CONVEXITY OF 2-CONVEX TRANSLATING SOLITONS TO THE MEAN CURVATURE FLOW IN $\mathbb{R}^{n+1}$

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ABSTRACT. We prove that any complete immersed globally orientable uniformly 2-convex translating soliton $\Sigma \subset \mathbb{R}^{n+1}$ for the mean curvature flow is locally strictly convex. It follows that a uniformly 2-convex entire graphical translating soliton in $\mathbb{R}^{n+1}$, $n \geq 3$ is the axisymmetric "bowl soliton". We also prove a convexity theorem for graphical translating solitons defined over strictly convex domains with constant boundary data.

1. INTRODUCTION

A solution of the mean curvature flow is a smooth one-parameter family $\{\Sigma_t\}$ of hypersurfaces $\Sigma_t \subset \mathbb{R}^{n+1}$ with normal velocity equal to the mean curvature vector. A translating soliton for the mean curvature flow is the one that evolves purely by translation: $\Sigma_t = \Sigma + te$ for some fixed vector $e \in \mathbb{R}^{n+1} \backslash \{0\}$ and for all times t. In this case, the time slices are all congruent to Σ and satisfy

$$\boldsymbol{H} = \langle \nu, e \rangle \nu = e^{\perp} \tag{1}$$

where ν is a choice of normal vector field for Σ and $\boldsymbol{H} = -H\nu = -(\operatorname{div} \nu)\nu$ is the corresponding mean curvature vector of Σ. We call Σ a *translator* in the direction e for short. After a rotation, one can always assume $e = e_{n+1}$ after we normalize the speed to be one.

Translating solitons form a special class of eternal solutions for the mean curvature flow, that besides having their own intrinsic interest, are models of slow singularity formation. Therefore there has been a great deal of effort in trying to classify them in the most accessible case $H > 0$.

For $n = 1$ the unique solution is the grim reaper curve $\Gamma : x_2 = \log \sec x_1$, $|x_1| < \pi$. For $n = 2$, Wang [26] proved that any entire convex graphical translating soliton must be rotationally symmetric; this solution is called the "bowl soliton" [2], [10]. Moreover

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he showed that complete convex graphical translators that were not entire, necessarily live over strips. In [22], Spruck and Xiao proved that any complete immersed two-sided translating soliton in $\mathbb{R}^3$ with $H > 0$ must be convex. Furthermore, they also classified the asymptotic behavior of possible solutions in a strip and conjectured the existence of a unique locally strictly convex translating soliton asymptotic to the "tilted grim reaper", $x_3 = \lambda^2 \log sec\frac{x_1}{\lambda} + \sqrt{\lambda^2-1}\, x_2$, associated to any strip of width $\lambda\pi,\ \lambda > 1$.

Bourni et al. [8] proved the existence of such convex translators with the correct asymptotics (in fact in slabs of width greater than π in any dimension $n \geq 2$). At about the same time, Hoffman et al. [18] proved existence and uniqueness of locally strictly convex solitons in strips, thus completing the classification of all mean convex translating solitons in $\mathbb{R}^3$, which consists of the standard grim reaper surface in a strip of width π, the tilted grim reaper in a strip of width $\pi\lambda > \pi$, the locally strictly convex "delta wings" asymptotic to the tilted grim reaper at $\pm\infty$, in a strip of width $\pi\lambda > \pi$, and the bowl soliton.

In higher dimensions, Haslhofer [14] proved the uniqueness of the bowl soliton in arbitrary dimensions under the assumption that the translating soliton Σ is α-noncollapsed and uniformly 2-convex. The α-noncollapsed condition means that for each $P \in \Sigma$, there are closed balls $B^{\pm}$ disjoint from $\Sigma - P$ of radius at least $\frac{\alpha}{H(P)}$ with $B^+ \cap B^- = \{P\}$. This condition figures prominently in the regularity theory for mean convex mean curvature flow [17], [23], [24], [25]. The uniformly 2-convex condition (automatic if $n = 2$) means that if $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ are the ordered principle curvatures of Σ, then $\kappa_1 + \kappa_2 \geq \beta H$ for some uniform $\beta > 0$. The α-noncollapsed condition is a deep and powerful property of weak solutions of the mean convex mean curvature flow [25], [15], which implies that any complete α-noncollapsed mean convex solition Σ is convex with uniformly bounded second fundamental form. For some related results, see [4], [5], [7]. The purpose of this paper is to extend the work of [22] to 2-convex translating solitons in all dimensions. The main result of this paper is the following

Theorem 1.1. *If Σ^n is a complete immersed two-sided, uniformly 2-convex translator in $\mathbb{R}^{n+1}$, $n \geq 3$, then Σ is locally strictly convex.*

Note that uniform 2-convexity plus convexity implies that there can be at most one zero curvature at a point of Σ^n. Hence we are claiming in Theorem 1.1, that Σ^n cannot split off a line. For if $\Sigma^n = \Sigma^{n-1} \times \mathbb{R}$, then the uniform 2-convexity implies that the second fundamental form of Σ^{n-1} satisfies $h_{ij} \geq \frac{\beta}{n} H$ and Σ^{n-1} is a complete, locally strictly convex graph over either a slab or all of $\mathbb{R}^{n-1}$. Hence by the main theorem of Hamilton

[13], Σ^{n-1} is compact, a contradiction. As a corollary, we obtain the following uniqueness theorem for the bowl soliton by appealing to Corollary 8.3 of a recent paper of Bourni, Langford and Tinaglia [9] that characterizes the bowl soliton.

Corollary 1.2. *The bowl soliton is the unique uniformly 2-convex entire translator in* $\mathbb{R}^{n+1}$, $n \geq 3$.

In the opposite direction, Hoffman, Ilmanen, Martin and White developed a beautiful existence theory for graphical translators defined over an ellipsoid with constant height on the boundary. Using these solutions and compactness arguments, they proved the existence of an $(n-2)$ parameter family of distinct entire solutions (see Corollary 8.2 of [18]). They also developed a similar result for slabs of width greater than π (see Corollary 11.2 of [18]). In section 4, we study these solutions and prove in Theorem 4.1 that *they are locally strictly convex*. In fact we prove the strict convexity of such solutions over any smooth strictly convex domain. This last result combined with the results of [18] demonstrates there are many entire locally strictly convex translating solitons.

Corollary 1.3. *There exist an* $(n-2)$ *parameter family of distinct entire locally strictly convex graphical translating solitons in* $\mathbb{R}^{n+1}$.

For $n \geq 3$, the class of complete 2-convex translating solitons Σ is rather restrictive. One elementary but important observation is that $|A^{\Sigma}|$ is uniformly bounded. In fact, all the principal curvatures have absolute value less than one by Lemma 2.4. In our proof of Theorem 1.1, we shall essentially utilize the property that $\kappa_1/(H-\kappa_1)$ satisfies a fully nonlinear elliptic equation. However, even though κ_1 is smooth in $\{\kappa_1 < 0\}$, the second fundamental form $h_{ij} = A(e_i, e_j)$ of Σ may not be differentiable in a local curvature frame if there are positive curvatures with multiplicity. To carry out our analysis, we utilize a special approximation $\mu^n(\kappa)$ of $\min(\kappa_1, \ldots, \kappa_n)$ (section 2) which enjoys many good properties due to our uniform 2-convex assumption. The $\mu^n(\kappa)$ depend on a parameter δ and are defined recursively. Moreover, $\mu^n(\kappa) \to \kappa_1$ as $\delta \to 0$. We then apply the maximum principle to show that the infimum of $\mu^n/(H-\mu^n)$ cannot be achieved at a finite point, if δ is small enough. Thus the infimum must be achieved at infinity. As in [22], we apply the Omari-Yau maximum principle to a minimizing sequence of points P_N tending to infinity. This argument is delicate and utilizes the special properties of our approximation μ^n. After translating the P_N back to the origin and passing to a subsequence, we again obtain a contradiction if δ small enough. Finally, this means μ^n can never be negative when δ is small enough. Therefore $\kappa_1 \geq 0$ and Σ is convex.

