1 Completeness of R.

Recall that the completeness axiom for the real numbers $\mathbf{R}$ says that if $S \subset \mathbf{R}$ is a nonempty set which is bounded above (i.e there is a positive real number $M > 0$ so that $x \leq M$ for all $x \in S$), then l.u.b. S exists. Note that we need not state the corresponding axiom for nonempty sets S which are bounded below, that g.l.b S exists. For we can apply the completeness axiom to the set $-S = \{-x : x \in S\}$.

A simple application of the completeness axiom gives the so called Archimedean property of $\mathbf{R}$:

Theorem 1.1. $\mathbf{N}$ *is unbounded.*

Proof. If $\mathbf{N}$ is bounded, then by the completeness axiom, b=l.u.b $\mathbf{N}$ exists. Since $b - 1 < b$ there is an integer $n \in \mathbf{N}$ so that $n > b - 1$ (otherwise b-1 would be an upper bound which is impossible). But then $n + 1 > b$, a contradiction.

Corollary 1.2. *For any $x \in \mathbf{R}$ there is a positive integer n so that $n > x$.*

Proof. If not, x would be an upper bound for $\mathbf{N}$ contradicting Theorem 1.1.

Corollary 1.3. *If $x > 0$ and $y \in \mathbf{R}$ there is a positive integer n so that $nx > y$.*

Proof. Apply Corollary 1.2 with x replaced by $\frac{y}{x}$.

Theorem 1.4. *Let $a < b$ be real numbers. Then there is a rational number $r,\ a < r < b$.*

Proof. By Corollary 1.3, choose n so that $\frac{1}{n} < b - a$. Consider the multiples $\frac{m}{n}, m = 1, 2, \ldots$. There is a first integer m so that $r = \frac{m}{n} > a$.
We claim $r < b$. If not, then since $\frac{m-1}{n} < a$ and $\frac{m}{n} \geq b$, we would have $b - a > \frac{1}{n}$, a contradiction.

Definition 1.5. *Let $\{x_n\}$ be a sequence of real numbers. We say $\lim_{n\to\infty} x_n = l$ if given $\varepsilon > 0$ there exists an integer $N > 1$so that $|x_n - l| < \epsilon$ if $n \geq N$.*

We next show that the rational numbers are dense, that is, each real number is the limit of a sequence of rational numbers.

Corollary 1.6. *The rationals* **Q** *are dense in* **R**.

Proof. Let x be an arbitrary real number and let $a = x - \frac{1}{n}$, $b = x + \frac{1}{n}$. Then by Theorem 1.4 there is a rational r_n in (a, b). Clearly, $\lim_{n\to\infty} r_n = x$.

Although the rationals **Q** are dense, they are *countable*, i.e. they can be enumerated $\{r_1, r_2, \ldots\}$. In other words, they are in one to one correspondence with **N**. Note that both the even positive integers and the odd positive integers are in one to one correspondence with **N** (write down this correspondence yourself). We will give an enumeration of the positive rationals and leave it as an exercise to write down an enumeration of all of **Q**. Identify the fraction $\frac{p}{q}$, $p, q \in \mathbf{N}$ with the pair (p, q) in the first quadrant. Starting with (1,1) move right to (2,1) and then diagonally up to (1,2). Then move up to (1,3) and back down the diagonal to (2,2) and (3,1). Move right again to (4,1) and up the diagonal to (3,2), (2,3), (1,4). Move up to (1,5) and continue as above. Now make the enumeration except omit fractions already in the list:

$$1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, \ldots\ldots$$

This means that **Q** although dense is rather sparse. Consider now the irrational numbers. It is easy to see that they are dense for given any rational number x, the sequence of irrational numbers $x_n = x + \frac{\sqrt{2}}{n}$ (prove this is irrational) converges to x. Now if x is irrational, the sequence of irrational $x_n = x + \frac{1}{n}$ converges to x. The father of modern set theory Georg Cantor (1845-1918) proved using his famous diagonalization procedure that the irrational numbers are not countable. It suffices to show **R** is uncountable (since if the irrational were countable then **R** would be countable). We can further simplify and show that the real numbers in (0,1) are not countable. Suppose for contradiction that

