Chapter 3 : 1. Line integrals

Complex

A parametrized curve is a function $z(t)$, $a \le t \le b$, which maps $[a, b] \to \mathbb{C}$

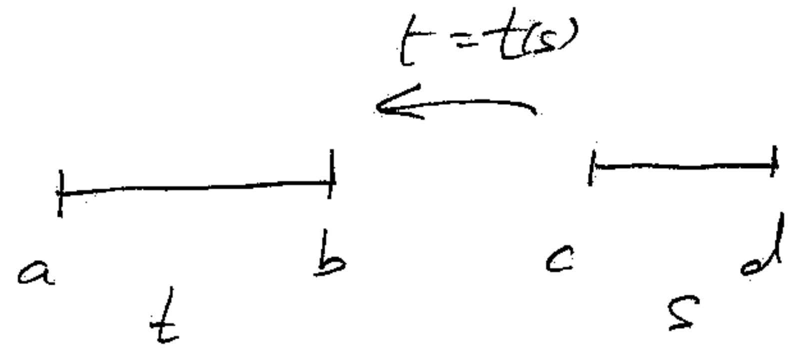
We say this curve is C^1 if z is continuously differentiable with $z'(t) \neq 0$ (regularity condition).

(At the endpts $t = a, b$, $z'(a), z'(b)$ are one-sided derivatives)

Similarly we say the parametrized curve is <u>piecewise smooth</u> if there is a partition $a = a_0 < a_1 < \dots < a_n = b$ where z is C' on (a_{i-1}, a_i) and the left and right hand derivatives exist at $t = a_i$ $(1 \le i \le n-1)$ but may be different

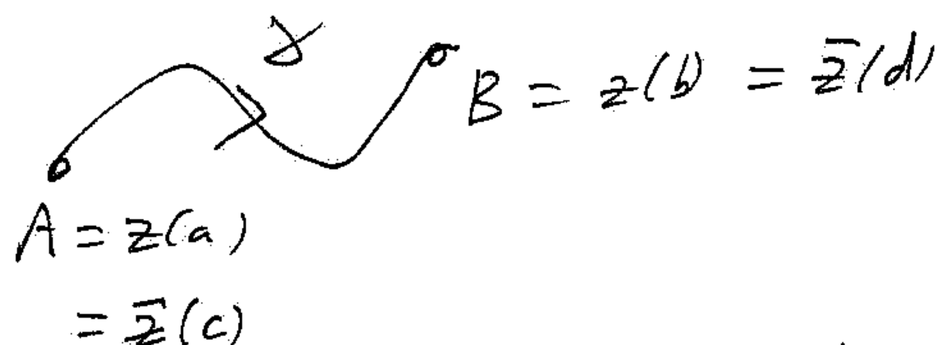
Def'n We say two parametrizations $z : [a,b] \to \mathbb{C}$ and $\overline{z} : [c,d] \to \mathbb{C}$ are <u>equivalent</u> if there exist a C^1 <u>bijection</u> $s \to t(s)$ from $[c,d]$ to $[a,b]$ so that $t'(s) > 0$ and

$$\overline{z}(s) = z(t(s))$$



The condition $t'(s) > 0$ says precisely that the orientation of the curve is preserved;

as s moves from c to d, $t(s)$ moves from a to b. Thus $\bar{z}(s)$ and $z(t)$ parametrize



the image γ in the same order.

given a parametrization of γ by $z(t)$ we can always "reverse the orientation" to obtain γ^- going from B to A

by defining $z^-(t) = z(b+a-t)$

The above notion extends to

piecewise C^1 curves.

<u>Defn</u> 1. We say a piecewise

~~smooth~~ C^1 curve is <u>closed</u>

if $z(a) = z(b)$ for any

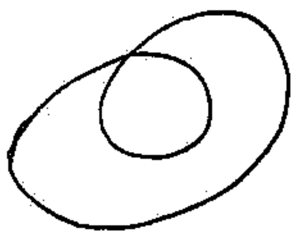
of its ('equivalent') parametrizations

(It is smooth if $z'(a) = z'(b)$)

2. We say a piecewise C^1

curve is <u>simple</u> if it has no <u>self-intersections</u>, i.e

$z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$.



closed but not simple.