The organization of the paper is as follows. In section 2 we construct an approximation of the minimum function and derive several essential properties of it that will be needed in the proof of Theorem 1.1. In section 3, we extend the method of [22] to prove the convexity of uniformly 2-convex translating solitons.

Finally in section 4, we study the family of graphical solutions (with boundary data zero) over strictly convex smooth domains, using the constant rank theorem of Bian-Guan [6] to prove Theorem 4.1.

2. Approximation of the minimum function

In this section we refine the iterative approximation of $\min\{x_1, \ldots, x_n\}$ of Heidusch [16] and Aarons [1] so that it can be used in a maximum principle argument to prove convexity of 2-convex translators for the mean curvature flow.

Definition 2.1. (i) The δ-approximation to the function

$$\min\{x_1, x_2\}$$

is given by

$$\mu(x_1, x_2) = \mu^2\,(x_1, x_2) = \frac{x_1 + x_2}{2} - \sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \delta x_1 x_2}$$

for any $\delta \in (0, \frac{1}{2})$.

(ii) For $n \geq 3$, define inductively the δ-approximation to

$$\min_n(x) := \min\{x_1, x_2, \cdots, x_n\}, \text{ for } x \in \mathbb{R}^n$$

by

$$\mu^n(x) := \frac{1}{n} \sum_{i=1}^{n} \mu(x_i, \mu^{n-1}(\bar{x}^i))$$

where $\bar{x}^i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$.

In order to accomondate our 2-convexity assumption and to show that the μ^n are well defined, we restrict μ^n to a convenient admissible domains $\mathcal{A}_n$, $\mathcal{A}_n^-$, $n \geq 2$, defined as follows:

$$\mathcal{A}_n = \left\{ x \in \mathbb{R}^n : |x_j| \leq 1, \forall\, j,\ x_k + x_l \geq \beta \sum_{i=1}^{n} x_i > 0 \ \forall\, k \neq l \right\}, \tag{2}$$

$$\mathcal{A}_n^- = \left\{ x \in \mathcal{A}_n : \min_n(x) \leq -\alpha \sum_{i=1}^{n} x_i \right\}, \tag{3}$$

for some fixed $\alpha, \beta \in (0,1)$. It is easy to see that if $x \in \mathcal{A}_n^-$, there is exactly one component of x with strictly negative minimum value and all the other components are strictly positive.

Lemma 2.2. *For $n \geq 3$, a necessary and sufficient condition that $\mathcal{A}_n^- \neq \emptyset$ is that*

$$\beta \leq \frac{1-(n-2)\alpha}{n-1}, \quad 0 < \alpha < \frac{1}{n-2}. \tag{4}$$

Proof. Without loss of generality, we may assume $x \in \mathcal{A}_n^-$ with $x_1 \leq x_2 \leq \cdots \leq x_n$. Then $x_1 < 0,\ x_2 > 0$ and

$$x_1 + x_2 \geq \beta \sum_{i=1}^{n} x_i > 0 \quad \text{and} \quad x_1 \leq \frac{-\alpha}{1+\alpha} \sum_{i=2}^{n} x_i.$$

or equivalently,

$$x_1 + x_2 \geq \frac{\beta}{1-\beta} \sum_{i=3}^{n} x_i\,,\ x_1 + \frac{\alpha}{1+\alpha} x_2 \leq \frac{-\alpha}{1+\alpha} \sum_{i=3}^{n} x_i,$$

which implies

$$\frac{1}{1+\alpha} x_2 \geq \left[\frac{\beta}{1-\beta} + \frac{\alpha}{1+\alpha}\right] \sum_{i=3}^{n} x_i \geq (n-2)\left[\frac{\beta}{1-\beta} + \frac{\alpha}{1+\alpha}\right] x_2.$$

Since $x_2 > 0$, we arrive at

$$\frac{1}{1+\alpha} \geq (n-2)\left[\frac{\beta}{1-\beta} + \frac{\alpha}{1+\alpha}\right]$$

which is equivalent to (4). Conversely for any $0 < \lambda \leq \frac{\alpha+1}{n-1}$, choose

$$\beta = \frac{1-(n-2)\alpha}{n-1},\ x_1 = -\frac{(n-1)\alpha\lambda}{1+\alpha},\ x_2 = x_3 = \cdots = x_n = \lambda.$$

Then we have

$$m := \sum_{i=1}^{n} x_i = \frac{(n-1)\lambda}{\alpha+1},\ x_1 = -\alpha m,\ x_1 + x_2 = \beta m$$

so $x \in \mathcal{A}_n^-$. $\square$

The function μ has the following important properties we will need to analyze μ^n.

Lemma 2.3. *For any $0 < \delta < \frac{1}{2}$ and $(x_1, x_2) \in \mathcal{A}_2$,*

i. μ is smooth and symmetric and if $x_1,\ x_2 > 0$, then $\mu(x) > 0$.

ii. μ *is monotonically increasing, concave and satisfies*

$$0 \leq \mu_{x_i} \leq 1, \ i = 1, 2.$$

iii. μ *is homogeneous of degree* 1 *and therefore satisfies* $\sum_{i=1}^2 x_i \mu_{x_i}(x) = \mu(x)$.

iv. $\mu(x) \leq \frac{1}{2}(x_1 + x_2)$ *and* $|\mu(x) - \min(x_1, x_2)| \leq 4\sqrt{\delta}\,(x_1 + x_2)$.

v. If $x \in \mathcal{A}_2^-$, *assuming* $x_1 \leq x_2$, *then*

$$\frac{1}{2}\left(1 + \frac{1 - 2\delta + 2\alpha(1-\delta)}{\sqrt{1 + 4\alpha(1+\alpha)(1-\delta)}}\right) \leq \mu_{x_1} < \frac{1}{2}\left(1 + \sqrt{1-\delta}\right) \tag{5}$$

and

$$\frac{1}{2}\left(1 - \frac{\alpha(1-2\delta) + 1 + \alpha}{\sqrt{1 + 4\alpha(1+\alpha)(1-\delta)}}\right) \leq \mu_{x_2} < \frac{1}{2}\left(1 - \sqrt{1-\delta}\right). \tag{6}$$

Proof. For i., it is easy to see that $\mu(x)$ is smooth except if $x_1 = x_2 = 0$, and $\mu = 0$ implies $\{x_1 = 0, x_2 \geq 0\}$ or $\{x_1 \geq 0, x_2 = 0\}$. If $x_1 > 0$ and $x_2 > 0$, then $\mu(x) > 0$.

For ii., a simple calculation show that

$$\mu_{x_1} = \frac{1}{2}\left(1 + \frac{(1-2\delta)x_2 - x_1}{\sqrt{(x_1 - x_2)^2 + 4\delta x_1 x_2}}\right) \in [0, 1] \tag{7}$$

and $\mu_{x_1} = 0$ if and only if $x_2 = 0$. Similarly

$$\mu_{x_2} = \frac{1}{2}\left(1 + \frac{(1-2\delta)x_1 - x_2}{\sqrt{(x_1 - x_2)^2 + 4\delta x_1 x_2}}\right) \in [0, 1] \tag{8}$$

and $\mu_{x_2} = 0$ if and only if $x_1 = 0$. Moreover,

$$(D^2\mu) = \frac{2\delta(1-\delta)}{\left[(x_1 - x_2)^2 + 4\delta x_1 x_2\right]^{\frac{3}{2}}} \begin{bmatrix} -x_2^2 & x_1 x_2 \\ x_1 x_2 & -x_1^2 \end{bmatrix} \tag{9}$$

is negative semi-definite. Hence μ is increasing and concave.

Statement iii. is obvious as is the first part of statement iv. We verify the second statement of iv.. By the homogeneity of μ, it suffices to restrict to $x_1 + x_2 = 1$ and we may also assume $x_1 \leq x_2$, $-1 \leq x_1 \leq 1/2$. Then

$$\begin{aligned} |\mu - x_1| =& \frac{1}{2}\left|\ \left((1 - 2x_1) - \sqrt{(1-\delta)(1-2x_1)^2 + \delta}\right)\right| \\ =& 2\delta\left|\frac{x_1^2 - x_1}{(1-2x_1) + \sqrt{(1-\delta)(1-2x_1)^2 + \delta}}\right| \leq 4\sqrt{\delta}, \end{aligned}$$

which gives $|\mu(x) - \min(x_1, x_2)| \leq 4\sqrt{\delta}$.

