Chapter 2 Analytic functions

1. Some preliminaries on continuous functions and compact sets.

Let f be a function defined on an open set $\Omega \subset \mathbb{C}$. We say f is continuous at $z_0 \in \Omega$ if for any $\varepsilon > 0$, $\exists\, \delta > 0$ s. that whenever $z \in \Omega$, $|z - z_0| < \delta$, then

$$|f(z) - f(z_0)| < \varepsilon.$$

Equivalently, for every sequence $\{z_n\} \subset \Omega$ with $\lim_{n\to\infty} z_n = z_0$, then

$$\lim_{n\to\infty} f(z_n) = f(z_0)$$

We then say f is continuous on Ω if it is continuous at each pt of Ω

Since convergence in $\mathbb{C}$ and convergence in $\mathbb{R}^2$ are equivalent,

f is continuous in $z = x+iy \iff$ f continuous as a function of $x y$

By the triangle inequality, if f continuous then the real function $z \mapsto |f(z)|$ is also continuous.

<u>Def'n</u> We say <u>f attains it maximum at $z_0 \in \Omega$</u> if

$$|f(z)| \leq |f(z_0)| \quad \forall z \in \Omega$$

(and similarly for f attaining its minimum). Recall

<u>Theorem</u> A continuous function on a compact set $K \subset \mathbb{C}$ is bounded and attains its max & min.

2. <u>Analytic</u> (holomorphic) <u>functions</u>

The central notion of complex analysis (which distinguishes itself from real analysis) is the existence of a <u>complex</u> <u>derivative</u>

<u>Def'n</u> We say the function f is differentiable (analytic or holomorphic)

at a pt $z_0 \in \Omega$ if the difference quotient

$$\frac{f(z_0+h) - f(z_0)}{h}$$

converges to a limit as $h \to 0$.
Here $h \in \mathbb{C}$, $h \neq 0$ and $z_0 + h \in \Omega$
(so the difference quotient is well-defined).
In this case we write

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

The important idea to grasp is that h is allowed to approach 0 from any direction so <u>that</u> <u>having</u> <u>a</u> <u>derivative</u> <u>is</u> <u>very</u> <u>special</u>.

<u>Example 1</u> Any polynomial

$p(z) = a_0 + a_1 z + \dots + a_n z^n$

is analytic. This follows from the proposition below or can easily be verified directly that

$$\frac{(z+h)^m - z^m}{h} = \frac{\cancel{h}}{\cancel{h}}\left((z+h)^{m-1} + (z+h)^{m-2} z + \dots + (z+h) z^{m-2} + z^{m-1}\right)$$

(using the identity $a^m - b^m = (a-b)(a^{m-1} + a^{m-2} b + \dots + a b^{m-2} + b^{m-1})$

so $\lim_{h \to 0} \frac{(z+h)^m - z^m}{h} = m z^{m-1}$

<u>Example 2</u> The function $f(z) = \bar{z}$ is <u>not</u> <u>differentiable</u>. For

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\bar{h}}{h}$$ which

has no limit as $h \to 0$, as we see by first taking <u>h real</u> and then <u>h purely imaginary</u>.

<u>Proposition</u> If f, g analytic on Ω, then

i) $f+g$ is also analytic on Ω and $(f+g)' = f'+g'$,

ii) fg is analytic on Ω and $(fg)' = f'g + fg'$.

iii) If $g(z_0) \neq 0$ f/g is analytic at z_0, and

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

iv) If f is analytic at z_0, then f is continuous at z_0.

Pf (exercise)

As in the real case, we also have the

<u>Theorem</u> (Chain Rule) Suppose $g(z)$ is analytic at z_0 and $f(w)$ is analytic at $w_0 = g(z_0)$. Then the composition $f \circ g\,(z) = f(g(z))$ is analytic at z_0 and (*) $(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$.

<u>Pf</u> The basic idea is that

(**) $$\frac{f(g(z)) - f(g(z_0))}{z - z_0} = \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}$$

and the first term on the right "should approach $f'(g(z_0))$ and the second term does) approach $g'(z_0)$ as $z \to z_0$.

However we must worry about $\{z : g(z) = g(z_0),\ z \neq z_0\}$. To avoid this problem we consider two cases:

<u>case 1</u> $\quad g'(z_0) \neq 0$

Then $\quad g(z) - g(z_0) = (z-z_0)\, g'(z_0) + o(|z-z_0|)$

error term that tends to 0 faster than $|z-z_0|$

so that

$$|g(z)-g(z_0)| = |z-z_0| \left(|g'(z_0)| + o(1) \right)$$

tends to 0 as $z \to z_0$

$$\geq \frac{|g'(z_0)|}{2} |z-z_0|$$

for $|z-z_0|$ small enough. Hence $g(z) \neq g(z_0)$ in this nbhd and we can pass to the limit in (**)

<u>Case 2</u> $g'(z_0) = 0$.