<u>Example</u> A basic example is the circle $C_r(z_0)$, center z_0 and radius r. As a set,

$$C_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| = r \}$$

The positive orientation (counterclockwise) of $C_r(z_0)$ is given by the standard param.

$$z(t) = z_0 + re^{it} \quad 0 \le t \le 2\pi$$

while the negative orientation (clockwise) is given by $z(t) = z_0 + re^{-it}$

$$0 \le t \le 2\pi .$$

Given a C^1 curve γ in $\mathbb{C}$ parametrized by $z : [a,b] \to \mathbb{C}$ and a continuous function f on γ, we define the line integral

of f along γ by

$$(*) \qquad \int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

For this to be well-defined

we must show the rhs of $(*)$

is independent of parametrization of γ

Suppose $\bar{z}$ is an equivalent

parametrization. Then

by the change of variable ⑨

formula of calculus,

$$\int_a^b f(z(t))\, z'(t)\, dt = \int_c^d f(z(t(s)))\, z'(t(s)) \cdot t'(s)\, ds$$

$t = t(s)$

$$= \int_c^d f(\bar{z}(s))\, \bar{z}'(s)\, ds$$

as required

If γ is only piecewise C'

we define

$$\int_\gamma f(z)\, dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))\, z'(t)\, dt$$

and everything works fine.

By definition, the <u>length</u> of γ is $\ell(\gamma) = \int_a^b |z'(t)|\,dt$

As above this is independent of parametrization (If γ is only piecewise C' we make the obvious extension)

Note that if we write

$z(t) = x(t) + i\,y(t)$,

$|z'(t)| = \sqrt{x'^2(t) + y'^2(t)}$ and

$$\ell(\gamma) = \int_a^b \sqrt{x'^2(t) + y'^2(t)}\, dt$$

is the usual formula of Calculus 2,3

obtained by Riemann sums of

inscribed polygons.

<u>Example</u> 1. $\int_C \frac{1}{z}\, dz = 2\pi i$

(where $C = \{|z| = 1\}$ positively oriented)

since

$$z = e^{it} \quad 0 \le t \le 2\pi$$

$$dz = i e^{it}$$

$$\frac{dz}{z} = i$$

hence $\int_C \frac{1}{z}\,dz = \int_0^{2\pi} i\,dt = 2\pi i$.

Example 2

$$\int_{C_R(z_0)} (z-z_0)^m\,dz = \begin{cases} 0 & m \neq -1 \\ 2\pi i & m = -1 \end{cases}$$

$C_R(z_0)$ positively oriented

We parametrize $C_R(z_0)$ by

$$z(t) = z_0 + Re^{it} \quad 0 \le t \le 2\pi$$

$$\int_{C_R(z_0)} (z-z_0)^m\,dz = \int_0^{2\pi} R^m e^{imt} \cdot Ri e^{it}\,dt$$

$$= R^{m+1} \cdot i \int_0^{2\pi} e^{i(m+1)t}\,dt$$

$=$ (if $m \neq -1$)

$$i R^{m+1} \frac{e^{i(m+1)t}}{m+1} \Big|_0^{2\pi} = 0$$

primitive of $e^{i(m+1)t}$ by periodicity

If $m = -1$ we obtain $2\pi i$

<u>Proposition</u> Integration of continuous functions over curves has the following properties:

i) (linearity) If $\alpha, \beta \in \mathbb{C}$ then

$$\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)$$

ii) $$\int_\gamma f(z)\, dz = -\int_{\gamma^-} f(z)\, dz$$

iii) $$\left|\int_\gamma f(z)\, dz\right| \leq \sup_{z\in\gamma} |f(z)|\; l(\gamma)$$

<u>Theorem</u> (fundamental theorem)

If a continuous function $f(z)$ has a primitive $F(z)$ in Ω $\frac{1}{2}$
(i.e. F is analytic in Ω and $F'(z) = f(z)$) and γ is a curve in Ω beginning at w_1 and ending at w_2, then

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1).$$

Pf Assume first that γ is C^1. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$= \int_a^b F(z(t))\, z'(t)\, dt$$

chain rule (check)

$$= \int_a^b \frac{d}{dt}\left(F(z(t)\right) dt$$

$$= F(z(b)) - F(z(a)) = F(w_2) - F(w_1).$$

If γ is only piecewise C^1

(exercise)

$$\int_\gamma f(z) = \sum_{k=0}^{n-1} \left(F(z(a_{k+1})) - F(z(a_k))\right) =$$

$$F(z(a_n)) - F(z(a_0))$$

$$= F(z(b)) - F(z(a)) = F(w_2) - F(w_1)$$

<u>Corollary</u> If γ is closed $\subset \Omega$ and f is continuous and has a primitive in Ω open, then

$$\int_\gamma f(z)\,dz = 0.$$

As we saw earlier with $f(z) = \frac{1}{z}$

$$\int_C \frac{1}{z}\,dz = 2\pi i \neq 0$$

because $\frac{1}{z}$ does not have a primitive in $\mathbb{C} \setminus \{0\}$