Practice problems for the Final Exam

1a. Give a precise statement of the mean value theorem.
b. Let f be a differentiable function such that f(0)=0 and $f' \leq 1$ for all x. Use the mean value theorem to show that $f(2) \neq 3$.

2. Prove by induction that $e^x > (1 + \frac{x}{n})^n$ for $x > 0$ and any integer $n \geq 1$. Hint: n=1 case. Let $f(x) = e^x - (1 + x)$ Show that $f(0) = 0$ and $f' > 0$ for $x > 0$.

3. Find the derivative of the following functions:
a. 2^{3x} b. $\sin(\cos^2 x)$ c. $f^{-1}(x)$ where $f(x) = (x + 1)^{10}$, $x > 0$.

3. Evaluate:
a. $\int \frac{1}{x^2\sqrt{x^2-1}}\ dx$
b. $\int \frac{x^2+1}{x^2+x-2}\ dx$
c. $\int \sin\sqrt{t}\ dt$

4a. Evaluate: $\lim_{h\to\infty} \frac{e^{4+3h}-e^4}{h}$
b. Determine the least positive integer N so that $a_n = \frac{2n}{n^3+1} < 0.01$ if $n > N$.

5. Estimate $\int_0^{\frac{1}{2}} \frac{1}{1+x^4}\ dx$ with an error less than 0.0001. Hint: Use the formula for the sum of a geometric series

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots + t^n + \frac{t^{n+1}}{1-t}$$

to find the Taylor series for $\frac{1}{1+x^4}$ and estimate the error.

6. Determine if the following sequences converge. If so, find the limit. Explain your reasoning.
a. $a_n = \frac{2^n}{n!}$
b.. $a_n = [\log(1 + \frac{1}{n}))]^n$
c. $a_n = n\cos\left(\frac{\pi}{2}e^{\frac{1}{n}}\right)$

7. Determine if the following series converge or diverge. Explain your reasoning.
a. $\sum_{k=1}^{\infty} \frac{4k^3}{3k^3+5}$

b. $\sum_{k=1}^{\infty} ke^{-k}$
c. $\sum_{k=1}^{\infty}(-1)^k(\sqrt{k+1}-\sqrt{k})$

8. Using the definition of a sum as the limit of the partial sums, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} \ .$$