Math 113 Exam 2 Solutions

1.(20pts) Let $I = \int_1^2 (2x-1)\ dx$
Here, f(x)=2x-1 on [1,2] and $x_i = 1 + \frac{i}{n},\ i = 0, 1, 2, \ldots, n$. Note $x_i - x_{i-1} = \frac{1}{n}$ and since f(x) is increasing, $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. So,

$$L(f,P) = \sum_{i=1}^{n} (2x_{i-1} - 1)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}(1 + \frac{2}{n}(i-1)) = 1 + \frac{2}{n^2}\sum_{i=1}^{n-1} i$$

$$= 1 + \frac{2}{n^2}\frac{(n-1)n}{2} = 1 + \frac{n-1}{n} = 2 - \frac{1}{n}$$

$$U(f,P) = \sum_{i=1}^{n}(2x_i - 1)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}(1 + \frac{2}{n}i) = 1 + \frac{2}{n^2}\frac{n(n+1)}{2} = 2 + \frac{1}{n}$$

Hence $2 - \frac{1}{n} < I < 2 + \frac{1}{n},\ \forall n$ so $I = 2$.

2. (20 pts) Let $f(x) = \frac{x}{1-x},\ x \neq 1$.

$$\frac{f(x+h) - f(x)}{h} = -\frac{\frac{x+h}{1-(x+h)} - \frac{x}{1-x}}{h} = -\frac{(1-x))(x+h) - x(1-(x+h))}{h(1-(x+h))(1-x)}$$

$$= \frac{h}{h(1-(x+h))(1-x)} = \frac{1}{(1-(x+h))(1-x)}$$

Hence

$$f'(x) = \lim_{h\to 0}\frac{1}{(1-(x+h))(1-x)} = \frac{1}{(1-x)^2}$$

3. (20pts) Let g(x) denote the piecewise linear function on [0,5] whose graph is given below. Let $G'(x) = g(x),\ G(1) = 2$. Find $G(0), G(3), G(5)$.

$$G(x) = \int_1^x g(t)\ dt + 2 \text{ since } G(1) = 2\ .$$

We can do the integration using the signed area under the graph, so

$G(0) = 2 - (-\frac{1}{2}) = \frac{5}{2}$. $G(3) = 2 + \frac{1}{2}(2)(2) = 4$, $G(5) = 4 + 1(2) + \frac{1}{2}(2) = 7$.

4a.(10pts) Let $F(x) = \int_{\sqrt{x}}^{x^2} (1+t^2)^{10}\, dt$. Then

$$F'(x) = (1+x^4)^{10} \cdot 2x - (1+x)^{10} \cdot \frac{1}{2\sqrt{x}}$$

4b. (10pts) Use the mean value theorem to show that $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$.
Let $f(x) = \sqrt{1+x}$, $f'(x) = \frac{1}{2\sqrt{1+x}}$. Then $f(x) - f(0) = xf'(c)$ where $c \in (0, x)$. Hence

$$\sqrt{1+x} - 1 = \frac{x}{2\sqrt{1+c}} < \frac{x}{2} \ .$$

That is, $\sqrt{1+x} < 1 + \frac{x}{2}$.
Alternatively, let $g(x) = 1 + \frac{x}{2} - \sqrt{1+x}$. Then g(0)=0 and

$$g'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} > 0 \text{ for } x > 0.$$

Hence by MVT, $g(x) > 0$ for $x > 0$.

5. (20pts) Find all critical points and local max or min of $f(x) = x^4 - 4x^3$. Where is f increasing and decreasing? Use this information to sketch the graph.
$f' = 4x^3 - 12x^2 = 4x^2(x-3)$ so f has critical points at x=0 and x=3. We also see that f is strictly increasing for $x > 3$ and decreasing for $x < 3$ (strictly decreasing for $x < 0$. Hence x=3 is a local min which is also the absolute min. The point x=0 is not a local max or min.
$f''(x) = 12x^2 - 24x = 12x(x-2)$ so f is convex (concave up) for $x > 2$ and $x < 0$. f is concave (concave down) for $0 < x < 2$ The points $x = 0, x = 2$ are called inflection points.