Math 417 Midterm 1 solutions

1. Suppose that $u(x,t)$ satisfies the inhomogeneous heat equation

$$u_t = u_{xx} - 2 \quad \text{for } 0 < x < 2,\ t > 0$$

with boundary conditions $u(0,t) = 4$, $u(2,t) = 0$ and initial condition $u(x,0) = 4 + x^2$.

a. (10pts) Find the steady state temperature (say $v(x)$).

$v_x x = 2$, $v(0) = 4$, $v(2) = 0 \implies v(x) = (x-2)^2$.

b. (15pts) Find $u(x,t)$.

Let $\tilde{u} = u(x,t) - v(x),$. Then

$$\tilde{u}_t = \tilde{u}_{xx} = 0,\ \tilde{u}(0,t) = \tilde{u}(2,t) = 0,\ \tilde{u}(x,0) = 4 + x^2 - (x-2)^2 = 4x\ .$$

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(\frac{n\pi}{2})^2 t}\ ,$$

where $B_n = \int_0^2 4x sin\frac{n\pi x}{2}\ dx = \frac{16}{n\pi}(-1)^{n+1}$. Therefore,

$$u(x,t) = (x-2)^2 + \sum_{n=1}^{\infty} \frac{16}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{2} e^{-(\frac{n\pi}{2})^2 t}$$

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2. Let $f(x) = (x-1)^2$ for $0 \le x \le 1$.

a. (11pts) Compute the Fourier cosine series of $f(x)$.

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$ where

$$a_0 = \int_0^1 (x-1)^2\ dx = \frac{1}{3} \quad \text{and}$$

$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x\ dx = \frac{4}{(n\pi)^2}\ .$$

b.(7pts) Draw a careful graph of the function to which your series converges to on the interval $-2 \le x \le 2$. (Note that *I am not* asking you to graph the Fourier cosine series!)

Extend $f(x)$ to (-1,0) as an even function $\tilde{f}$, i.e. $\tilde{f}(x) = f(-x)$ for $x \in (-1,0)$ and then extend $\tilde{f}$ everywhere as a periodic function of period 2. The fourier cosine series converges to this extension.

c. (7pts) *Do not compute* the Fourier sine series of $f(x)$ but draw a careful graph of the function to which the Fourier sine series converges on the interval $-2 \le x \le 2$. (Note that *I am not* asking you to graph the Fourier sine series!)

Similarly extend $f(x)$ to (-1,0) as an odd function $g(x)$, i.e. $g(x) = -g(-x)$ for $x \in (-1,0)$ and then extend g everywhere as a periodic function of period 2. The fourier sine series converges to this extension.

3. (20pts) Solve $\Delta u = 0$ in the unit disk $B_1(0) \subset R^2$ (using Fourier series in polar coordinates) with boundary condition $u(1,\theta) = |\theta|$, $-\pi \leq \theta \leq \pi$.

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \ .$$

When $r = 1$, $u = |\theta|$ on $(-\pi, \pi)$ (and extended periodic) which is even so $B_n = 0$ and

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_0^\pi \theta \ d\theta = \frac{\pi}{2} \ , \\ A_n &= \frac{2}{\pi}\int_0^\pi \theta \cos n\theta \ d\theta \\ &= \frac{2}{\pi}\{\theta \frac{\sin n\theta}{n}|_0^\pi - \int_0^\pi \frac{\sin n\theta}{n} \ d\theta\} \\ &= \frac{2}{\pi}\frac{\cos n\theta}{n^2}|_0^\pi = \frac{2}{n^2\pi}\left\{ \begin{array}{ll} 0 & \text{if n even} \\ -2 & \text{if n odd} \end{array} \right. \end{aligned}$$

Hence

$$u(r,\theta) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\text{n odd}} r^n \frac{\cos n\theta}{n^2} \ .$$

4. (30pts) Solve using the D'Alembert formula and Duhamel's principle:

$u_{tt} = u_{xx} + xt$ on $-\infty < x < \infty$, $t > 0$, $u(x,0) = e^x$, $u_t(x,0) = 0$.

By superposition and D'Alembert, $u(x,t) = \frac{1}{2}(e^{x+t} + e^{x-t}) + v(x,t)$ where v satisfies

$$v_{tt} = v_{xx} + xt \ , \ v(x,0) = v_t(x,0) = 0 \ .$$

By Duhamel's principle, $v(x,t) = \int_0^t w(x,t;s) \ ds$, where

$$w_{tt} = w_{xx} \ , \ w(x,s) = 0 \ , \ w_t(x,s) = xs \ .$$

Hence

$$w(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} sy \ dy = \frac{s}{4}[(x+(t-s))^2 - (x-(t-s))^2] = sx(t-s) \ .$$

This gives

$$v(x,t) = \int_0^t xs(t-s) \ ds = x\int_0^t (ts - s^2) \ ds = x(\frac{t^3}{2} - \frac{t^3}{3}) = x\frac{t^3}{6} \ .$$