$$x_1 = 0.a_1, a_2, a_3, \ldots$$

$$x_2 = 0.b_1, b_2, b_3, \ldots$$

$$x_3 = 0.c_1, c_2, c_3, \ldots$$

$$.$$

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is an enumeration of the decimal expansions of the irrationals in (0,1). Consider the real number $x = (n_1, n_2, n_3, \ldots \ldots)$ where n_1 is different from a_1 and $n_1 \neq 9$. Then $x \neq x_1$. Now choose $n_2 \neq b_2$ and $n_2 \neq 9$. Then $x \neq x_2$. Continue in this fashion. Then $x \in (0,1)$ but x is not in the list, a contradiction!

2 Monotone sequences and nested intervals

A first important result about existence of limits concerns monotone sequences.

Theorem 2.1. *Let $\{x_n\}$ be a monotone increasing (respectively decreasing) sequence of real numbers which is bounded above (respectively below). Then $\lim_{n\to\infty} x_n$ exists.*

Proof. We give the proof in the monotone increasing case; do the other case yourself. Let

$$S = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots\}$$

which is bounded above by assumption. Then by the completeness axiom, l=lub S exists. Then $x_n \leq l$ for all $n \geq 1$ and given $\varepsilon > 0$ there is a positive integer $N > 1$ so that $x_N > l - \varepsilon$ (otherwise $l - \varepsilon$ would be a smaller upper bound for the sequence). But then $x_n > l - \varepsilon$ for all $n \geq N$ (since the sequence is monotone increasing). Hence $l - \varepsilon < x_n \leq l$ for all $n \geq N$. In particular

$$|x_n - l| < \epsilon \ \text{ if } n > N \text{ i.e. } \lim_{n\to\infty} x_n = l \ .$$

A very useful application of this result is the so called Nested Interval theorem which we state after the next definition.he

Definition 2.2. *We say $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed nested intervals if $I_1 \supset I_2 \ldots I_n \supset I_{n+1} \supset \ldots$ and their lengths $|I_n|$ tend to zero (i.e. $\lim_{n\to\infty} |I_n| = 0$.)*

Theorem 2.3. *Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed nested intervals. Then there is a unique point c common to all the I_n, i.e. $\cap_{n=1}^{\infty} I_n = \{c\}$.*

Proof. Let $I_n = [a_n, b_n],\ n = 1, 2, \ldots$ Then $a_n \leq \alpha_{n+1} \leq b_1$ and $a_1 \leq b_{n+1} \leq b_n\ , n = 1, 2, \ldots$. By Theorem 2.1, the limits $\lim_{n\to\infty} a_n = a\ , \lim_{n\to\infty} b_n = b$ exist. However,

$$\lim_{n\to\infty} |I_n| = \lim_{n\to\infty} (b_n - a_n) = 0$$

so a=b=c.

It is useful (for later reference) to have a convergence criteria for non-monotone sequences.

Definition 2.4. *A sequence $\{x_n\}$ is said to be a Cauchy sequence if given $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ so that $|x_j - x_k| < \varepsilon$ if $j, k > N$.*

The importance of Cauchy sequences is due to the

Theorem 2.5. *A sequence $\{x_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. Suppose $\lim_{n\to\infty} = l$. Then given $\varepsilon > 0$ choose N so that $|x_n - l| < \frac{\varepsilon}{2}$ if $n > N$. Then by the triangle inequality, if $j, k > N$,

$$|x_j - x_k| = |(x_j - l) + (l - x_k| \leq |x_j - l| + |x_k - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

