Homework 8 Solutions

9.3.1 $u'' = f(x)$, $0 < x < L$, $u(0) = u(L) = 0$.

$$G(x, x_0) = \begin{cases} \frac{-x(L-x_0)}{L} & 0 < x < x_0 \\ \frac{-x_0(L-x)}{L} & x_0 < x < L \end{cases}$$

Then $G(x, x_0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$ where

$$A_n = -\frac{2}{L^2}\{(L - x_0)\int_0^{x_0} x \sin \frac{n\pi x}{L}\ dx + x_0 \int_{x_0}^{L} (L - x) \sin \frac{n\pi x}{L}\ dx\} = -\frac{2}{L} \frac{\sin \frac{n\pi x_0}{L}}{(\frac{n\pi}{L})^2}$$

after integration by parts and simplification.

9.3.3 a,b Start with the homogeneous problem

$$w_t = kw_{xx}, 0 < x < L,\ t > 0,\ w(0, t) = w'(L, t) = 0.$$

Separation of variables leads to the eigenvalue problem

$$\phi'' + \lambda\phi = 0,\ 0 < x < L,\ \phi(0) = 0,\ \phi'(L) = 0.$$

The solution is $\phi_n(x) = \sin \sqrt{\lambda_n} x$, $\lambda_n = (\frac{\pi}{L})^2 (n - \frac{1}{2})^2$, $n = 1, 2, \ldots$.
The method of eigenfunction expansion seeks a solution of the original problem by decomposing $Q(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$ where

$$q_n(t) = \frac{2}{L}\int_0^L Q(y, t) \sin \frac{\pi}{L}(n - \frac{1}{2})y\ dy$$

(since it still is true that $\int_0^L \phi_n(x)\phi_m(x)dx = \frac{L}{2}\delta_{mn}$.)

$$\text{Set } u(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) = \sum_{n=1}^{\infty} a_n(t) \sin \sqrt{\lambda_n} x\ ,$$

Plugging these series into the equation gives and ode for $a_n(t)$:
Using the method of Duhamel, we find

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(s)e^{-k\lambda_n(t-s)}\ ds$$

$$= a_n(0)e^{-k\lambda_n t} + \frac{2}{L}\int_0^t \int_0^L e^{-k\lambda_n(t-s)} Q(y, s) \sin \sqrt{\lambda_n} y\ dyds$$

where we must take

$$a_n(0) = \frac{2}{L}\int_0^L g(t) \sin \sqrt{\lambda_n} t dt\ .$$

to satisfy the initial conditon $u(x,0) = g(x)$.

b. Putting all of this together gives the Green's function $\tilde{G}(x,t;x_0,t_0)$ for the time dependent problem. Here we have both space and time parameters (x_0,t_0). Set $Q(x,t) = \delta(x-x_0)\delta(t-t_0)$, $\tilde{a}_n(0) = 0$ and find

$$\tilde{a}_n(t) = \frac{2}{L} e^{-k\lambda_n(t-t_0)} \sin\sqrt{\lambda_n} x_0 \ .$$

Then

$$\tilde{G}(x,t;x_0,t_0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\sqrt{\lambda_n} x_0 \sin\sqrt{\lambda_n} x, \ t > t_0 \ .$$

9.3.6a We look for

$$G(x,x_0) = \begin{cases} ax+b & 0 < x < x_0 \\ cx+d & x_0 < x < L \end{cases}$$

Using the boundary conditions $G(0,x_0) = G_x(L,x_0) = 0$, we must have $b = c = 0$. Now we use continuity and the jump condition at x_0:

$$0 = G(x_0^+,x_0) - G(x_0^-,x_0) = d - ax_0, \ 1 = G_x(x_0^+,x_0) - G_x(x_0^-,x_0) = 0 - a$$

to conclude $a = -1$, $d = -x_0$. Hence the desired solution is

$$G(x,x_0) = \begin{cases} -x & 0 < x < x_0 \\ -x_0 & x_0 < x < L \end{cases}$$

9.3.11a $G_{xx} + G = \delta(x-x_0)$, $G(0,x_0) = 0$, $G(L,x_0) = 0$.
If $L = n\pi$ then 1 is an eigenvalue with eigenfunction $\sin x$ and as we have seen in class the direct method of construction fails as does the method of eigenfunction expansion. Below we see that $\sin L$ appears in the denominator. G must be of the form

$$G(x,x_0) = \begin{cases} A\sin x & 0 < x < x_0 \\ B\sin(x-L) & x_0 < x < L \end{cases}$$

