Worked Examples

1. Show by mathematical induction that $(1+x)^n \geq 1+nx$ for $x > -1$.

Solution. For n=1 we have equality as both sides are equal to $1+x$. Assume that the inequality holds for n (induction step) and we show it holds for n+1. That is, for $x+1>0$,

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+nx+x+nx^2$$

$$= 1+(n+1)x+nx^2 \geq 1+(n+1)x$$

(since $x^2 \geq 0$). hence by the principle of mathematical induction, the inequality is valid for all n.

2. True or false; justify.

a. There is a positive real number x so that $x = \frac{3}{x}$.

True. This is equivalent to $x^2 = 3$ or $x = \sqrt{3}$ exists. We gave on proof in class based on the completeness axiom. Here we will use the intermediate value theorem for the continuous function $f(x) = x^2 - 3$ on the interval [0,2]. We have $f(0) = -3 < 0$ and $f(2) = 1 > 0$. By the Intermediate value theorem there is an $x \in (0,2)$ satisfying $f(x) = 0$ or $x^2 = 3$.

b. There is a rational number $x > 0$ so that $x = \frac{3}{x}$.

False. There is no positive rational solution. We use that every positive integer can be written as $3k-2,\ 3k-1,\ 3k$ for $k \in \mathbf{N}$ (that is it is divisible by 3 or has remainder 1 or 2. Moreover,

$$(3k-2)^2 = 9k^2 - 12k + 4 = 3(3k^2 - 4k + 1) + 1\ ,$$

$$(3k-1)^2 = 9k^2 - 6k + 1 = 3(3k^2 - 2k) + 1\ ,$$

$$(3k)^2 = 9k^2 = 3(3k^2)\ .$$

Hence if $p \in \mathbf{N}$ and p^2 is divisble by 3, then p is divisible by 3. Now suppose for contradiction that there is a rational number $\frac{p}{q}$ such that $(\frac{p}{q})^2 = 3$ or equivalently, $p^2 = 3q^2$. We may assume that p and q have no common factors (except 1). We will show that this leads to a contradicition. For p^2 is divisible by 3 so p is divisible by 3. Write $p = 3k$. Then $p^2 = 9k^2 = 3q^2$ so $q^2 = 3k^2$. Hence q is also a multiple of 3, contradiciton.

3. a. State the ϵ, δ definition of the continuity of $f(x)$ at $x = a$.
b. Use part a. to show that $f(x) = \frac{1}{x-1}$ is continuous at $x = 3$.

a. We say $f(x)$ is continuous at $x = a$ if given $\epsilon > 0$ there is a $\delta > 0$ so that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

b. $f(3) = \frac{1}{2}$ so we must consider

$$|f(x) - f(3)| = |\frac{1}{x-1} - \frac{1}{2}| = |\frac{2-(x-1)}{2(x-1)} = |\frac{3-x}{2(x-1)}| = \frac{|x-3|}{2|x-1|} \ .$$

We first choose $0 < \delta < 1$ so that $2 < x < 4$ if $|x - 3| < \delta$. Then $|x - 1| > 1$ so

$$\frac{|x-3|}{2|x-1|} < \frac{|x-3|}{2} < \frac{\delta}{2} \ .$$

Finally choose $\delta = \min(2\epsilon, 1)$. Then

$$|f(x) - f(3)| = |\frac{1}{x-1} - \frac{1}{2}| < \epsilon \ .$$

4. Suppose $f(x)$ is continuous on [-1,1] and satisfies $x^2 + (f(x))^2 = 1$. Show that either $f(x) = +\sqrt{1-x^2}$ or $f(x) = -\sqrt{1-x^2}$.
Solution.
Note that if $f(x) = 0$, then $x = \pm 1$ and also that $f(0) = \pm 1$. Suppose $f(0) = 1$. Then f(x) must be positive on (-1,1), for if f(c) is negative at some $c \in (-1, 1)$ then by the Intermediate value theorem f must vanish at some point between c and 0, a contradiciton. Therefore, $f(x) = +\sqrt{1-x^2}$. Similarly if $f(0) = -1$, f must be negative on (-1,1) and so $f(x) = -\sqrt{1-x^2}$.

5. Let $f(x)$ be defined for all x, and continuous except for $x = -1$ and $x = 3$. Let

$$g(x) = \begin{cases} x^2 + 1 & \text{for} \quad x > 0 \\ x - 3 & \text{for} \quad x \leq 0 \end{cases}$$

For what values of x can you be certain that $f(g(x))$ is continuous? Explain.
Solution.
Note that $g(x)$ is not continous at $x = 0$ so $f(g(x)$ need not be continuous at x=0. (It would be continuous if f(1)=f(-3)). We also have a problem if g(x)=-1 (never happens) or if g(x)=3 (this happens when $x = \sqrt{2}$. Thus

$f(g(x)$ need not be continuous at $x = 0$ and $x = \sqrt{2}$.