Conversely (this is the hard part of the proof) assume that the sequence $\{x_n\}$ is Cauchy. We first show the sequence is bounded. Take $\varepsilon = 1$; then there is a positive integer N so that $|x_n - x_{N+1}| < 1$ for $n > N$. Hence $|x_n| < |x_{N+1}| + 1$ for $n > N$. Therefore for all j, $|x_j| < M$ where $M = max|x_1|, |x_2|, \ldots, |x_N|, |x_{N+1}| + 1$. Now define

$$S = \{x \in \mathbf{R} : x < x_n \text{ for all but finitely many n }\}$$

Note that $-M$ is in S so $S \neq \emptyset$ and that S is bounded above by M. Hence l=l.u.b.S exists by the completeness axiom.
Claim: $l = \lim_{n\to\infty} x_n$.
To see this, given $\varepsilon > 0$ choose N (nothing to do with the N in the proof of boundedness) so that $|x_j - x_k| < \frac{\varepsilon}{2}$ if $j, k > N$. In particular,

$$a_{N+1} - \frac{\varepsilon}{2} < a_n < a_{N+1} + \frac{\varepsilon}{2} \ \text{ for } \ n > N\ .$$

So $a_{N+1} - \frac{\varepsilon}{2} \in S$ and $a_{N+1} + \frac{\varepsilon}{2} \notin S$ and hence

$$a_{N+1} - \frac{\varepsilon}{2} \leq l \leq a_{N+1} + \frac{\varepsilon}{2}\ .$$

Finally for $n > N$,

$$|x_n - l| \leq |x_n - x_{N+1}| + |x_{N+1} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which completes the proof.

3 Convergence of series

Definition 3.1. *A series* $\sum_{k=0}^{\infty} a_k$ *is said to converge to* S *(its sum) if the sequence of partial sums* $\{S_n\}$, $S_n = \sum_{k=1}^{n} a_k$ *converges to* S.

Example 3.2. *(geometric series)* $\sum_{n=0}^{\infty} r^n$, $|r| < 1$. *Then*

$$S_n = 1 + r + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

converges to $\frac{1}{1-r}$.

More sophisticated examples concern the (transcendental) number e.

Example 3.3. *Define* $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ *We must actually show the series is convergent to justify the definition. But this is easy since*

$$S_{n+1} = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots + \frac{1}{1 \cdot 2 \cdots n} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

By the previous example, $S_{n+1} < 3$ *so by Theorem 2.1, the series converges.*

Using similar reasoning we can show

Theorem 3.4. *e is irrational*

Proof. For $n > m$,

$$S_{n+1} = S_{m+1} + \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \ldots + \frac{1}{n!}$$

$$= S_{m+1} + \frac{1}{(m+1)!}(1 + \frac{1}{m+1} + \frac{1}{(m+2)(m+3)} + \ldots$$

$$< S_{m+1} + \frac{1}{(m+1)!}(1 + \frac{1}{m+1} + \frac{1}{(m+1)^2} + \ldots$$

$$= S_{m+1} + \frac{1}{(m+1)!}\,\frac{1}{1-\frac{1}{m+1}} = S_{m+1} + \frac{1}{m}\frac{1}{m!}$$

So for $n > m$,

$$S_{m+1} < S_{n+1} < S_{m+1} + \frac{1}{m}\frac{1}{m!}\ .$$

Now suppose e is rational (recall $2 < e < 3$). Write $e = \frac{p}{m}$ with $m \geq 2$. Then letting n tend to ∞ in the inequality above gives

$$S_{m+1} < \frac{p}{m} < S_{m+1} + \frac{1}{m}\frac{1}{m!}$$

Multiplying both sides by $m!$ gives

$$m!S_{m+1} < p(m-1)! < m!S_{m+1} + \frac{1}{m} < m!S_{m+1} + 1\ .$$

But $m!S_{m+1} = m! + m! + \frac{m!}{2} + \frac{m!}{3!} + \ldots + \frac{m!}{m!}$ is an integer since each individual term is. This is a contradiction since the integer $p(m-1)!$ lies strictly between two consecutive integers.

We next show that the more elementary definition of e is also valid.

Theorem 3.5. $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$.

Proof. Let $T_n = (1+\frac{1}{n})^n$ $n = 1, 2, \ldots$ Then by the binomial theorem,

$$T_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \ldots + \frac{n(n-1)\cdots 2\cdot 1}{n!}\frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\ldots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n})$$

So $T_n < S_{n+1} < 3$ and T_n is increasing (convince yourself) so $\lim_{n\to\infty} T_n$ exists by Theorem 2.1. To see that the limit is e, observe that for $m > n$,

$$T_m > T_n > 1 + 1 + \frac{1}{2!}(1-\frac{1}{m}) + \ldots + \frac{1}{n!}(1-\frac{1}{m})(1-\frac{2}{m})\ldots(1-\frac{n-1}{m})$$

Fix n and let m tend to ∞. Then $T \geq S_{n+1} \geq T_n$ so $T = \lim_{n\to\infty} S_n = e$.

