Math113 Exam 2 Practice Solutions

1. let $f(x) = \sqrt{x^2+1}$. Use the definition of the derivative to compute f'(x).

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1)} - \sqrt{x^2+1}}{h} = \lim_{h\to 0} \frac{((x+h)^2+1) - (x^2+1)}{h(\sqrt{(x+h)^2+1)} + \sqrt{x^2+1}}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h(\sqrt{(x+h)^2+1)} + \sqrt{x^2+1})} = \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1)} + \sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}} \ .$$

2. Compute the area of the region between the graphs of $f(x) = x$ and $g(x) = \frac{x^3}{4}$ on the interval [-1,2].
The graphs intersect at x=0 and $x = \pm 2$ with the graph of $f(x) = x$ above the graph of $g(x) = \frac{x^3}{4}$ on (0,2) and below on (-2,0). Hence,

$$A = \int_{-1}^{0} (\frac{x^3}{4} - x)\ dx + \int_{0}^{2} (x - \frac{x^3}{4})\ dx = (\frac{x^4}{16} - \frac{x^2}{2})|_{-1}^{0} + (\frac{x^2}{2} - \frac{x^4}{16})|_{0}^{2} = \frac{23}{16} \ .$$

3. Find the rectangle of largest area that can be inscribed in the unit circle.
Let the vertices of the rectangle be (a,b),(a,-b),(-a,b),(-a,-b) with $a > 0\ , b > 0$ Then we want to maximize 4ab subject to the constraint $a^2 + b^2 = 1$ (which says that the rectangle is inscribed in the unit circle). Since $b = \sqrt{1-a^2}$ we want to maximize

$$f(a) = 4a\sqrt{1-a^2} \text{ on } (0,1) \ .$$

Note that f(0)=f(1)=0 and f is differentiable with

$$f'(a) = 4(\sqrt{1-a^2} - \frac{a^2}{\sqrt{1-a^2}}) = \frac{4(1-2a^2)}{\sqrt{1-a^2}} \ .$$

Hence f(a) has a unique critical point at $a = \frac{1}{\sqrt{2}} = b$ (which is therefore the global maximum). So the inscribed rectangle of maximum area is a square of side $\sqrt{2}$.

4. Calculate $\frac{d}{dx}\sqrt{\sin\sqrt{x}}$ for $x > 0$.
Use the chain rule several times:

$$\frac{d}{dx}\sqrt{\sin\sqrt{x}} = \frac{1}{2\sqrt{\sin\sqrt{x}}} \cdot \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \ .$$

5. Let $f(x) = \frac{x}{\sqrt{x^2+1}}$
a. Show that f^{-1} exists and find the domain and range of f^{-1}.

$$f'(x) = \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} = (x^2+1)^{-\frac{3}{2}} \ ,$$

so f is strictly increasing and hence f^{-1} exists. Write $y = \frac{x}{\sqrt{x^2+1}}$ so $y^2 = \frac{x^2}{x^2+1}$ and we can solve $x^2 = \frac{y^2}{1-y^2}$. Since y has the same sign as x, $x = \frac{y}{\sqrt{1-y^2}}$ which has domain (-1,1) and range all values of x.

b. Evaluate $\int_0^2 f(x)\ dx$.
From question 1 we see $f(x) = g'(x)$ for $g(x) = \sqrt{x^2+1}$. Hence by the fundamental theorem,

$$\int_0^2 f(x)\ dx = g(2) - g(0) = \sqrt{5} - 1 \ .$$

6. Suppose that f(x) is a continuous function defined for all x with the property that $|f(x_1) - f(x_2)| \leq |x_1 - x_2|^2$
a. Write down a partition of the interval [a,b] into N subintervals of equal length(this is the regular partition).
The partition points are $x_i = a + i\frac{b-a}{N}$, $0 \leq i \leq N$.
b. Express $f(b) - f(a)$ as a telescoping sum using the partition.

$$f(b) - f(a) = \sum_{i=1}^{N}(f(x_i) - f(x_{i-1}) \ .$$

c. Show that f(x) is constant using part b and the stated property of f.

$$|f(b)-f(a)| \leq \sum_{i=1}^{N}|(f(x_i)-f(x_{i-1})| \leq \sum_{i=1}^{N}(x_i-x_{i-1})^2 = N\cdot(\frac{b-a}{N})^2 = \frac{(b-a)^2}{N} \ .$$

Now let $N \to \infty$ to conclude $|f(b) - f(a)| \leq 0$, i.e f(b)=f(a). Since a and b are arbitrary, f is constant.

7. Let $F(x) = \int_2^{x^2} (3t^2+1)^3 \, dt$. Find $F(\sqrt{2}), F'(\sqrt{2}), F''(\sqrt{2})$.
$F(\sqrt{2}) = 0$ by properties of the integral. By the fundamental theorem and the chain rule,

$$F'(x) = (3(x^2)^2+1)^3 \cdot (2x) = 2x(3x^4+1)^3 \ ,$$

$$F''(x) = 2(3x^4+1)^3 + (2x) \cdot 3(3x^4+1)^2 \cdot (12x^3)$$

Therefore $F'(\sqrt{2}) = 2\sqrt{2}(13)^3$, $F''(\sqrt{2}) = \ 2(13)^3 + (6\sqrt{2})(13)^2(24\sqrt{2})$.

8. Consider the integral $\int_0^1 (1-x) \, dx$ and let P be the regular partition of [0,1] into N equal subintervals.
a. Write down L(f,P) and U(f,P) ($f(x) = 1-x$) and explicitly evaluate them.
let $x_i = \frac{i}{N}$, $0 \le i \le N$. Then since f is decreasing,

$$L(f,P) = \sum_{i=1}^{N} f(x_i)\frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N}(1-\frac{i}{N})$$

$$U(f,P) = \sum_{i=1}^{N} f(x_{i-1})\frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N}(1-\frac{i-1}{N})$$

b. Use part a to evaluate the integral.

$$L(f,P) = 1 - \frac{1}{N^2}\frac{N(N+1)}{2} = \frac{1}{2} - \frac{1}{2N}$$

$$U(f,P) = 1 - \frac{1}{N^2}\frac{(N-1)N}{2} = \frac{1}{2} + \frac{1}{2N}$$

Hence $\frac{1}{2} - \frac{1}{2N} < \frac{1}{2} < \frac{1}{2} + \frac{1}{2N}$ for all N so the integral is $\frac{1}{2}$.