Even More Worked Examples

1. Let $f(x) = x^2 - \cos x$.
a. Show that f(x) has exactly two zeros.
Note that f(x) is an even function so we need to show there is exactly one positive root. Now f(0)=-1 and $f(x) > 0$ for $x \geq 1$. So by the intermediate value theorem there is at least one root $x_0 \in (0,1)$. But $f'(x) = 2x + \sin x > 0$ for $x \in (0,1)$, hence f(x) is strictly increasing and there is exactly one positive root.
b. Use the Taylor expansion of f(x) about x=0 to calculate the approximation r of the positive root . Estimate the error $|x_0 - r|$ you make.

$$f(x) = x^2 - (1 - \frac{x^2}{2} + R_3) = -1 + \frac{3}{2}x^2 - R_3 \text{ where } 0 < R_3 < \frac{x^4}{4!} \; .$$

So $x_0 = \sqrt{\frac{2}{3}}\,\sqrt{1 + R_3(x_0)}$. Therefore the approximate positive root r satisfies $r = \sqrt{\frac{2}{3}}$ and

$$\sqrt{\frac{2}{3}} < x_0 < \sqrt{\frac{2}{3}}\,\sqrt{1+R_3} < \sqrt{\frac{2}{3}}\,(1 + \frac{1}{2}R_3) < \sqrt{\frac{2}{3}}\,(1 + \frac{1}{48})$$

Hence $0 < x_0 - r < \frac{1}{48}\,\sqrt{\frac{2}{3}} = 0.017$.

2. Estimate $\int_0^{\frac{1}{\sqrt{2}}} \sin x^2 \; dx$
For $0 \leq x \leq \frac{1}{2}$, $\sin x = x - \frac{x^3}{3!} + R_4$ where $|R_4| \leq \frac{(\frac{1}{2})^5}{5!}$. So,

$$\int_0^{\frac{1}{\sqrt{2}}} \sin x^2 \; dx = \int_0^{\frac{1}{\sqrt{2}}} (x^2 - \frac{x^6}{6} + R_4) \; dx = \frac{(\frac{1}{\sqrt{2}})^3}{3} - \frac{(\frac{1}{\sqrt{2}})^7}{42} + E = \frac{55}{336\sqrt{2}} + E$$

where $|E| \leq \frac{1}{\sqrt{2}} \frac{(\frac{1}{2})^5}{5!} = \frac{1}{\sqrt{2}} \frac{1}{3840} = 1.84 \cdot 10^{-4}$.

3. The sequence $\{a_n\}$ is defined recursively by

$$a_1 = 4, \; a_{n+1} = 4 - \frac{3}{a_n} \; .$$

a. Show by induction that $a_n \geq 3$.
$a_1 = 4 > 3$. Assume $a_n \geq 3$. Then since $\frac{3}{a_n} \leq 1$, $a_{n+1} \geq 4 - 1 = 3$ completing

the induction.
b. Show that $a_{n+1} \leq a_n$.

$$a_{n+1} - a_n = 4 - \frac{3}{a_n} - a_n = -\frac{1}{a_n}((a_n)^2 - 4a_n + 3) = -\frac{1}{a_n}(a_n - 3)(a_n - 1) < 0 \ .$$

c. Explain why $\lim_{n\to\infty} a_n$ exists and find the limit.
Then sequence $\{a_n\}$ is decreasing and bounded below and so $a = \lim_{n\to\infty} a_n$ exists. let n tend to infinity in the recursion $a_{n+1} = 4 - \frac{3}{a_n}$ to obtain $a = 4 - \frac{3}{a}$ (which has roots a=3 and a=1). Since $a \geq 3$, a=3 is the unique limit.

4. Determine if the following series converge or diverge. Explain the reason in each case.
a. $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$
Use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)!)^2}{2^{(n+1)^2}}}{\frac{(n!)^2}{2^{n^2}}} = (\frac{n+1}{2^{n+1}})^2 \to 0 \ .$$

(Here we have used that $e^x > Ax^p$ for any$A, p > 0$ for x large enough.) Since the limit is less than 1, the series converges.
b. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$
Here we use the Leibnitz theorem for conditional convergence for alternating series: $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the a_n are nonnegative and decrease to zero. Here $a_n = \frac{\sqrt{n}}{n+100}$ is clearly nonnegative and is decreasing for $n \geq 100$ (this is good enough) since $f(x) = \frac{\sqrt{x}}{x+100}$ satisfies $f'(x) \leq 0$ for $x \geq 100$.
c. $\sum_{n=1}^{\infty} \frac{1}{(\log n)^3}$
Since $n > (\log n)^3$ for n large ($e^x > x^3$ for x large so put $x = \log n$)

$$\sum_{n=1}^{\infty} \frac{1}{(\log n)^3} > \sum_{n=N} \frac{1}{n} = \infty \ ,$$

so the series diverges.