To prove v., let $x_1 = \min_2(x)$ and observe that $x \in \mathcal{A}_2^-$ is equivalent to

$$-1 < t := \frac{x_1}{x_2} \leq -\frac{\alpha}{1+\alpha}.$$

Moreover a simple calculation shows that both

$$\mu_{x_1} = \frac{1}{2}\left(1 + \frac{(1-2\delta) - t}{\sqrt{(t-1)^2 + 4\delta t}}\right), \tag{10}$$

and

$$\mu_{x_2} = \frac{1}{2}\left(1 + \frac{(1-2\delta)t - 1}{\sqrt{(t-1)^2 + 4\delta t}}\right), \tag{11}$$

are decreasing functions of t on $(-1, -\frac{\alpha}{1+\alpha}]$ and then (5), (6) follow by evaluation at the appropriate endpoints. $\square$

We next show that μ^n is well-defined on $\mathcal{A}_n$ and has nice properties.

Lemma 2.4. *Let $x \in \mathcal{A}_n$, $n \geq 3$ and set $m := \sum_{j=1}^n x_j$. Then*

i. $|\min_n(x)| \leq 1 - \frac{(n-2)\beta}{1-\beta}$ *if* $\min_n(x) < 0$,
ii. $|\min_n(x)| \leq \frac{1-\beta}{n-2}m,\ \max_i x_i \leq (1-\beta)m$,
iii. $x \in \mathcal{A}_n \Rightarrow \overline{x}^i \in \mathcal{A}_{n-1},\ i = 1, \ldots, n$,
iv. $|\mu^n(x) - \min_n(x)| \leq c(n)\sqrt{\delta}\, m$,
v. For δ sufficiently small, $x \in \mathcal{A}_n \Rightarrow (x_i, \mu^{n-1}(\overline{x}^i)) \in \mathcal{A}_2,\ i = 1, \ldots, n$,
vi. For any i such that $x_i \neq \min_n(x)$, $x \in \mathcal{A}_n^- \Rightarrow \overline{x}^i \in \mathcal{A}_{n-1}^-$,
vii. For δ sufficiently small, $x \in \mathcal{A}_n^- \Rightarrow (x_i, \mu^{n-1}(\overline{x}^i)) \in \mathcal{A}_2^-,\ i = 1, \ldots, n$.

Proof. We prove i., ii. together. Let $x \in \mathcal{A}_n$ and assume $x_1 \leq x_2 \leq \ldots \leq x_n$. Then

$$\sum_{i=1}^n x_i = m \text{ with } x_1 + x_2 \geq \beta m\,.$$

In particular,

$$(1-\beta)(x_1 + x_2) \geq \beta \sum_{i=3}^n x_i \geq (n-2)\beta x_2,$$

so

$$|x_1| \leq \frac{1-(n-1)\beta}{1-\beta} x_2 \leq 1 - \frac{(n-2)\beta}{1-\beta} \text{ if } x_1 < 0.$$

In addition, $\beta m + x_3 + \ldots + x_n \leq m$. Then $|x_1| \leq x_2 \leq x_3 \leq \frac{1-\beta}{n-2}m$ and it follows by induction that $x_n \leq (1-\beta)m$, proving i, ii.. Next observe that iii. is trivial if $i \geq 2$ or $i = 1$

and $x_1 \geq 0$. So assume $i = 1$ and $x_1 < 0$. Then we must show $x_2 + x_3 \geq \beta(\sum_{j=1}^{n} x_j - x_1)$ or equivalently $\beta x_1 + x_2 + x_3 \geq \beta \sum_{j=1}^{n} x_j = \beta m$. Observe that

$$\beta x_1 + x_2 + x_3 \geq (x_2 - (1-\beta)x_1) + x_1 + x_2 \geq \\ (x_2 - (1-\beta)x_1) + \beta \sum_{j=1}^{n} x_j > \beta m \, ,$$

proving iii. (since we already know $|x_j| \leq 1$ for all j).
We next prove iv. by induction; we assume $x_1 < x_2 \leq \ldots \leq x_n$. Since μ is homogenous of degree one, we will also assume $m = 1$. The case $n = 2$ is Lemma 2.3 part iv. Now assume we have proved iv. for $n = 2, \ldots, k-1$. Then by the monotonicity of μ,

$$\mu^k(x) = \frac{1}{k} \sum_{j=1}^{k} \mu(x_j, \mu^{k-1}(\overline{x}^j)) = \frac{1}{k}(kx_1 + O(\sqrt{\delta})) = x_1 + O(\sqrt{\delta}),$$

which gives $|\mu^k(x) - \min_k(x)| \leq c(k)\sqrt{\delta}$ completing the induction.
We first prove v. for $n = 3$. Let $x \in \mathcal{A}_3$ and note that if $x_1 \geq 0$, then by part iii. and the definition of μ, $0 \leq \mu(x_i, \overline{x}^i) < 1$, $i = 1, 2, 3$. If $-1 \leq x_1 < 0$, then

$$\mu(x_1, x_2) - x_1 = \frac{x_2 - x_1}{2} - \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \delta x_1 x_2} > \frac{x_2 - x_1}{2} - \frac{x_2 - x_1}{2} = 0.$$

Hence $\mu(x_1, x_2) \geq x_1 \geq -1$ completing the proof for $n = 3$. Now suppose v. holds for $n = k - 1$. Then

$$|\mu^{k-1}(x)| \leq \frac{1}{k-1} \sum_{j=1}^{k-1} |\mu(x_j, \mu^{k-2}(\overline{x}^j))| \leq 1$$

since by induction, $|\mu^{k-2}(\overline{x}^j)| \leq 1$ and then $|\mu(x_j, \mu^{k-2}(\overline{x}^j)| \leq 1$ by the $n = 3$ case. It remains to show $x_i + \mu^{k-1}(\overline{x}^i) > 0$ for all i. But again using part iv., we find (as in the proof of part iii.)

$$x_i + \mu^{k-1}(\overline{x}^i) \geq x_1 + x_2 - c(n)\sqrt{\delta} \sum_{j=1}^{n} x_j \geq (\beta - c(n)\sqrt{\delta}) \sum_{j=1}^{n} x_j > 0$$

for δ small enough, completing the proof of part v.
To prove vi. it suffices by part iii. to show that for $i \geq 2$, $x_1 \leq -\alpha(m - x_i)$ or equivalently, $x_1 + \alpha m \leq \alpha x_i$. This holds trivially since the left hand side is nonpositive since $x \in \mathcal{A}_n^-$.

Finally we prove vii. Again assume $m = 1$. By part iv.,

$$\min(x_i, \mu^{n-1}(\overline{x}^i)) \leq \min_n(x) + c(n)\sqrt{\delta} = x_1 + c(n)\sqrt{\delta}$$
$$x_i + \mu^{n-1}(\overline{x}^i) \leq x_n + x_1 + c(n)\sqrt{\delta}.$$

Thus it suffices to show that

$$x_1 + c(n)\sqrt{\delta} \leq -\alpha(x_n + x_1 + c(n)\sqrt{\delta}) = -\alpha(1 - (x_2 + \ldots + x_{n-1}) + c(n)\sqrt{\delta}).$$

Since by hypothesis, $x \in \mathcal{A}_n^-$, $x_1 \leq -\alpha m = -\alpha$, we need only show that

$$x_2 \geq (1 + \frac{1}{\alpha})c(n)\sqrt{\delta}.$$

But $x_1 + x_2 \geq \beta m = \beta$, hence $x_2 \geq \beta - x_1 \geq \beta + \alpha > (1 + \alpha^{-1})c(n)\sqrt{\delta}$ for δ sufficiently small. □

The following additional properties of the μ^n follow easily from Lemmas 2.3, 2.4 by induction.

