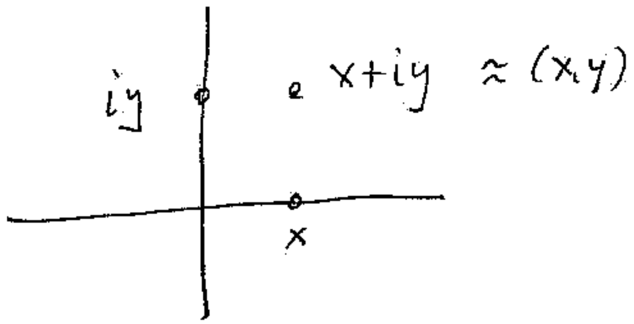
Chapter 1 <u>Section 1</u>

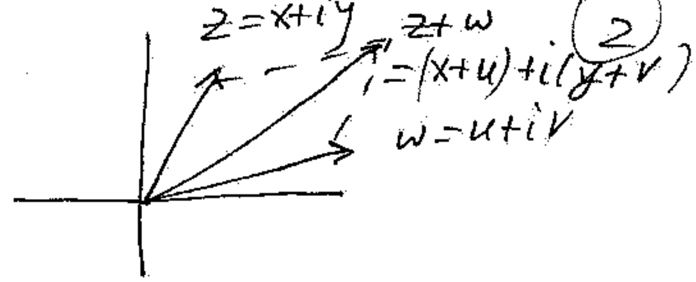
A complex number is an expression of the form $z = x+iy$ where $x, y \in \mathbb{R}$, $x = \operatorname{Re} z$ $y = \operatorname{Im} z$

The set of complex numbers is denoted by $\mathbb{C}$ and we think of $\mathbb{R} = \{z : \operatorname{Im} z = 0\} \subseteq \mathbb{C}$

We call the complex no's of the form iy "purely imaginary"

We can also uniquely identify $z = x+iy$ with the point $(x,y) \in \mathbb{R}$ (1-1, onto correspondence)





addition of complex no's

We can make $\mathbb{C}$ into a complete field with operations $+, \cdot$

$$z+w = (\mathrm{Re}\, z + \mathrm{Re}\, w) + i\,(\mathrm{Im}\, z + \mathrm{Im}\, w)$$

For multiplication we distribute the ordinary multiplication with the proviso $i^2=-1$. Namely

$$(x+iy)(u+iv) = (xu - yv) + i\,(xv + yu)$$

One checks directly that the usual algebraic properties hold for $\cdot$, , that is

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$
$$z_1 z_2 = z_2 z_1$$
$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

For $z \neq 0$ there is a multiplicative inverse $\frac{1}{z} = \frac{x - iy}{x^2 + y^2}$.

Ex. $\frac{1}{i} = -i$

It is useful to introduce the complex conjugate $\bar{z} = x - iy$

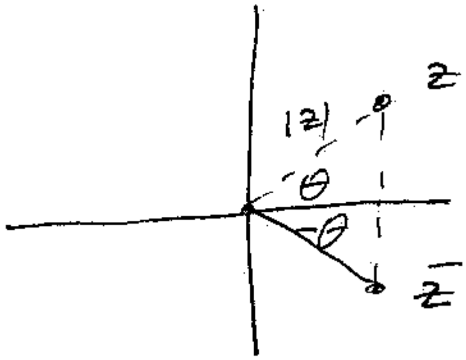
Note $\overline{(\bar{z})} = z$. It has the nice property that

$$\overline{z + w} = \bar{z} + \bar{w}$$
$$\overline{zw} = \bar{z}\,\bar{w}$$

$$|z|^2 = x^2 + y^2 = z\bar{z} = |\bar{z}|^2$$

Note $|zw|^2 = (zw)\overline{(zw)} = (zw)\bar{z}\,\bar{w}$
$= (z\bar{z})(w\bar{w}) = |z|^2\,|w|^2$

