Math 113 Final Exam Solutions

1a. (10pts) Find the upper sum for $\int_0^2 x^2 dx$ using a uniform partition of [0,2] into 4 equal subintervals.

$$x_i = \frac{1}{2}i,\ i = 0, 1, 2, 3, 4.\ U = \frac{1}{2}\sum_{i=1}^{4} \frac{i^2}{4} = \frac{1}{8}(1 + 4 + 9 + 16) = \frac{15}{4}$$

b. (10pts) Let $F'(x) = x\sqrt{x-3}$ for $x \geq 3$. Find F(x) if F(4)=0.

$$F(x) = \int_4^x t\sqrt{t-3}\ dt = \int_1^{\sqrt{x-3}} (3 + u^2)u \cdot 2u\ du =$$

$$= (2u^3 + \frac{2}{5}u^5)|_1^{\sqrt{x-3}} = 2(x-3)^{\frac{3}{2}} + \frac{2}{5}(x-3)^{\frac{5}{2}} - \frac{12}{5}$$

Alternatively you can find $F(x) = \frac{2}{3}(x(x-3)^{\frac{3}{2}} - \frac{2}{5}(x-3)^{\frac{5}{2}} - \frac{18}{5})$ using integration by parts. This answer is equivalent.

c. (10pts) Let $f(x) = \frac{1}{1+4x^2}$. Find $f^{12}(0)$.
Using the geometric series $\frac{1}{1-t} = 1 + t + t^2 + ... + t^n + ...$ with $t = -4x^2$ gives

$$f(x) = \sum_{k=0}^{\infty} (-1)^k 4^k x^{2k}\ .$$

Hence taking k=6, $f^{12}(0) = (12!)4^6$.

2a. (15pts) Find the area of the region in the first quadrant bounded by the graphs $y = x$ and $y = \frac{x}{\sqrt{10-x^2}}$.

$$A = \int_0^3 (x - \frac{x}{\sqrt{10-x^2}})\ dx = (\frac{x^2}{2} + (10 - x^2)^{\frac{1}{2}})|_0^3 = \frac{11}{2} - \sqrt{10}\ .$$

b. (15pts) Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x \to 4} \sqrt{x} = 2\ .$$

Take $\delta < 1$; then if $0 < |x - 4| < 1,\ 3 < x < 5$ and

$$|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2} \leq \frac{|x-4|}{2+\sqrt{3}} < \epsilon \ \text{ if } \delta = \min(1, (2+\sqrt{3})\epsilon)\ .$$

3. Let $f(x) = \int_0^x \sqrt{4 + \cos^2 t}\, dt$ and $A = f(\frac{\pi}{2})$.
a. (7pts)Explain why f(x) has an inverse.
$f'(x) = \sqrt{4 + \cos^2 x} > 2$ so f(x) is strictly increasing (so f is one to one) and therefore has an inverse.
b. (8pts) Find $(f^{-1})'(A)$.
$(f^{-1})'(A) = \frac{1}{f'(\frac{\pi}{2})} = \frac{1}{2}$ by part a.

4. (20 pts) Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$. Find all critcal points of f, intervals where f is increasing and decreasing and intervals where f is concave up (convex) and concave down (concave). Characterize the critical points as local or absolute max or min (or neither) and make a careful graph of f.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$$

Hence x=-1,0,2 are critical points and f is decreasing on $(-\infty, -1)$ and (0,2) and increasing on (-1,0) and $(2, \infty)$. Hence x=-1 is a local min, x=0 is a local max and x=2 is a local min. Since f(x) tends to $+\infty$ as x tends to $\pm\infty$ we see that x=2 is the absolute min (f(2)=-29). Also,

$$f"(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

which vanishes at $\frac{1}{3}(1 \pm \sqrt{7})$. Hence f is concave up on $(-\infty, \frac{1-\sqrt{7}}{3})$ and $(\frac{1+\sqrt{7}}{3}, \infty)$ and concave down on $(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3})$.

5. Let $f(x) = 4x^3 - \sin x$ Clearly f(0)=0 but there is another zero of f in (0,1).
a. (10 pts) Using the linear approximation of $\sin x$ about x=0, what is your first guess for this zero? Is this guess too big or too small?
Using $sinx \approx x$, $f(x) \approx 4x^3 - x = 0$ gives $x = \frac{1}{2}$ the initial guess. This is too big since $\sin x < x$ on (0,1).
b. (10 pts) Use the Taylor polynomial of degree 4 about x=0 (it really is degree 3 but I tell you this so you will better estimate the error) to get a better approximation for the positive zero x_0 of f. Estimate the error $|f(x_0)|$.
We use $\sin x = x - \frac{x^3}{6} + R_4(x)$ where $|R_4(x)| < \frac{x^5}{5!}$ on $(0, \frac{1}{2})$. So $f(x) = \frac{25}{6}x^3 - x - R_4$ and $x_0 = \frac{\sqrt{6}}{5} = 0.489$ is the better estimate with $|f(x_0)| < |R_4| < \frac{(\frac{\sqrt{6}}{5})^5}{120}$.

6. Evaluate the following (10 pts each):
a. $\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4} = \lim_{x\to 4} \frac{9-(x+5)}{(x-4)(3+\sqrt{x+5})} = \lim_{x\to 4} \frac{-1}{3+\sqrt{x+5}} = -\frac{1}{6}$

b. $\lim_{t\to 0}(1+2t)^{\frac{2}{t}} = \lim_{t\to 0}[(1+2t)^{\frac{1}{2t}}]^4 = e^4$.

c. $\lim_{x\to 0}\frac{(8+x)^{\frac{1}{3}}-8^{\frac{1}{3}}}{x} = f'(8) = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$ (where $f(x) = x^{\frac{1}{3}}$.)

7. Determine whether the series converges or diverges and justify (10pts each) .

a. $\sum_{n=1}^{\infty}(\frac{n}{n+1} - \frac{n-1}{n})$

$(\frac{n}{n+1} - \frac{n-1}{n}) = \frac{1}{n(n+1)} < \frac{1}{n^2}$ so the series converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$. It is also telescoping and the partial sums converge to 1.

b. $\sum_{n=1}^{\infty}\frac{n+1}{n^2}$

$\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ so the series diverges since the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

c. $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n}$

This series diverges since $|a_n| = \frac{n+1}{n}$ tends to 1 as n tends to infinity.