4 Continuous functions and the "hard theorems"

We will use the Nested Intervals Theorem 2.3 to prove the Intermediate value theorem, the boundedness of continuous functions on closed interval from which the Extreme value theorem follows easily.

Theorem 4.1. *Let $f(x)$ be continuous on [a,b] and suppose $f(a) \neq f(b)$. Then for any value k intermediate to $f(a)$ and $f(b)$, there is at least one point $c \in [a, b]$ so that $f(c) = k$.*

Proof. For definiteness, assume $f(a) < k < f(b)$ (do the other case yourself) and set $g(x) = f(x) - k$. Then g is continuous on [a,b] and $g(a) < 0 < g(b)$. We need to show there is a point $c \in [a, b]$ with $g(c) = 0$. Suppose for contradiction that g is never zero. We use the method of bisection. Let $I_1 = [a, b]$ and bisect the interval by the midpoint P into two closed subintervals. If $g(P) < 0$ let I_2 be the right subinterval while if $g(P) > 0$, let I_2 be the left subinterval. In either case, g is negative on the left endpoint and positive on the right endpoint. Continue this bisection process to obtain a sequence of nested intervals $\{I_n\} = [a_n, b_n]$ of length $|I_n| = \frac{b-a}{2^{n-1}}$. By Theorem 2.3, there is a limit point $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ common to all the I_n. Now by continuity of g, there is an interval J of the form $(c - \delta, c + \delta)$ (if c is interior to [a,b]), $[a, a + \delta)$ (if c=a) or $(b - \delta, b]$ (if c=b) on which g is of one sign. But for n large, I_n is contained in J which is a contradiction, since by construction g is negative at the left endpoint of I_n and positive at the right endpoint of I_n.

Theorem 4.2. *Let f be continuous on [a,b]. Then f is bounded on [a,b].*

Proof. Suppose for contradiction that the theorem is false, Again we use the method of bisection. Let $I_1 = [a, b]$ and bisect the interval by the midpoint P into two closed subintervals. Then f must be unbounded on at least one of the two subintervals. If f is unbounded only one subinterval let I_2 be that subinterval, Otherwise choose I_2 to be the left subinterval. Continue this bisection process to obtain a sequence of nested intervals $\{I_n\} = [a_n, b_n]$ of length $|I_n| = \frac{b-a}{2^{n-1}}$. By Theorem 2.3, there is a limit point $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ common to all the I_n. Now by continuity of g, there is an interval J of the form $(c - \delta, c + \delta)$ (if c is interior to [a,b]), $[a, a + \delta)$ (if c=a) or

$(b-\delta, b]$ (if c=b) on which f is bounded. But for n large, I_n is contained in J which is a contradiction, since by construction f is unbounded on each I_n.

By Theorem 4.2, $M = \sup\{f(x) : x \in [a,b]\}$ and $m = \inf\{f(x) : x \in [a,b]\}$ exist. These are the so called extremes of f(x).

Theorem 4.3. *Let f be continuous on [a,b]. Then there are points $c_1, c_2 \in [a,b]$ so that $f(c_1) = M,\ f(c_2) = m$.*

Proof. We prove the theorem for M (do the other case yourself). If the theorem is false, then $f(x) \neq M$ and therefore $g(x) = \frac{1}{M-f(x)}$ is well-defined , positive and continuous on [a,b]. By Theorem 4.2 g is bounded, say $0 < g(x) \leq A$ on [a,b]. But then $M - f(x) > \frac{1}{A}$ on [a,b], that is, $f(x) < M - \frac{1}{A}$ contradicting the definition of M.

5 Uniform continuity

In this section we discuss uniform continuity and prove the important result that a continous function on a closed interval is uniformly continuous. This last result is needed to show that the integral of a continuous function always exists.