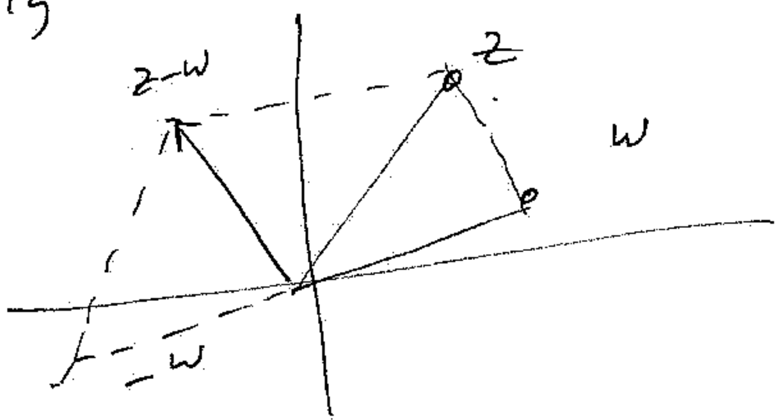
so $|zw| = |z|\,|w|$



$\bar{z}$ is the reflection of z in the real axis

and $|z| = \text{distance}(z, 0)$

Similarly $|z-w| = \text{distance}(z, w)$



This notion of distance allows us to define Cauchy sequences $\{z_n\}$ (same def'n as for $\mathbb{R}$)

<u>Exercise</u> $\{z_n = x_n + iy_n\}$ is Cauchy iff $\{x_n\}, \{y_n\}$ is Cauchy.

<u>Corollary</u> $\mathbb{C}$ is complete.

<u>Def'n</u> A complex polynomial of degree n is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

where $a_n \neq 0$, $a_n \in \mathbb{C}$

Later we will prove <u>the fundamental theorem of algebra</u>: Every complex polynomial $p(z)$ of degree n, $n \geq 1$ has the factorization

$$p(z) = c(z-z_1)^{m_1} \dots (z-z_k)^{m_k}$$

where the z_i's are distinct, $c \neq 0$, $m_j \geq 1$ and $n = \sum_{j=1}^{k} m_j$

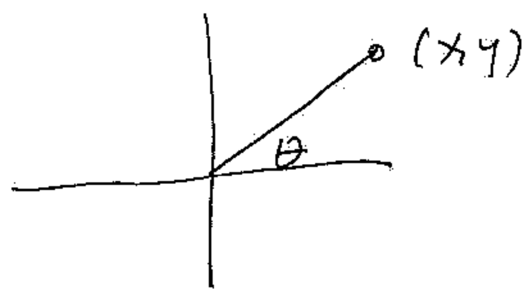
We will give a number of different proofs later.

Note the z_i are the roots of $p(z)$ i.e. $p(z_i) = 0$ and so there are n roots counting multiplicity. It is easy to check that the factorization is unique (up to permutation of the factors). So the heart of the matter is to prove that every complex polynomial has a root.

Section 2 Polar representation

Recall the standard polar coordinates in $\mathbb{R}^2$:

$$x = r\cos\theta$$
$$y = r\sin\theta$$



where $r = \sqrt{x^2+y^2}$, $\theta = \measuredangle$ between $\overline{O(x,y)}$ and the positive x axis