Corollary 2.5. *For any $0 < \delta < \frac{1}{2}$, on $\mathcal{A}_n$, $n \geq 2$ we have:*

i. *μ^n is smooth and symmetric and if $x_i > 0$, $\forall\, i$, then $\mu_n(x) > 0$.*
ii. *μ^n is monotonically increasing and concave and satisfies*

$$0 \leq \mu^n_{x_i} \leq 1 \ \textit{for}\ i = 1, \cdots, n.$$

iii. *μ^n is homogeneous of degree 1 and therefore $\sum_{i=1}^n x_i \mu^n_{x_i}(x) = \mu^n(x)$.*
iv. *$\mu^n(x) \leq \frac{1}{n}\sum_{j=1}^n x_j$.*

Next we shall prove some important properties related to the derivatives of μ^n.

Lemma 2.6. *For $n \geq 2$, there exist $\delta_n = \delta_n(\alpha, \beta, n)$ such that the following properties hold for $0 < \delta < \delta_n$:*

i. *If $x \in \mathcal{A}_n$ and $x_i < x_j$, then $\mu^n_{x_i} > \mu^n_{x_j}$.*
ii. *If $x \in \mathcal{A}_n^-$, then*

$$\mu^n_{x_i}(x) \to \begin{cases} 1 & \textit{if } x_i = \min_n(x) \\ 0 & \textit{otherwise} \end{cases}$$

uniformly as $\delta \to 0^+$.
iii. *If $x \in \mathcal{A}_n^-$, then $\mu^n_{x_i x_i}(x) < 0$.*

Proof. We first verify properties i., ii., iii. for $n = 2$. From the explicit formulas (7) and (8), property (1) is obvious. If $(x_1, x_2) \in \mathcal{A}_2^-$, then as in Lemma 2.3 part (5) for $x_1 = \min_2(x)$, $t := \frac{x_1}{x_2} \in (-1, -\frac{\alpha}{1+\alpha}]$ and one sees that $\mu_{x_1} \to 1$ by (10) and $\mu_{x_2} \to 0$ by (11) as $\delta \to 0^+$. Therefore property ii. is established and property iii. follows from (9).

For $n \geq 3$ we prove properties i., ii., iii. by induction. Assume they hold for μ^{n-1}. For i., it is enough to show that if $x_1 < x_2$, then $\mu^n_{x_1} > \mu^n_{x_2}$. Write

$$\mu^n(x) = d^n(x) + r^n(x) \tag{12}$$

where

$$\begin{aligned} d^n(x) =& \frac{1}{n}\mu(x_1, \mu^{n-1}(\bar{x}^1)) + \frac{1}{n}\mu(x_2, \mu^{n-1}(\bar{x}^2)) \\ r^n(x) =& \frac{1}{n}\sum_{i=3}^{n} \mu(x_i, \mu^{n-1}(\bar{x}^i)). \end{aligned}$$

One sees easily that $d^n(x)$ and $r^n(x)$ are both symmetric and concave in the variables x_1 and x_2. Consequently $d^n_{x_1} \geq d^n_{x_2}$ and $r^n_{x_1} \geq r^n_{x_2}$. Furthermore,

$$r^n_{x_1}(x) - r^n_{x_2}(x) = \frac{1}{n}\sum_{i=3}^{n} \mu_{y_2}(x_i, \mu^{n-1}(\bar{x}^i))[\mu^{n-1}_{x_1} - \mu^{n-1}_{x_2}](\bar{x}^i). \tag{13}$$

Here μ_{y_2} means the partial derivative with respect to the second variable. It follows from Lemma 2.4 that $\bar{x}^i \in \mathcal{A}_{n-1}$ and $(x_i, \mu^{n-1}(\bar{x}^i)) \in \mathcal{A}_2$. Therefore,

$$\mu_{y_2}(x_i, \mu^{n-1}(\bar{x}^i)) \geq 0,$$

with equality for such an i only when $x_i = 0$. By our assumption on μ^{n-1}, one knows

$$\mu^{n-1}_{x_1}(\bar{x}^i) > \mu^{n-1}_{x_2}(\bar{x}^i).$$

Since $x \in \mathcal{A}_n$, x cannot have two entries which are both equal to 0. Therefore $r^n_{x_1}(x) - r^n_{x_2}(x) > 0$ except possibly if $n = 3$ and $x_3 = 0$. However, in this case one must have $x_1, x_2 > 0$ and consequently $\mu^3(x) = \mu(x_1, x_2)$, a case already verified. The proof of property i. is complete.

To prove property ii., we may assume $x_1 = \min_n(x)$. Since $x \in \mathcal{A}_n^-$, $x_1 < 0$. Note that

$$\mu^n_{x_1}(x) = \frac{1}{n}\mu_{y_1}(x_1, \mu^{n-1}(\bar{x}^1)) + \frac{1}{n}\sum_{i=2}^{n} \mu_{y_2}(x_i, \mu^{n-1}(\bar{x}^i))\mu^{n-1}_{x_1}(\bar{x}^i).$$

It follows from Lemma 2.4 that $(x_i, \mu^{n-1}(\bar{x}^i)) \in \mathcal{A}_2^-$ for any $1 \leq i \leq n$ and $\bar{x}^i \in \mathcal{A}_{n-1}^-$ for any $i \geq 2$. Again by our assumption on μ^{n-1}, we see that $\mu^n_{x_1} \to 1$ as $\delta \to 0$. One can prove similarly that $\mu^n_{x_i} \to 0$ for $i \geq 2$.

To prove iii., we split μ^n as

$$\mu^n(x) = \frac{1}{n}\mu(x_i, \mu^{n-1}(\bar{x}^i)) + \frac{1}{n}\sum_{j\neq i} \mu(x_j, \mu^{n-1}(\bar{x}^j))$$

The last term is concave in x_i and its second pure derivative in x_i is non-positive. Thus we need to show

$$\mu_{x_i x_i}(x_i, \mu^{n-1}(\bar{x}^i)) = \mu_{y_1 y_1}(x_i, \mu^{n-1}(\bar{x}^i)) < 0. \tag{14}$$

However, this can be seen from $(x_i, \mu^{n-1}(\bar{x}^i)) \in \mathcal{A}_2^-$. The proof by induction is now complete. □

Recall from Lemma 2.4 part v. that if $x \in \mathcal{A}_2^-$ with $x_1 < 0 < x_2$,

$$\mu_{x_2} \geq \Lambda = \Lambda(\alpha, \delta) := \frac{1}{2}\left(1 - \frac{\alpha(1-2\delta)+1+\alpha}{\sqrt{1+4\alpha(1+\alpha)(1-\delta)}}\right) > 0.$$

Lemma 2.7. *Suppose $n \geq 3$. There exists $\delta_n = \delta_n(\alpha, \beta, n)$ such that if $0 < \delta < \delta_n$ and $x \in \mathcal{A}_n^-$, then*

$$\frac{\mu^n_{x_i} - \mu^n_{x_j}}{x_i - x_j} \leq -\frac{\Lambda^{n-2}}{2\cdot n!}\frac{1}{|x_i - x_j| + \sqrt{x_i x_j}} \tag{15}$$

for any $x_i \neq x_j$ and $x_i > 0, x_j > 0$.

Proof. We first examine the case $n = 2$. If $x_1 \neq x_2$ and $x_1, x_2 > 0$, then it follows from (7) and (8) that

$$\frac{\mu_{x_1} - \mu_{x_2}}{x_1 - x_2} = -\frac{1-\delta}{2\sqrt{(x_1-x_2)^2 + 4\delta x_1 x_2}} \leq -\frac{1}{4|x_1 - x_2| + 4\sqrt{x_1 x_2}} \tag{16}$$

provided $\delta < \frac{1}{4}$. Now consider $n \geq 3$, using the decomposition $\mu^n = d^n + r^n$ as in (12). Then

$$\frac{\mu^n_{x_i} - \mu^n_{x_j}}{x_i - x_j} \leq \frac{r^n_{x_i} - r^n_{x_j}}{x_i - x_j}$$

We may assume $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_1 < 0$ since $x \in \mathcal{A}_n^-$. Notice we are assuming $x_i, x_j > 0$, which implies $i > 1$, $j > 1$.