Example 5.1. *Let $f(x) = 5x + 3$ Suppose that we want to make $|f(x) - f(a)| = 5|x-a| < \varepsilon$ if $|x-a| < \delta$. Then we can take $\delta = \frac{\varepsilon}{5}$ The function $\delta(\varepsilon)$ is called a modulus of continuity for f at a.*

In the case of a linear function, the modulus is independent of the point $x = a$ This is not the case in our next example.

Example 5.2. *Let $f(x) = x^2$. Then $|f(x)-f(a)| = |x^2-a^2| = |x+a||x-a| < (|x|+|a|)|x-a| < (\delta+2|a|)\delta$ if $|x-a| < \delta$. Thus to make $|f(x)-f(a)| < \varepsilon$, the modulus δ must depend on ε and a. For example, we can take $\delta = \frac{\varepsilon}{1+\varepsilon+2|a|}$ (check that this works). Thus if a is large δ is much smaller than if a is close to 0.*

Definition 5.3. *We say that a function f is uniformly continuous if given $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ so that whenever $|x-y| < \delta,\ |f(x) - f(y)| < \varepsilon$.*

In other words, f is uniformly continuous if it has a uniform modulus of continuity. We will show in a minute that a continuous function on a closed interval is uniformly continuous. First we need a

Definition 5.4. *Let f be continuous on [a,b]. Then the span of f on [a,b] (sometimes called oscillation) is* $\sup_{[a,b]} f - \inf_{[a,b]}$. *A partition P of [a,b] is a subdivision* $a = t_0 < t_1 < t_2 < \dots t_n = b$ *into subintervals* (t_{k-1}, t_k) .

We look for a partition so that the span of f on each (t_{k-1}, t_k) is small.

Lemma 5.5. *let f be continuous on* $[a, b]$ *and let* $c \in [a, b]$ *Then given* $\varepsilon > 0$ *there is an interval J containing c of the form* $(c - \delta, c + \delta)$ *(if c is interior to [a,b]),* $[a, a + \delta)$ *(if c=a) or* $(b - \delta, b]$ *(if c=b) on which the span of f is less than* ε.

Proof. By continuity we can choose J so that $|f(x) - f(c)| < \frac{\varepsilon}{2}$ if $x \in J$. Then for $x, y \in J$, $|f(x) - f(y)| = |((f(x) - f(c)) + (f(c) - f(y)|$
$< |f(x) - f(c)| + |f(y) - f(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Theorem 5.6. *Let f be continuous on [a,b]. Then given* $\varepsilon > 0$ *there is a partition P of [a,b] so that the span of f on each* (t_{k-1}, t_k) *is smaller than* ε.

Proof. Suppose the theorem is false, that is, there is some $\varepsilon > 0$ so that no matter how we partition [a,b], the span of f on some subinterval of the partition is at least ε. Let $I_1 = [a, b]$ and bisect the interval by the midpoint P into two closed subintervals. Note that the theorem must be false on at least one of the two closed subinterval (possibly both), otherwise it would be true for I_1. Let I_2 be either the unique subinterval on which the theorem is false or the left half. Continue this bisection process to obtain a sequence of nested intervals $\{I_n\} = [a_n, b_n]$ of length $|I_n| = \frac{b-a}{2^{n-1}}$ on which the theorem fails for ε. By Theorem 2.3, there is a limit point $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ common to all the I_n. Now by Lemma 5.5, there is an interval J of the form $(c - \delta, c + \delta)$ (if c is interior to [a,b]), $[a, a + \delta)$ (if c=a) or $(b - \delta, b]$ (if c=b) on which the span of f is less than ε. . But for n large, I_n is contained in J which is a contradiction, since by construction the span of f on I_n is at least ε.

Corollary 5.7. *(uniform continuity) Let f be continuous on [a,b]. Then f is uniformly continuous on [a,b].*

Proof. Given $\varepsilon > 0$ there is by Theorem 5.6 a partition P so that the span of f on each (t_{k-1}, t_k) is smaller than $\frac{\varepsilon}{2}$. Let $\delta = \min(t_k - t_{k-1})$. Then if $|x - y| < \delta$, x and y must lie in two consecutive subintervals on which the span of f is at most ε