Exam 1 Solutions

1a. A set S is inductive if $1 \in S$ and if $k \in S$ then $k+1 \in S$. The principle of mathematical induction states that if S is an inductive set, then S contains the natural numbers $\mathbf{N}$.
b. Let $S_n = 1 + 3 + 5 + \ldots + (2n-1)$. We want to show that the set S of natural numbers n so that $S_n = n^2$ is an inductive set and thus $S_n = n^2$ is valid all n. Note $1 \in S$ since $S_1 = 1 = 1^2$. Suppose $k \in S$, i.e $S_k = k^2$. Then

$$S_{k+1} = S_k + (2k+1) = k^2 + 2k + 1 = (k+1)^2 \ .$$

Thus $k+1 \in S$ and so S is inductive.

2a. S is bounded above if there is a real number M so that if $x \in S$, then $x \leq M$ The real number M is called an upper bound for S.
2b. c=lub S if c is an upper bound for S and if M is any upper bound for S, then $c \leq M$.
2c. Note that $\frac{2n-3}{3n+1} < \frac{2n}{3n=1} < \frac{2}{3}$ (so S is bounded above by $\frac{2}{3}$ and

$$\lim_{n\to\infty} \frac{2n-3}{3n+1} = \frac{2}{3} \ .$$

This means precisely that $\frac{2}{3}$= lub S.

3a. $\lim_{x\to a} f(x) = l$ if given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |x-a| < \epsilon$, then $|f(x) - l| < \epsilon$.
3b. Note that the value of f at x=2 is irrelevant and the limit must be 1. We calculate

$$|\frac{2x+1}{x^2+1} - 1| = |\frac{(2x+1)-(x^2+1)}{x^2+1}| = |\frac{2x-x^2}{x^2+1}| = \frac{|x||x-2|}{x^2+1} \ .$$

Note that the denominator $x^2+1 \geq 1$ so we must estimate $|x|$. Suppose first that $0 < |x-2| < \delta < 1$. Then $|x| < 2+1 = 3$ so

$$|f(x) - 1| = |\frac{2x+1}{x^2+1} - 1| = |\frac{|x||x-2|}{x^2+1} < 3\delta < \epsilon \ .$$

if we choose $\delta = \min(1, \frac{\epsilon}{3})$.

4a, True. If $f(2) \neq 0$ then $f(2)$ and $f(4)$ have opposite signs so the Intermediate value theorem says there is a $c \in [2,4]$ with $f(c) = 0$. Otherwise $f(2) = f(4) = 0$ so c=2,4 works.
4b. True. The quotient of the continuous functions f and x is continuous since $x \neq 0$ on [2,4].
4c. True. Since $|f| < 1$ on (-1,1), $|xf(x)| < |x|$ which tends to 0 as $x \to 0$.
4d. False. $f(f(x))$ need not be defined on [1,2]. For example take $f(x) = \sqrt{x} - 10$. Then $f(x) < 0$ on [1,2] so $f(f(x)) = \sqrt{\sqrt{x} - 10} - 10$ is not defined.