Supposing $n = 3$ and recalling (13), one obtains

$$\begin{aligned}\frac{r^3_{x_2} - r^3_{x_3}}{x_2 - x_3} \leq & \frac{1}{3}\mu_{y_2}(x_1, \mu(\bar{x}^1)) \cdot \frac{\mu_{x_2}(\bar{x}^1) - \mu_{x_3}(\bar{x}^1)}{x_2 - x_3} \\ \leq & -\frac{\Lambda}{2\cdot 3!}\frac{1}{|x_2 - x_3| + \sqrt{x_2 x_3}}\end{aligned}$$

We use induction to prove the cases $n > 3$. Let l be an number in $\{2, \cdots, n\}$ with $l \neq i, j$. Again using (13),

$$\frac{r^n_{x_i} - r^n_{x_j}}{x_i - x_j} \leq \frac{1}{n}\mu_{y_2}(x_l, \mu^{n-1}(\bar{x}^l)) \cdot \frac{\mu^{n-1}_{x_i}(\bar{x}^l) - \mu^{n-1}_{x_j}(\bar{x}^l)}{x_i - x_j}$$

By Lemma 2.6, $(x_l, \mu^{n-1}(\bar{x}^l)) \in \mathcal{A}_2^-$, so it follows from (8) that

$$\frac{r^n_{x_i} - r^n_{x_j}}{x_i - x_j} \leq \frac{\Lambda}{n}\frac{\mu^{n-1}_{x_i}(\bar{x}^l) - \mu^{n-1}_{x_l}(\bar{x}^1)}{x_i - x_j} \leq -\frac{\Lambda^{n-2}}{2 \cdot n!}\frac{1}{|x_i - x_j| + \sqrt{x_i x_j}},$$

where the last inequality follows from our induction hypothesis and $\bar{x}^l \in \mathcal{A}^-_{n-1}$.

□

3. 2-CONVEX TRANSLATORS

Suppose e_{n+1} is the direction of the translation. The mean curvature $H = \mathrm{tr}A$ and second fundamental form $A = (h_{ij})$ satisfy the following equations, for instance see [19]

$$\Delta H + \nabla_{e_{n+1}} H + |A|^2 H = 0, \tag{17}$$

$$\Delta A + \nabla_{e_{n+1}} A + |A|^2 A = 0. \tag{18}$$

Define $L = \Delta + \nabla_{e_{n+1}}$, which is so called drift laplacian. Suppose $\kappa_1, \cdots, \kappa_n$ are the principle curvatures of Σ and $\tau_1, \cdots, \tau_n$ is a smooth orthonormal frame. We write $\mu^n = \mu^n(\kappa_1, \cdots, \kappa_n)$ for short.

Suppose Σ is uniformly 2-convex. More precisely, assuming $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$, we have $\kappa_1 + \kappa_2 \geq \beta H > 0$ for some $\beta > 0$. Then $(\kappa_1, \cdots, \kappa_n) \in \mathcal{A}_n$. By Corollary 2.5, μ^n is smooth on Σ. Since μ^n is a symmetric function of the principle curvatures, we can express μ^n as a fully nonlinear equation of the second fundamental form, namely $\mu^n(\kappa_1, \cdots, \kappa_n) = F((h^i_j))$. Here we point out that since μ^n is a smooth function and symmetric on its arguments, then F is smooth on the second fundamental forms. Denote $F_{ij} = \frac{\partial F}{\partial h_{ij}}$ and $F_{ij,rs} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{rs}}$. Then a standard calculation gives

$$\begin{aligned}
\Delta\mu^n =& F^{ij} h_{ijkk} + F^{ij,rs} h_{ijk} h_{rsk} \\
=& F^{ij}(-|A|^2 h_{ij} - \nabla_{e_{n+1}} h_{ij}) + F^{ij,rs} h_{ijk} h_{rsk} \\
=& -|A|^2 F^{ij} h_{ij} - \nabla_{e_{n+1}} \mu + F^{ij,rs} h_{ijk} h_{rsk}
\end{aligned}$$

where we have used (18). Recall that by the definition of L,

$$L\mu^n = -|A|^2 F^{ij} h_{ij} + F^{ij,rs} h_{ijk} h_{rsk}.$$

Define $Q_\delta = \frac{\mu^n}{H-\mu^n}$. Then a simple calculation gives

$$\Delta Q_\delta = \frac{H\Delta\mu^n - \mu^n \Delta H}{(H-\mu^n)^2} - 2\left\langle \frac{\nabla(H-\mu^n)}{H-\mu^n}, \nabla Q_\delta \right\rangle$$

which means

$$LQ_\delta + 2\frac{\langle \nabla(H-\mu^n), \nabla Q_\delta\rangle}{H-\mu^n} = \frac{HL\mu^n - \mu^n LH}{(H-\mu^n)^2}. \tag{19}$$

The previous calculation of $L\mu^n$ and (17) shows that

$$HL\mu^n - \mu^n LH = H|A|^2(\mu^n - F^{ij}h_{ij}) + HF^{ij,pq}h_{ijk}h_{pqk} = HF^{ij,pq}h_{ijk}h_{pqk}, \tag{20}$$

since $F^{ij}h_{ij} = \mu^n$ by the homogeneity property of μ^n. Furthermore, the concavity of μ^n implies $F^{ij,rs}$ is negative definite. Since $H > 0$, it follows from (19), (20) that

$$LQ_\delta + 2\frac{\langle \nabla(H-\mu^n), \nabla Q_\delta\rangle}{H-\mu^n} = \frac{HF^{ij,pq}h_{ijk}h_{pqk}}{(H-\mu^n)^2} \le 0 \tag{21}$$

We can restate our main result as

Theorem 3.1. *If Σ^n is a uniformly 2-convex translator with $n \ge 3$, then*

$$\lim_{\delta\to 0^+} \inf_{\Sigma} Q_\delta \ge 0.$$

As a consequence, Σ must be convex.

Proof. We will prove the theorem by contradiction. Recall that $\mu^n < \frac{1}{n}H$ and $H > 0$. It is easy to see that $Q_\delta > -1$ on Σ. Assume $\inf_\Sigma Q_\delta < -\varepsilon_0 < 0$ for any $\delta > 0$ small. From now on, we choose $\tau_1, \tau_2, \cdots, \tau_n$ to be principle directions corresponding to the ordered principle curvatures $\kappa_1 \le \cdots \le \kappa_n$.

Suppose Q_δ attains its infimum at some interior point $P \in \Sigma$. Applying the strong maximum principle to (21) yields $Q_\delta \equiv Q_\delta(P) < -\varepsilon_0 < 0$ and $F^{ij,pq}h_{ijk}h_{pqk} = 0$. In particular, $\mu^n = \frac{Q_\delta}{1+Q_\delta}H$. It follows from Lemma 2.4 (iv) that,

$$\kappa_1 \le \mu^n + c(n)\sqrt{\delta}H = \frac{Q_\delta}{1+Q_\delta}H + c(n)\sqrt{\delta}H \le -\alpha H \tag{22}$$

for some $\alpha > 0$ small, if δ is taken small enough. We fix $\alpha, \beta > 0$ small enough that (4) holds. Consequently, $(\kappa_1, \cdots, \kappa_n) \in \mathcal{A}^-_{n-1}$ for such α and β and all points on Σ.

Claim 1. If Q_δ has an interior infimum, then $h_{ijk} \equiv 0$, that is $\nabla A \equiv 0$.

