We half proved half the Theorem; for the other half read Gamelin II.3.

<u>Example</u> In Chapter 1 we (tentatively) defined

$$e^z = e^{x+iy} = e^x e^{iy} =$$
$$e^x(\cos y + i \sin y) =: u + iv$$

So
$$u(x,y) = e^x \cos y$$
$$v(x,y) = e^x \sin y$$

$$u_x = e^x \cos y = \frac{\partial}{\partial y}(e^x \sin y) = v_y$$
$$u_y = -e^x \sin y = -\frac{\partial}{\partial x}(e^x \sin y) = -v_x$$

Thus $e^z$ is analytic and

$$\frac{d}{dz} e^{z} = u_x - i u_y = e^x \cos y + i e^x \sin y$$
$$= e^{z}$$

<u>Corollary 1</u> If $f$ analytic on $\Omega$ (connected) and $f'(z) \equiv 0$ on $\Omega$, then $f \equiv$ constant

Pf (exercise).

<u>Corollary 2</u> If $f$ analytic on $\Omega$ and $f$ real then $f \equiv$ constant.

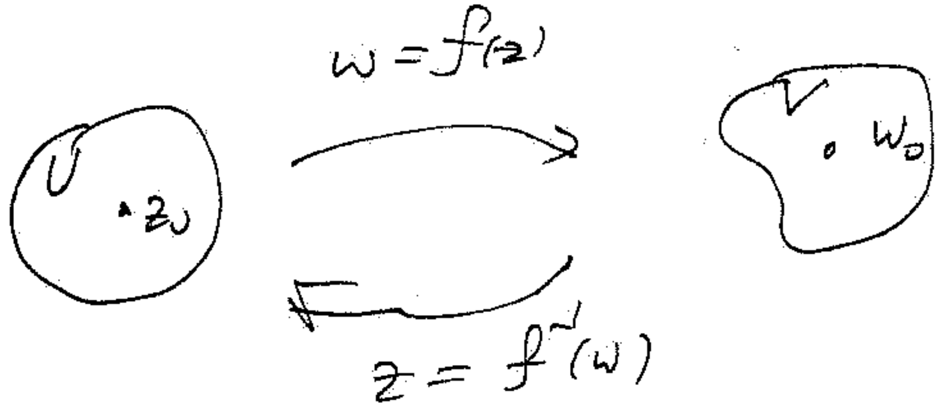
# 4. Inverse mapping and the Jacobian

We continue our discussion of analytic functions as mappings.

**Theorem** Suppose $f$ is analytic and continuous on $D$, $z_0 \in D$ and $f'(z_0) \neq 0$. Then there exists a small disk $U \subset D$ containing $z_0$ s.t. that the image $V = f(U)$ is open and the inverse function $f^{-1}: V \to U$ exists and is analytic.

Moreover,

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}, \quad z \in U$$




<u>Pf</u> Since $\det J_f(z_0) = |f'(z_0)|^2 > 0$, the inverse function theorem gives the existence of a real continuously differentiable inverse $F^{-1}(x,y) =: f^{-1}(w)$.

To check that $g = f^{-1}$ is analytic on $V$, fix $w, w_1 \in V$ with $w \neq w_1$.

Q1

Set $z = g(w)$, $z_1 = g(w_1)$.

Then,

$$\frac{g(w) - g(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{\frac{f(z) - f(z_1)}{z - z_1}}$$

As $w \to w_1$, $z \to z_1$ and the rhs tends to $\frac{1}{f'(z_1)}$ (note that $|f'(z_1)| = \det J_F(x_1, y_1) > 0$ in $U$ small. If we write $w = f(z)$ we have shown

$$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}},$$

the "usual formula" for the derivative of an inverse function //.

Remark We shall soon prove Goursat's Theorem which implies that every analytic function locally has a primitive i.e. given $f$ analytic in $B_\varepsilon(z_0)$ $\exists$ $F$ analytic in $B_\varepsilon(z_0)$ with $F'(z) = f(z)$.

This will allow us to prove the Cauchy integral formula, which in its simplest form states that if $f$ is analytic in an open set $\Omega$ containing a

circle $C$ in its interior,
then for all $z$ inside $C$

$$(*) \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

We then will deduce many other properties and formulas. In particular $(*)$ will imply that $f'(z)$ is also analytic in $\Omega$ so by induction $f^{(k)}(z)$ exists for all $k$. Thus the hypothesis that $f'(z)$ continuous is [illegible]. superfluous.

