Math 417 Take-home Final Exam, Spring 2015

This is an open book exam meaning that you may use class notes, text and homework, but no collaboration and no internet. It should be returned to me by noon on Monday May 11.

1. (30pts) Let $u(x,t)$ denote the concentration of a chemical which satisfies

$$u_t = 2u_{xx} + \frac{2}{5}x, \qquad 0 < x < 4,\ t > 0$$

$$u_x(0,t) = 0,\ u_x(4,t) = 1,\ u(x,0) = 2x\ .$$

Let $M(t) := \int_0^4 u(x,t)dx$ denote the total amount of chemical at time t.
a. (10pts) What is the physical meaning of the boundary condition $u_x(4,t) = 1$?
b. (10pts) Compute $\frac{dM}{dt}(t)$ in simplest form.
c. (10pts) Find M(t).

2. (40pts) Solve the heat equation $u_t = u_{xx} + u_{yy}$ on a thin $45°$ sector of radius 1, i.e $0 < r < 1$, $0 < \theta < \frac{\pi}{4}$ with boundary conditions

$$u(r,0,t) = u(r,\frac{\pi}{4},t) = 0, u_r(1,\theta,t) = 0,\ 0 < \theta < \frac{\pi}{4},\ u(r,\theta,0) = F(r,\theta)$$

by separation of variables $u = h(t)\phi(r)g(\theta)$.
Hint: Review similar problems in a circular domain (in section 7.7 of the text) involving Bessel's equation and modify the eigenvalue problem for $g(\theta)$ to fit the given geometry.

3a. (15pts) Use the method of characteristics to find the general solution of

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0\ ,\ \ -\infty < x < \infty\ ,\ t > 0\ .$$

Hint: The equation can be "factored" as $(\partial_t - \partial_x)(\partial_t + 3\partial_x)u = 0$. Follow the method for the wave equation using my lectures or section 12.3 of the text.

3b. (15pts) Use part a. to find the (D'Alembert) solution of

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0\ ,\ \ -\infty < x < \infty\ ,\ t > 0\ .$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$