**More Worked Examples**

1. Let $f(x) = \begin{cases} \frac{3-x^2}{2} & \text{if } x \le 1 \\ \frac{1}{x} & \text{if } x \ge 1 \end{cases}$

Show that f(x) satisfies the conditions of the mean value theorem on [0,2] and determine all the points x provided by the theorem.

f(x) is continuous everywhere (you need to check $x = 1$) and clearly differentiable except possibly at x=1 where the one sided derivatives exist:
$f'(x) = \begin{cases} -x & \text{if } x \le 1 \\ -\frac{1}{x^2} & \text{if } x \ge 1 \end{cases}$

The point is that $f'_-(1) = f'_+(1) = -1$ so the derivative exists everywhere. Moreover, the MVT says there is at least one value of x in [0,2] where

$$f(2) - f(0) = \frac{1}{2} - \frac{3}{2} = -1 = 2f'(x) \ ,$$

i.e where $f'(x) = -\frac{1}{2}$. From the definition of f'(x) we see that $x = \frac{1}{2}$ or $x = \sqrt{2}$.

2. Let $g'(x)$ be defined by $g'(x) = \begin{cases} -\frac{x}{5} & \text{if } 0 \le x \le 5 \\ \frac{x}{5} - 2 & \text{if } x \ge 5 \end{cases}$

Assume g(0)=12. Find g(5), g(10).

$g(x) = \int_0^x g'(t)\ dt + 12 = \begin{cases} -\frac{x^2}{10} + 12 & \text{if } 0 \le x \le 5 \\ \frac{x^2}{10} - 2x + c & \text{if } x \ge 5 \end{cases}$

To determine c, we want

$$(\frac{x^2}{10} - 2x + c)|_{x=5} = (-\frac{x^2}{10} + 12)|x = 5 = g(5) = \frac{19}{2}$$

(since g is continuous) which gives $c = 17$. Then $g(10) = 7$.

3. For any $a > 0$ show that the tangent line to the graph of $f(x) = x^3$ at (a, f(a)) intersects the graph at a unique point (b,f(b)), $b \neq a$ and $f'(b) = 4f'(a)$.

The equation of the tangent line is (since $f'(a) = 3a^2$)

$$y - a^3 = 3a^2(x - a) \ .$$

This line meets the graph (at $(b, b^3)$) precisely when

$$b^3 - a^3 = 3a^2(b - a)$$

Since we are assuming $b \neq a$ this is equivalent to (dividing both sides by b-a):

$$b^2 + ba + a^2 = 3a^2 \text{ or } b^2 + ba - 2a^2 = 0 \ .$$

Factoring gives $(b - a)(b + 2a) = 0$ or b=-2a. Hence $f'(b) = 3b^2 = 12a^2 = 4f'(a)$.

4. Suppose f(x) is everywhere differentiable. Show directly from the definition that

$$\frac{d}{dx}(f^2(x)) = 2f(x)f'(x) \ .$$

Let $g(x) = f^2(x)$. Then

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))(f(x+h) + f(x))}{h}$$

$$= \lim_{h\to 0}(f(x+h) + f(x)) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 2f(x)f'(x) \ .$$