Review problems for Midterm 1

1. Review the solution from this week lecture of the simple transport equation

$$u_t + bu_x = 0 \ , \ u(x,0) = g(x) \ .$$

Now figure out how to solve (by a change of dependent variable u(x,t) goes to v(x,t))

$$u_t + bu_x + cu = 0 \ , \ u(x,0) = g(x) \ .$$

Here b and c are constants.

2. Solve the inhomogeneous heat equation $u_t = u_{xx} - \sin x$ on $(0, \pi)$ with

$$u(0,t) = 0 \ , \ u_x(\pi, t) = 0 \ , \ u(x,0) = 2\sin 2x - x \ .$$

3. A thin rectangular plate bounded by the lines x=0, x=a, y=0, y=b whose surface is impervious to heat flow is given an initial temperature distribution $\sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. Its four edges are kept at zero temperature. Find the temperature distribution at later times.

4. Find the steady state temperature $u(r,\theta)$ of a thin plate over the sector

$$\Omega = \{(r,\theta) : 0 < r < 1 \ , \ 0 < \theta < \frac{\pi}{3}\}$$

given that $u(r,0) = 0$, $u_\theta(r, \frac{\pi}{3}) = 0$ and $u(1,\theta) = \sin \frac{3}{2}\theta$. You may assume that u is bounded.

5. Find the solution of Laplace's equation in a disk of radius R with boundary values 1 on the upper semicircle and 0 on the lower semicircle boundaries.

6. Let $f(x)$ be defined on the real line by $f(x) = \begin{cases} 2 - x & \text{if } 0 < x < 3 \\ 0 & \text{if } -3 < x < 0 \\ f(x+6) = f(x) & \text{otherwise} \end{cases}$

Find and plot the Fourier series of f on (-6,6) .

7. Find a cosine series to represent $f(x) = e^x$ in $0 \le x < \pi$. Sketch the series over the range $(-2\pi, 2\pi)$.

8. Solve by separation of variables:

$$u_{tt} = u_{xx} \ , \ 0 < x < \pi \ , \ t > 0 \ , \text{ where}$$

$$u_x(0,t) = u_x(\pi,t) = 0 \ , \ u(x,0) = x \ , \ u_t(x,0) = 0 \ .$$

9. Solve by separation of variables:

$$u_{tt} + 2u_t = 4u_{xx} \ , \ 0 < x < \pi \ , \ t > 0 \ , \text{ where}$$

$$u(0,t) = u(\pi,t) = 0 \ , \ u(x,0) = 0 \ , \ u_t(x,0) = 1 \ .$$

10. Consider the inhomogeneous heat equation:

$$u_t = u_{xx} + Q(x,t) \ , \ 0 < x < \pi \ , \ t > 0 \text{ with } u(x,0) = u(0,t) = u(\pi,t) = 0 \ .$$

Solve using Duhamel's principle, that is,

$$u(x,t) = \int_0^t w(x,t;\tau) \ d\tau \ , \text{ where}$$

$$w(x,\tau) = Q(x,\tau) \ , \ w(0,t) = w(\pi,t) = 0 \ .$$

Solve for w by separation of variables.