Proof. With the notation that $\mu_i^n = \frac{\partial \mu^n}{\partial \kappa_i}$ and $\mu_{ij}^n = \frac{\partial^2 \mu^n}{\partial \kappa_i \partial \kappa_j}$, it is well known that $F^{ij,pq} h_{ijk} h_{pqk}$ can be calculated (see for example, [21] or [3]):

$$F^{ij,pq} h_{ijk} h_{pqk} = \sum_{i,j,k} \mu_{ij}^n h_{iik} h_{jjk} + \sum_{\{i,j:\kappa_i \neq \kappa_j\}} \sum_k \frac{\mu_i^n - \mu_j^n}{\kappa_i - \kappa_j} h_{ijk} h_{ijk}. \tag{23}$$

By concavity, both terms on the right hand side of (23) are nonpositive. Since Q_δ is constant, then (21) implies $F^{ij,pq} h_{ijk} h_{pqk} = 0$. Therefore both of the above terms must be 0. Because of Lemma 2.6,

$$\frac{\mu_i^n - \mu_j^n}{\kappa_i - \kappa_j} < 0 \quad \text{for } \kappa_i \neq \kappa_j.$$

Thus necessarily, for each i and j such that $\kappa_i \neq \kappa_j$

$$h_{ijk} = 0, \quad \forall\, k.$$

However, if $i \neq j$ and $\kappa_i = \kappa_j$, then $h_{ijk} = 0$ for any k. Indeed, for a dense open set of points, we can choose a smooth orthonormal frame $\tau_1, \ldots, \tau_n$ of eigenvectors for the ordered principal curvatures $\kappa_1 \leq \ldots \leq \kappa_n$, see Theorems 2 and 3 of [20]. Then (see formula (9) of [8]),

$$\begin{aligned} 0 &= \tau_k h_{ij} = \nabla_{\tau_k} h_{ij} + h(\nabla_{\tau_k} \tau_i, \tau_j) + h(\tau_i, \nabla_{\tau_k} \tau_j) \\ &= h_{ijk} + \Gamma_{ki}^j \kappa_j + \Gamma_{kj}^i \kappa_i = h_{ijk} + \Gamma_{ki}^j (\kappa_j - \kappa_i) = h_{ijk}. \end{aligned} \tag{24}$$

Summing up the above analysis, one has $h_{ijk} \equiv 0$ for $i \neq j$ and any k, for a dense open set of points of Σ. Since $A = (h_{ij})$ is a Codazzi tensor, it remains to show $h_{iii} = 0$ for all i. Recall that we also know $\sum_{i,j,k} \mu_{ij}^n h_{iik} h_{jjk} = 0$, which now reduces to $\sum_i \mu_{ii}^n h_{iii}^2 = 0$. Since $\mu_{ii}^n < 0$ as a consequence of Lemma 2.6 iii., it follows that $h_{iii} = 0$ for all i. Claim 1 is established for a dense open set of points and hence all points by continuity. $\square$

Now if $\nabla A \equiv 0$, then $\nabla H \equiv 0$. It follows that

$$0 = \nabla_{\tau_l} H = \kappa_l \langle \tau_l, e_{n+1} \rangle.$$

Since $\kappa_l \neq 0$ for any $1 \leq l \leq n$, then the unit normal of Σ is $\nu = e_{n+1}$ and $H \equiv 1$ which is impossible. Therefore Q_δ cannot have an interior infimum.

Therefore the infimum of Q_δ is achieved at infinity and we can apply the Omori-Yau maximum principle. That is, there exists a sequence$P_{\delta,N} \to \infty$ such that

$$Q_\delta(P_{\delta,N}) \to \inf_\Sigma Q_\delta, \quad |\nabla Q_\delta(P_{\delta,N})| < \frac{1}{N}, \quad \Delta Q_\delta(P_{\delta,N}) > -\frac{1}{N} \tag{25}$$

Moreover, we can perturb the $P_{\delta,N} \to \infty$ slightly so that (23) and (24) are also satisfied.

If $H(P_{\delta,N})$ does not tend to zero, we can choose a subsequence (which we still denote by $P_{\delta,N}$) and consider $\Sigma_N = \Sigma - P_{\delta,N}$. We know from Lemma 2.4 that Σ has bounded principle curvatures, so the same is true of Σ_N. Then a subsequence of the Σ_N will converge smoothly to Σ_∞, which is again a mean convex translating soliton with $H(0) > 0$. However,

$$\inf_{\Sigma_\infty} \frac{\mu^n}{H-\mu^n} = \frac{\mu^n}{H-\mu^n}(0) \leq -\varepsilon_0$$

This contradicts the fact that Q_δ has no interior negative minimum.

Therefore, we must have $H(P_{\delta,N}) \to 0$ as $N \to \infty$.

Claim 2. If $n \geq 3$, then $\kappa_i(P_{\delta,N}) \to 0$, for any $1 \leq i \leq n$ as $N \to \infty$. Moreover, there exists $C(n) > 1$ such that for all l,

$$C(n)^{-1} \leq \left|\frac{\kappa_l}{H-\mu^n}\right|(P_{\delta,N}) \leq C(n) \tag{26}$$

$$C(n)^{-1} \leq \frac{H}{H-\mu^n}(P_{\delta,N}) \leq C(n). \tag{27}$$

provided δ small and N large enough.

Proof. Order the principal curvatures as before: $\kappa_1 \leq \cdots \leq \kappa_n$. Since Σ is uniformly 2-convex,

$$H = \sum_{j=1}^n \kappa_j \geq \sum_{j=3}^n \kappa_j + \beta H.$$

Therefore $\kappa_j(P_{\delta,N}) \to 0$ for $j \geq 3$ and $(\kappa_1+\kappa_2)(P_{\delta,N})$ as $N \to \infty$. By Lemma 2.4 ii., we also have $\kappa_j(P_{\delta,N}) \to 0,\ j = 1,2$ at the same time. Using Lemma 2.4 iv., at each $P_{\delta,N}$,

$$0 \geq \frac{\mu^n}{H} \geq \frac{\kappa_1 - c(n)\sqrt{\delta}\,H}{H} \geq -c(n)\sqrt{\delta} - \frac{1}{n-2}$$

which implies

$$1 \geq \frac{H}{H-\mu^n} = \frac{1}{1-\mu^n/H} \geq \frac{n-2}{n-1+(n-2)c(n)(\delta)} > 0$$

Suppose δ is small enough such that at each $P_{\delta,N}$, (22) holds. Then if N is large enough,

$$-1 \leq \frac{\kappa_1}{H-\mu^n} \leq \frac{\mu^n + c(n)\sqrt{\delta}H}{H-\mu^n} \leq Q_\delta + c(n)\sqrt{\delta} \leq -\frac{1}{2}\varepsilon_0 + c(n)\sqrt{\delta} \leq -\frac{1}{4}\varepsilon_0.$$

For $l \geq 2$, it is easy to see

$$\frac{\beta H}{H-\mu^n} \leq \frac{\kappa_l}{H-\mu^n} \leq 1.$$

Our Claim 2 follows from taking δ small enough and N large enough $\square$

It follows from Claim 2 that

$$-\frac{\mu^n \nabla_l H}{(H-\mu^n)^2} = \frac{-\mu^n}{H-\mu^n}\frac{\kappa_l \langle \tau_l, e_{n+1}\rangle}{H-\mu^n}$$

is uniformly bounded. Furthermore, by (25) and

$$\nabla Q_\delta = \frac{(H-\mu^n)\nabla\mu^n - \mu^n(\nabla H - \nabla\mu^n)}{(H-\mu^n)^2} = -\frac{\mu^n \nabla H}{(H-\mu^n)^2} + \frac{H\nabla\mu^n}{(H-\mu^n)^2},$$

we see that $\frac{H\nabla\mu^n}{(H-\mu^n)^2}$ is uniformly bounded at each point $P_{\delta,N}$. Since $\frac{H}{H-\mu^n}$ is bounded away from 0 by Claim 2, we conclude that $\frac{\nabla\mu^n}{H-\mu^n}$ is also uniformly bounded at each point $P_{\delta,N}$. Adopting the notation $\nabla_l = \nabla_{\tau_l}$,

Claim 3. We have

$$\frac{\nabla_l H}{H-\mu^n}(P_{\delta,N}) \to 0 \quad \text{as } N\to\infty$$

for any $1 \le l \le n$.