Example The principal logarithm $w = \operatorname{Log} z$ is a continuous inverse fn $z = e^w$, $-\pi < \arg w < \pi$.

Since $e^w$ is analytic and $(e^w)' \neq 0$ the last theorem applies (with $z$ and $w$ interchanged.) Hence $\operatorname{Log} z$ analytic and since $z = e^{\operatorname{Log} z}$ we obtain

$$1 = e^{\operatorname{Log} z} \frac{d}{dz} \operatorname{Log} z = z \frac{d}{dz} \operatorname{Log} z.$$

Thus

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$$

Since any other continuous branch of logarithm differs from the principal branch by a constant, it has the same derivative.

## 5. Harmonic functions

The Laplace operator for a real function $u(x_1, \dots, x_n)$ is

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

In particular for $u = u(x, y)$

$$\Delta u = u_{xx} + u_{yy}$$

and solutions of $\Delta u = 0$ are called harmonic functions

Theorem If $f = u + iv$ is analytic with $u, v \in C^2$ then $u, v$ are harmonic.

<u>Remark</u> As before we will show the hypothesis $u, v \in C^2$ is not needed.

<u>Pf</u> Since $u, v$ satisfy

(CR) $u_{xx} = v_{yx}$

$u_{yy} = -v_{xy}$

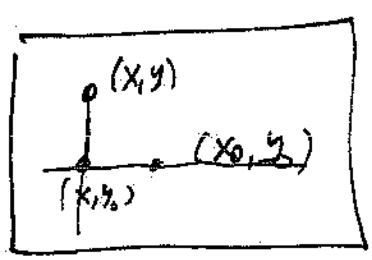
so $v \in C^2 \Rightarrow v_{yx} = v_{xy}$ and

so $\Delta u = 0$. Similarly $\Delta v = 0$. //

<u>Defn</u> If $u, v$ satisfy (CR) the $u, v$ called <u>harmonic conjugates</u>

given a rectangle $R$ with sides parallel to the $x, y$ axes and a harmonic function on $R$, we integrate (CR) from any fixed $(x_0, y_0) \in R$. Namely "integrate vertically" from $y_0$ to $y$ keeping $x$ fixed:

$$v(x,y) = \int_{y_0}^{y} u_x(x,t)\,dt + h(x)$$



"where $h(x)$ is the constant of integration". The second (CR) eqn becomes

$$u_y(x,y) = -v_x(x,y) =$$

$$-\frac{\partial}{\partial x}\int_{y_0}^{y} u_x(x,t)\,dt - h'(x)$$

valid if $u \in C^2$

$$= -\int_{y_0}^{y} u_{xx}(x,t)\,dt - h'(x)$$

since $-u_{xx} = u_{yy}$

$$= \int_{y_0}^{y} u_{yy}(x,t)\,dt - h'(x)$$

$$= u_y(x,y) - u_y(x,y_0) - h'(x).$$

Hence $\quad h'(x) = -u_y(x, y_0)$

which has the solution

$$h(x) = -\int_{x_0}^{x} u_y(s, y_0)\,ds + C$$

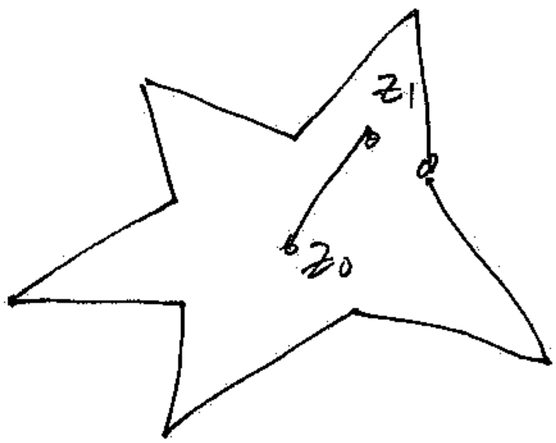
constant

Hence

(*) $$v(x,y) = \int_{y_0}^{y} u_x(x,t)\,dt - \int_{x_0}^{x} u_y(s,y_0)\,ds + C$$

The formula (*) is valid for D an open disk a rectangle or $\mathbb{C}$. We will later show it is valid for very general domains including "starshaped" domains.

(a domain D is starshaped with respect to $z_0=(x_0,y_0) \in D$ if for any $z_1 \in D$, the ray joining $z_0 z_1$ is contained in D.. Note the




cut plane, $\mathbb{C} \setminus (-\infty, 0)$ is starshaped with respect to any point on the positive real axis.)