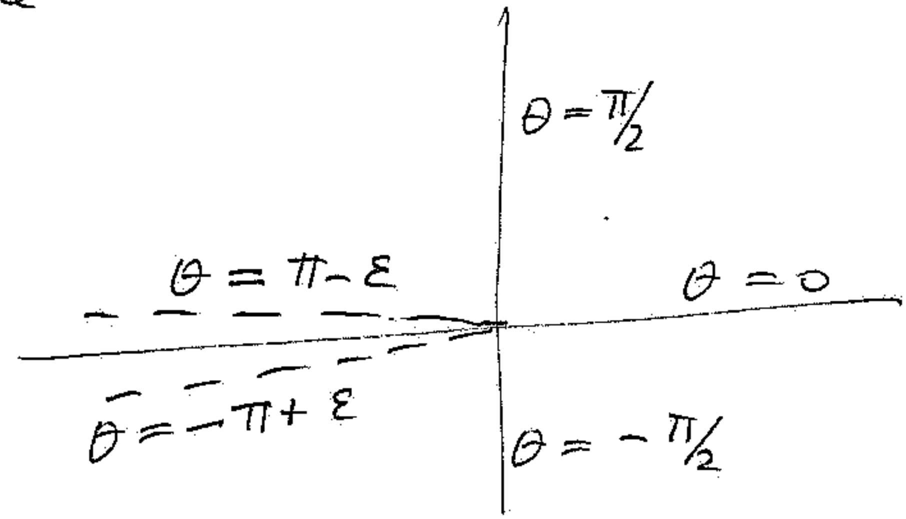
In complex notation

$$z = x+iy = r(\cos\theta + i\sin\theta)$$

$r = |z|$ and we write $\theta = \arg z$, a "multi-valued function" defined for $z \neq 0$

We define the principal value

Arg z = unique θ in the interval $(-\pi, \pi]$

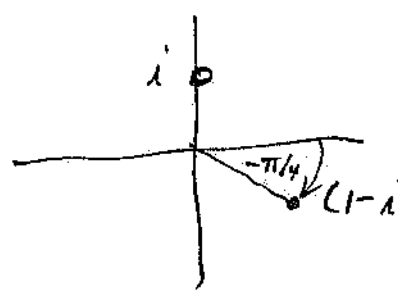


$\theta = -\pi$ is a "branch cut"

(we can similarly define $\mathrm{Arg}_\tau z$ = unique θ in the interval $(\tau, \tau + 2\pi]$)

<u>Ex</u> $\mathrm{Arg}\, i = \pi/2$

$\mathrm{Arg}\, 1-i = -\pi/4$



Following Euler, we introduce the notation $e^{i\theta} = \cos\theta + i\sin\theta$;

then $z = r e^{i\theta}$

$r = |z|$, $\theta = \arg z$

$$e^{i(\theta + 2\pi k)} = e^{i\theta}, \quad k = 0, \pm 1, \pm 2, \dots$$

Note the identities

$$e^{2\pi k i} = 1 \qquad k = 0, \pm 1, \dots$$

$$|e^{i\theta}| = 1$$

$$\overline{e^{i\theta}} = e^{-i\theta} \quad \longleftrightarrow \quad \arg \bar{z} = -\arg z$$

$$\frac{1}{e^{i\theta}} = e^{-i\theta} \qquad \arg \frac{1}{z} = -\arg z$$

The most important property is the addition formula

$$e^{i(\theta+\varphi)} = e^{i\theta} e^{i\varphi} \longleftrightarrow \begin{array}{l}\arg(z_1 z_2) \\ = \arg z_1 + \arg z_2 \\ (\mathrm{mod}\ 2\pi)\end{array}$$

which is equivalent to

$$\cos(\theta+\varphi) + i\sin(\theta+\varphi) =$$

$$(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$

which (after multiplication) is equivalent to

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

$$\sin(\theta+\varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi,$$

the "addition formulas" for sine & cosine.

Cor. $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

$$= e^{in\theta} = \cos n\theta + i\sin n\theta$$

<u>Example</u> $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\Leftrightarrow \begin{array}{l} \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \end{array}$$