Proof. It follows from (21) and (25) that

$$-\frac{2}{N} + 2\frac{\langle \nabla(H-\mu^n), \nabla Q_\delta\rangle}{H-\mu^n} \le \frac{HF^{ij,pq}h_{ijk}h_{pqk}}{(H-\mu^n)^2} \le 0 \tag{28}$$

holds at each $P_{\delta,N}$. From the previous analysis, one knows

$$\frac{\nabla(H-\mu^n)}{H-\mu^n}(P_{\delta,N})$$

is uniformly bounded as $N\to\infty$. Therefore, the left hand side of (28) converges to 0 as $N\to\infty$. Consequently (since $\frac{H}{H-\mu^n}$ is bounded away from zero)

$$\frac{F^{ij,pq}h_{ijk}h_{pqk}}{H-\mu^n} \to 0, \quad \text{as } N\to\infty.$$

By (23), one has

$$\frac{F^{ij,pq}h_{ijk}h_{pqk}}{H-\mu^n} \le \frac{1}{H-\mu^n}\sum_{\{i,j:\kappa_i\neq\kappa_j\}}\sum_k \frac{\mu_i^n-\mu_j^n}{\kappa_i-\kappa_j}h_{ijk}^2 \le 0 \tag{29}$$

and each term in the summation is nonpositive. Therefore all the terms converge to zero as $N\to\infty$. Suppose δ is small enough. Taking $i=1$ and $j>1$ and using Lemma 2.6 ii., one obtains $(H-\mu^n)^{-1}|h_{1jk}| \to 0$ as $N\to\infty$ for any k. In particular,

$$\frac{|h_{1jj}|+|h_{11j}|}{H-\mu^n} \to 0, \quad \text{as } N\to\infty, \quad \text{for } j>1. \tag{30}$$

Taking both $i, j > 1$ and $i \geq j$, if $\kappa_i = \kappa_j$, then $h_{ijk} = 0$ by (24). If $\kappa_i \neq \kappa_j$, we use Lemma 2.7 to conclude

$$\begin{aligned}\frac{\Lambda^{n-2}}{2\cdot n!}\frac{h_{ijk}^2}{(H-\mu^n)^2} \leq& \frac{\Lambda^{n-2}}{2\cdot n!}\frac{1}{H-\mu^n}\frac{h_{ijk}^2}{|\kappa_i-\kappa_j|+\sqrt{\kappa_i\kappa_j}}\\ \leq& -\frac{1}{H-\mu^n}\frac{\mu^i-\mu^j}{\kappa_i-\kappa_j}h_{ijk}^2 \leq \left|\frac{F^{ij,pq}h_{ijk}h_{pqk}}{H-\mu^n}\right|\end{aligned}$$

In either case, we must have

$$\frac{|h_{ijk}|}{H-\mu^n} \to 0 \quad \text{as } N \to \infty, \quad \text{for } i \neq j. \tag{31}$$

Since $\nabla_l \mu^n = \sum_i \mu_i^n \nabla_l \kappa_i = \sum_i \mu_i^n h_{iil}$, we can rewrite $\nabla_l Q_\delta$ as

$$\begin{aligned}\nabla_l Q_\delta = \frac{H\nabla_l\mu^n - \mu^n\nabla_l H}{(H-\mu^n)^2} =& \left[1+\frac{H(\mu_l^n-1)}{H-\mu^n}\right]\frac{h_{lll}}{H-\mu^n}\\ +& \sum_{i\neq l}\frac{H\mu_l^n h_{iil}-\mu^n h_{iil}}{(H-\mu^n)^2}\end{aligned} \tag{32}$$

When δ is small enough, it follows from Lemma 2.6 ii. that $\mu_l^n \to 0$ if $l > 1$ and $\mu_l^n \to$ if $l = 1$, as $N \to \infty$, uniformly as δ decreases to zero. Combined with the uniform estimates of Claim 2 and (31), the last term on the right hand side of (32) tends to zero as $N \to \infty$. Moreover, $1 + H(\mu_l^n - 1)(H - \mu^n)^{-1}$ is uniformly bounded away from zero for any l. Since $\nabla_l Q_\delta \to 0$ as $N \to \infty$, we conclude from (32) that

$$\frac{h_{lll}}{H-\mu^n} \text{ as } N \to \infty. \tag{33}$$

Therefore (31) and (33) imply

$$\frac{\nabla_l H}{H-\mu^n} = \sum_{i=1}^n \frac{h_{iil}}{H-\mu^n} \to 0 \text{ as } N \to \infty,$$

proving Claim 3.

□

Again consider $\Sigma_N = \Sigma - P_{\delta,N}$. A subsequence of the Σ_N converges locally smoothly to a translator Σ_∞ . Since $H = \langle \nu, e_{n+1}\rangle$,

$$\frac{\nabla_l H}{H-\mu^n}(P_{\delta,N}) = \frac{\kappa_l\langle\tau_l, e_{n+1}\rangle}{H-\mu^n}(P_{\delta,N}) \to 0, \quad \text{as } N \to \infty.$$

But then by (26),

$$\langle\tau_l, e_{n+1}\rangle(P_{\delta,N}) \to 0 \text{ as } N \to \infty. \tag{34}$$

This means that at the origin on Σ_∞, we have $\langle \tau_l, e_{n+1}\rangle(0) = 0$. In other words, $\nu = e_{n+1}$ and $H = 1$ at 0. However,

$$\inf_{\Sigma_\infty} \frac{\mu^n}{H - \mu^n} = \frac{\mu^n}{H - \mu^n}(0) \leq -\varepsilon_0$$

This again contradicts the fact that Q_δ cannot have an interior negative minimum. This completes the proof of Theorem 3.1 and consequently Theorem 1.1 is proven.

□

4. STRICT CONVEXITY OF GRAPHICAL TRANSLATORS WITH BOUNDARY

In this section we prove a convexity theorem for graphical solutions of the translator equation in a smooth strictly convex domain with constant boundary values. As an application we show the existence of a multi-parameter family of (distinct) locally strictly convex entire translators.

For the graph of u, the induced metric g_{ij}, its inverse matrix g^{ij}, and its second fundamental form h_{ij} are given by, respectively,

$$g_{ij} = \delta_{ij} + u_i u_j, \quad g^{ij} = \delta_{ij} - \frac{u_i u_j}{W^2}$$

and

$$h_{ij} = \frac{u_{ij}}{W}, \quad W = \sqrt{1 + |Du|^2}.$$

Moreover, the principal curvatures of the graph of u are the eigenvalues of the symmetric matrix $A[u] = (A_{ij})$:

$$A_{ij} = \gamma^{ik} \frac{u_{kl}}{W} \gamma^{lj}, \tag{35}$$

where (γ^{ik}) and its inverse matrix (γ_{ik}) are given respectively, by

$$\gamma^{ik} = \delta_{ik} - \frac{u_i u_k}{W(1+W)}, \ \gamma_{ik} = \delta_{ik} + \frac{u_i u_k}{1+W}.$$

Geometrically, (γ_{ik}) is the square root of the metric, i.e., $\gamma_{ik}\gamma_{kj} = g_{ij}$.

Let $\lambda_1 \leq \ldots \leq \lambda_n$ and $\kappa_1 \leq \ldots \leq \kappa_n$ be the ordered eigenvalues of D^2u and $A[u]$, respectively. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we have [12]

$$u_{ij}\xi_i\xi_j = W a_{kl}\gamma_{ik}\gamma_{lj}\xi_i\xi_j = W a_{kl}\xi'_k\xi'_l, \tag{36}$$

where $\xi'_i = \gamma_{ik}\xi_k = \xi_i + \frac{(\xi\cdot Du)u_i}{1+W}$. Note that

$$|\xi|^2 \leq |\xi'|^2 = |\xi|^2 + |\xi \cdot Du|^2 \leq W^2|\xi|^2,$$

where $\xi' = (\xi'_1, \dots, \xi'_n)$. If both D^2u and $A[u]$ are positive semidefinite, it follows from (36) and the minimax characterization of eigenvalues that

$$W\kappa_k \le \lambda_k \le W^3\kappa_k,\ 1 \le k \le n. \tag{37}$$

In particular κ_1 vanishes if and only if λ_1 vanishes.

