(x)$, $0 < x < \pi$, $y(0) = y(\pi) = 0$.
a. Find the eigenfunctions and eigenvalues of $y'' + k^2 y + \lambda y = 0$, $0 < x < \pi$, $y(0) = y(\pi) = 0$.
b. Express $f(x)$ as a Fourier series and solve the original problem. What assumptions are needed?
c. Use your answer from part b. to immediately write down the Green's function $G(x, x_0)$ for the original problem.