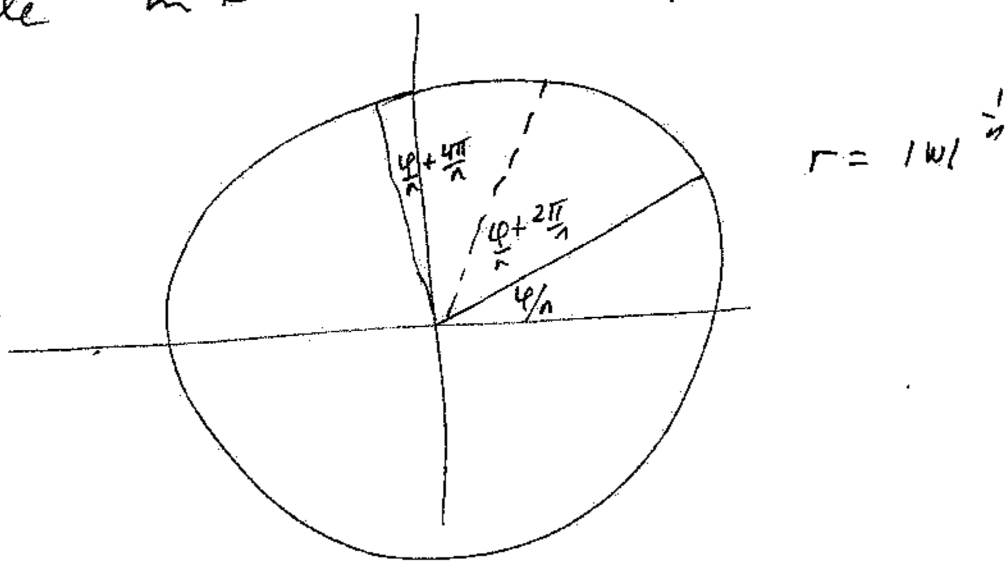
<u>Defn</u> A soln of $z^n = w$ is called an n^{th} root of $w \neq 0$.

Write $w = \rho e^{i\varphi}$. Then $z = re^{i\theta}$ must satisfy $r^n e^{in\theta} = \rho e^{i\varphi}$, so

$$r = \rho^{1/n} > 0.$$

$$\theta = \frac{\varphi}{n} + \frac{2\pi k}{n} \quad k = 0, 1, \dots, n-1$$

The roots lie on the circle $|w|^{\frac{1}{n}}$ and divide the circle into n equal arcs:



We postpone I.3, I.4

Section 5 The exponential function

We extend the defn $e^{i\theta}$ to all values of z:

Defn $e^z = e^x e^{iy} =$
$e^x(\cos y + i \sin y)$

Thus $|e^z| = e^x$
$\arg e^z = y$

If $z = x$ real, we have the usual exponential function of calculus while if z is imaginary, we have Euler's exponential function

Note that e^z is periodic of period $2\pi i$:

$$e^{z+2\pi i} = e^z \qquad z \in \mathbb{C}$$

(so also $e^{z+2\pi k i} = e^z$)

It is also easy to see that

$$(*) \qquad e^{z+w} = e^z e^w$$

(since $e^{(x+iy)+(u+iv)} = e^{(x+u)+i(y+v)}$

$= e^x e^u e^{iy} e^{iv} = e^z e^w$)

From $(*)$ $\quad e^z e^{-z} = e^0 = 1 \quad$ so

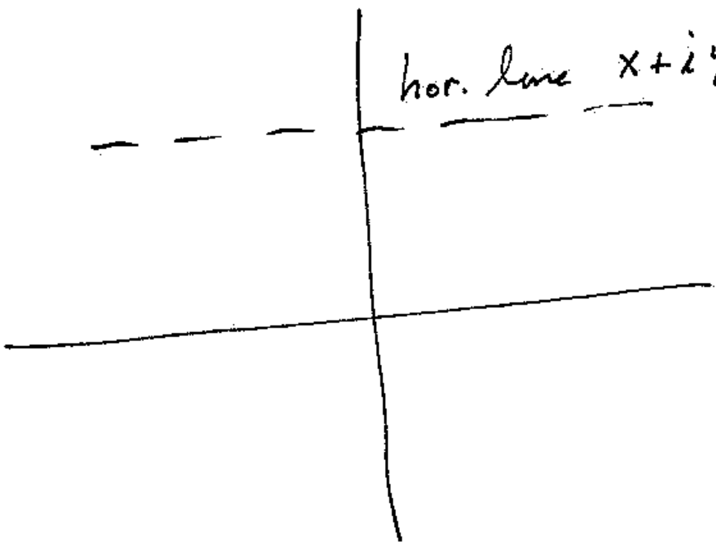
$\frac{1}{e^z} = e^{-z}$ is the

inverse of e^z.