Continuity at x_0 gives $A\sin x_0 = B\sin(x_0 - L)$ and the jump condition gives $1 = B\cos(x_0 - L) - A\cos x_0$ which implies (using the standard trig identities)
$A = \frac{\sin(x_0-L)}{\sin L}$, $B = \frac{\sin x_0}{\sin L}$, hence

$$G(x,x_0) = \begin{cases} \frac{\sin(x_0-L)\sin x}{\sin L} & 0 < x < x_0 \\ \frac{\sin x_0 \sin(x-L)}{\sin L} & x_0 < x < L \end{cases}$$

9.3.14b $Lu := (pu')' + qu = f(x)$, $u'(0) = \alpha$, $u'(L) = \beta$.

$$G(x,x_0) = \begin{cases} A\sin x & 0 < x < x_0 \\ B\sin(x-L) & x_0 < x < L \end{cases}$$

Let $G(x,x_0)$ be the Green's function satisfyng

$$LG = p(G_x)_x + qG = \delta(x-x_0),\ G_x(0,x_0) = G_x(L,x_0) = 0\ .$$

Using the Lagrange identity we find

$$\int_0^{x_0} GLu\ dx = p(x)(Gu' - uG_x)_0^{x_0^-}\ ,$$

$$\int_{x_0}^{L} GLu\ dx = p(x)(Gu' - uG_x)_{x_0^+}^{L}\ .$$

Adding gives

$$\int_0 G(x,x_0)f(x)\ dx = (p(L)G(L,x_0)\beta - p(0)G(0,x_0)\alpha) + p(x_0)u(x_0)(G_x(x_0^+,x_0) - G_x(x_0^-,x_0)$$

$$= u(x_0) - (\alpha p(0)G(0,x_0) - \beta p(L))G(L,x_0))\ .$$

Hence $u(x_0) = (\alpha p(0)G(0,x_0) - \beta p(L)) + \int_0 G(x,x_0)f(x)\ dx$.

9.3.15a

$$LG = p(G_x)_x + qG = \delta(x-x_0),\ G(0,x_0) = G(L,x_0) = 0\ .$$

We assume we have linearly independent solutions $Ly_i = 0$, $y_i(0) = 0$, $y_i'(0) = 1$, $i = 1,2$. We look for G of the form

$$G(x,x_0) = \begin{cases} Ay_1(x) & 0 < x < x_0 \\ By_2(x) & x_0 < x < L \end{cases}$$

Then $Ay_1(x_0) - By_2(x_0) = 0$, $By_2'(x_0) - Ay_1'(x_0) = \frac{1}{p(x_0)}$. Solving $A = \frac{y_2(x_0)}{p(x_0)W(y_1,y_2)(x_0)}$, $\frac{y_1(x_0)}{p(x_0)W(y_1,y_2)(x_0)}$ where $W(y_1,y_2)(x)$ is the Wronskian which satisfies $p(x)W(y_1,y_2)(x) \equiv k$ constant. Hence,

$$G(x,x_0) = \begin{cases} \frac{1}{k}y_2(x_0)y_1(x) & 0 < x < x_0 \\ \frac{1}{k}y_1(x_0)y_2(x) & x_0 < x < L \end{cases}$$

9.3.18 $Lu = f(x)$, $u(0) = u(L) = 0$.
Let $u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$ where $L\phi_n + \lambda_n\sigma\phi_n = 0$, $\phi_n(0) = \phi_n(L) = 0$.
Then $Lu = -\sum_{n=1}^{\infty} a_n\lambda_n\sigma(x)\phi_n(x) = f(x)$. Hence

$$-a_n\lambda_n = \frac{\int_0^L f(x_0)\phi_n(x)dx}{\int_0^L \phi_n^2(x_0)\sigma(x_0)dx_0},\ u(x) = \int_0^L f(x_0)\sum_{n=1}^{\infty}\left(\frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n\int_0^L \phi_n^2(x)\sigma(x)dx}\right)dx_0$$

$$= \int_0^L f(x_0)G(x,x_0)dx_0,\ G(x,x_0) = \sum_{n=1}^{\infty}\frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n\int_0^L \phi_n^2(t)\sigma(t)dt}\ .$$