Then since $f(w)$ is analytic at w_0

$$\left| \frac{f(w)-f(w_0)}{w-w_0} \right| \le C, \quad 0 < |w-w_0| < \varepsilon.$$

So $|f(g(z)) - f(g(z_0))| \le C\,|g(z)-g(z_0)|$

for z near z_0, and therefore

$$\left| \frac{f(g(z)) - f(g(z_0))}{z-z_0} \right| \le C \left| \frac{g(z)-g(z_0)}{z-z_0} \right| \to 0$$

as $z \to z_0$. So both sides of $(*)$ are 0 and the Chain Rule holds in this case as well. //

3. Complex valued functions as mappings and the Cauchy-Riemann equations

We clarify the relationship between real and complex derivative. Our example $f(z) = \bar{z}$ corresponds to the real mapping $F(x,y) = (x, -y)$ which is differentiable in the real sense. It's derivative at a pt is the linear mapping DF given by the Jacobian matrix (in the standard basis of $\mathbb{R}^2$), a 2×2 matrix of partial derivative. In this case F is in fact infinitely differentiable in the real sense.

Write $f = u + iv$ and consider the mapping $F(x,y) = (u(x,y), v(x,y))$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Recall F is differentiable at $P_0 = (x_0, y_0)$ if $\exists$ a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$

s.t. that

$$\frac{|F(P_0+H) - F(P_0) - JH|}{|H|} \to 0$$

as $|H| \to 0$ $\quad H \in \mathbb{R}^2$ (think of H as a vector)

Equivalently

$$F(P_0+H) - F(P_0) = J(H) + |H|\, o(H)$$

as $|H| \to 0$.

The linear transformation is unique and is called $DF(P_0)$

If F differentiable, the partial derivatives of (u, v) exist at (x_0, y_0)

and $J = J_F(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

In the case of complex differentiability, <u>$f'(z_0)$</u> <u>is</u> <u>a</u> <u>complex number</u>. <u>To</u> <u>see</u> <u>the</u> <u>relationship</u> between these two notions, take first h real, and write $z = x + iy$, $z_0 = x_0 + iy_0$ and $f(z) = f(x, y)$. Then

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0) - f(x_0, y_0)}{h_1}$$

$$= \frac{\partial f}{\partial x}(x_0, y_0) \quad (y_0 \text{ fixed in the limit})$$

Now take h purely imaginary, say $h = i h_2 \Longrightarrow$

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0+h_2) - f(x_0, y_0)}{i h_2}$$

$$= \frac{1}{i} \frac{\partial f}{\partial y}(x_0, y_0)$$

Hence $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y} = -i \dfrac{\partial f}{\partial y}$.

Writing $f = u + iv$ and separating real and imaginary parts (check) gives

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{ Cauchy-Riemann equations (CR)}$$

It is useful to introduce two "differential operators"

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Proposition If f analytic at z_0, then

$\frac{\partial f}{\partial \bar{z}}(z_0) = 0$ and

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \qquad = 2\frac{\partial u}{\partial z}$$

If we write $F(x,y) = f(z)$ then

F is differentiable in the real sense and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

Pf Taking real and imaginary parts it is easy to see that the CR equations are equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$. By our earlier calculation,

$$f'(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right) = \frac{\partial f}{\partial z}(z_0)$$

exercise $u_x - i u_y = 2\, u_z(z_0)$

To show F differentiable, write $H = (h_1, h_2)$, $h = h_1 + i h_2$. Then

(CR) $\Rightarrow$

$$J_F(x_0, y_0)(H) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + i h_2)$$

$$= f'(z_0)\, h$$

(where we identify a complex number $a+ib$ with the pair (a, b))

Then (using (CR) again)

$$\det J_F(x_0, y_0) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$= u_x^2 + u_y^2 = \left|2\frac{\partial u}{\partial z}\right|^2 = |f'(z_0)|^2$$

<u>Theorem</u> Suppose $f = u + iv$ on Ω. ~~If~~ u, v are continuously differentiable (C^1) and satisfy (CR) on Ω, then f is analytic on Ω and $f'(z) = \frac{\partial f}{\partial z}$.