It is instructive to view

$w = e^z$ as a mapping

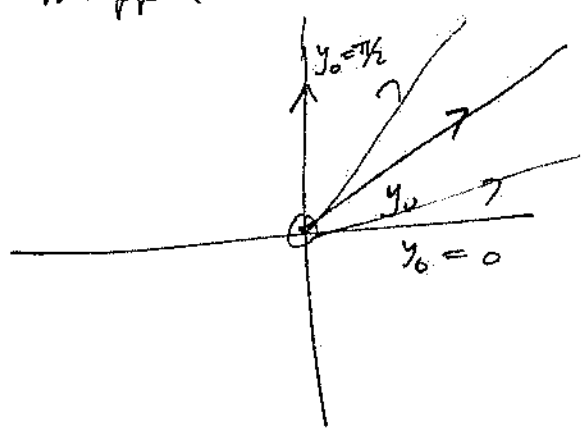
the z-plane to the w-plane



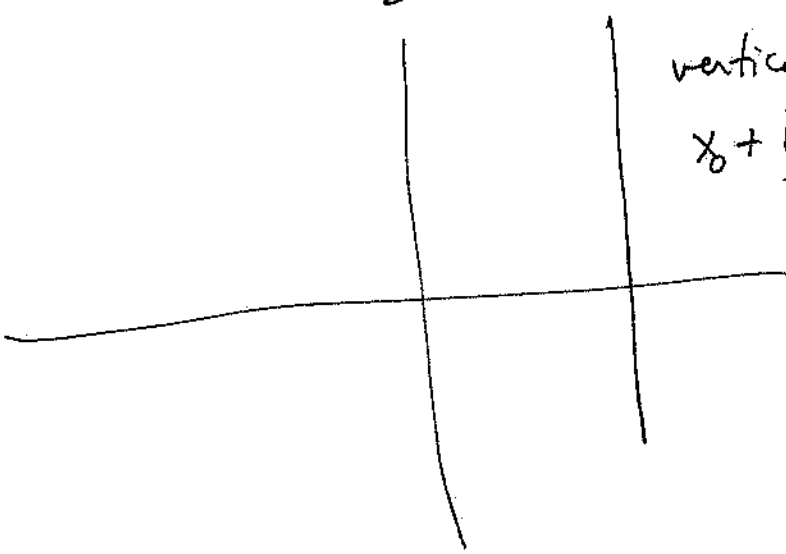
z

$-\infty < x < \infty$

mapped to $e^x e^{iy_0}$



w



z

mapped to circle of radius e^{x_0}

wrapped infinitely around, once as y increases by 2π

I Section 6, the logarithm

For $z \neq 0$ we define the (multivalued) function

$$\log z = \log|z| + i \arg z$$

$$= \log|z| + i \,\mathrm{Arg}\, z + 2\pi m i$$

$$m = 0, \pm 1, \pm 2, \ldots$$

Thus the values of $\log z$ are precisely the complex numbers w such that $e^w = z$, since

$$e^w = e^{\log|z|} e^{i \,\mathrm{Arg}\, z} e^{2\pi i m}$$

$$= |z| e^{i \,\mathrm{Arg}\, z} = z$$

On the other hand if

$e^w = z$ then z has the

polar representation $z = e^u e^{iv}$

so $u = \log |z|$ and $v = \text{Arg}\, z + 2\pi m$

Define $\text{Log}\, z = \log z + i \,\text{Arg}\, z$, $z \neq 0$

as the principal value of $\log z$,

a single-valued inverse fn w

$\underline{Ex}$ $\text{Log}(1+i) = \log|1+i| + i\,\text{Arg}(1+i)$

$= \log\sqrt{2} + i\,\pi/4$

We can regard $w = \operatorname{Log} z$ as a map from the "slit z-plane" $\mathbb{C} \setminus (-\infty, 0]$ to the w-plane