Theorem 4.1. *Let Ω be a strictly convex domain with $\Gamma := \partial\Omega$ analytic. Then there exists a unique strictly convex graphical translating soliton u in $\bar{\Omega}$ with $u = 0$ on $\partial\Omega$.*

Proof. Let Ω^t be a foliation of Ω by strictly convex domains with $\Gamma^t := \partial\Omega^t$ analytic that shrinks to a point (the origin say) and becomes asymptotically spherical. For example, we can let Γ^t be the mean curvature flow of Γ. We reindex so that $\Gamma^0 = \{0\}$ and $\Gamma^1 = \Gamma$. It is standard that we there is a unique solution of the Dirichlet problem

$$a^{ij}u_{ij} = (\delta_{ij} - \frac{u_iu_j}{1+|\nabla u|^2})u_{ij} = 1 \text{ in } \Omega^t,\ u = 0 \text{ on } \Gamma^t, \tag{38}$$

which yields for $t > 0$, a graphical translator u^t in Ω^t, $u^t = 0$ on Γ^t. Moreover since for t very small, Γ^t becomes spherical exponentially fast, u^t is asymptotically a small piece of the bowl soliton centered at the origin, and therefore strictly convex. Let $t^* = \sup\{t : u^t \text{ is strictly convex in } \Omega^t\}$ and suppose for contradiction that $t^* < 1$. Set $u^* = u^{t^*}$, $\Omega^* = \Omega^{t^*}$, $\Gamma^* = \partial\Omega^*$. Firstly, by maximum principle, one has $u^* < 0$ in Ω^*. Furthermore by Hopf boundary point lemma, $|\nabla u| \neq 0$ on $\partial\Omega^*$.

Claim 4. $\max_{\Omega^*} |\nabla u^*|$ is achieved at a point $Q \in \Gamma^*$. Moreover $\det(u^*_{ij}(Q)) > 0$.

By the convexity of u^*, the maximum of $|\nabla u^*|$ is achieved on Γ^*, say at Q. For any direction e, set $w = \nabla u^* \cdot e$. Then differentiating (38), we see that w satisfies

$$(a^*)^{ij}w_{ij} - 2(a^*)^{jk}\frac{u^*_{ij}u^*_i}{W^2}w_k = 0.$$

Hence by the Hopf boundary point lemma, $|\nabla w(Q)| > 0$. Choose a curvature frame $\{e_1, \dots, e_n\}$ at Q with e_n the outer unit normal. Then $u^*_{nk}(Q) = 0,\ k < n,\ u^*_{nn}(Q) > 0$ and (see below) $u_{kk}(Q) > 0,\ k < n$. Hence $\det(u^*_{ij}(Q)) = \Pi_{l=1}^n u_{ll}(Q) > 0$. Hence Claim 4 is proved.

By the constant rank theorem of Bian and Guan (see Corollary 1.3 of [6]), the rank of (u^*_{ij}) is n in Ω^* by Claim 4. Thereforeby the definition of t^*, we must have $\det u^*_{ij}(P) = 0$ for some $P \in \Gamma^*$. Choose a curvature frame $\{e_1, \dots, e_n\}$ at P with e_n the outer unit normal. Then for $k < n$, $u^*_{kk} = |\nabla u^*|\lambda^*_k$, where $\lambda^*_k > 0$ is a principal curvature of Γ^* at P.

Note also that $|\nabla u^*| > 0$ on Γ^* by the Hopf boundary point lemma, so $u^*_{kk}(P) > 0,\ k < n$. Then,

$$(u^*_{ij}(P)) = \begin{pmatrix} u^*_{11}(P) & & \mathbf{0} & u^*_{1n}(P) \\ & \ddots & & \vdots \\ \mathbf{0} & & u^*_{n-1n-1}(P) & u^*_{(n-1)n}(P) \\ u^*_{n1}(P) & \cdots & u^*_{n(n-1)}(P) & u^*_{nn}(P) \end{pmatrix} \tag{39}$$

and so

$$0 = \det u^*_{ij}(P) = u^*_{nn}(P)\Pi_{k<n}u^*_{kk}(P) - \sum_{k<n}(u^*)^2_{kn}(P)\,\Pi_{l\neq k}u^*_{ll}(P).$$

It follows that $u^*_{nn}(P) = \sum_{k<n}\frac{(u^*)^2_{kn}(P)}{u^*_{kk}(P)}$.

Claim 5. The multiplicity of $\lambda_1(u^*_{ij}(P))$ is one.

<u>Case</u> 1: $u^*_{nn}(P) = u^*_{kn}(P) = 0,\ k < n$.
Then $u^*_{ij}(P)\xi_i\xi_j = \sum_{k<n} u^*_{kk}(P)\xi_k^2$ is minimized ($|\xi| = 1$) when $\xi_k = 0,\ k < n$ and $\xi_n = 1$. Hence the dimension of the eigenspace for $\lambda_1^*(P) = 0$ is one.
<u>Case</u> 2: $u^*_{nn}(P) > 0$.
Then

$$\begin{aligned} u^*_{ij}(P)\xi_i\xi_j &= \sum_{k<n}\left(u^*_{kk}(P)\xi_k^2 + 2u^*_{kn}(P)\xi_k\xi_n + \frac{(u^*)^2_{nk}(P)}{u^*_{kk}(P)}\xi_n^2\right) \\ &= \sum_{k<n}\left(\sqrt{u^*_{kk}(P)}\xi_k + \frac{u^*_{nk}(P)}{\sqrt{u^*_{kk}(P)}}\,\xi_n\right)^2, \end{aligned}$$

is minimized ($|\xi| = 1$) when $\xi_k = -\frac{u^*_{kn}(P)}{u^*_{kk}(P)}\xi_n$ and $\xi_n = \left(1 + \sum_{k<n}\frac{(u^*)^2_{kn}(P)}{(u^*)^2_{kk}(P)}\right)^{-1}$. Thus the claim is proven.

Since Γ^* is analytic, u^* extends to a solution in a small neighborhood B of P by the Cauchy-Kowalewski theorem. where as before $\kappa_1^*(y) = \min_j \kappa_j^*(y)$. Then by Claim 5, κ_1^* is locally smooth and

$$\Delta^f\kappa_1^* + |A^*|^2\kappa_1^* = -2\sum_{l=1}^n\sum_{k=2}^n\frac{(\nabla_l h^*_{1l})^2}{\kappa_k^* - \kappa_1^*} \leq 0,$$

where $\Delta^f := \Delta^\Sigma + \langle \nabla^\Sigma, \nabla^\Sigma u\rangle$ is the so called drift Laplacian on Σ. Note that in graph coordinates, $\Delta^f = a^{ij}\partial_i\partial_j$ since

$$\Delta^\Sigma = a^{ij}\partial_i\partial_j - H\frac{u_k}{W}\partial_k = a^{ij}\partial_i\partial_j - \frac{u_k}{W^2}\partial_k = a^{ij}\partial_i\partial_j - \langle \nabla^\Sigma, \nabla^\Sigma u\rangle.$$

Therefore $\kappa_1^* > 0$ satisfies

$$(a^*)^{ij}\partial_i\partial_j\kappa_1^* \le 0 \text{ in } B \cap \Omega^*.$$

Since $\kappa_1^*(P) = 0$, the Hopf boundary point lemma implies that $|\nabla\kappa_1^*(P)| \neq 0$. Therefore $\kappa_1^* = 0$ on a smooth hypersurface Λ passing through P. Moreover, Λ must be transversal to Γ^* (since $u^*_{kk}(P) > 0$) contradicting that the rank of (u^*_{ij}) is n in Ω^*. Thus $t^* = 1$ and $u^1 = u$ is strictly convex in $\overline{\Omega}$. $\square$

Remark 4.2. By approximation from the outside by strictly convex analytic hypersurfaces, Theorem 4.1 holds for $\Gamma \in C^2$ in the sense that the solution is strictly convex in Ω.

We now apply Theorem 4.1 to the beautiful existence theory of Hoffman, Ilmanen, Martin and White developed in [18] for Ω an ellipsoid. In particular combining Theorem 4.1 and Corollary 8.2 of [18] gives Corollary 1.3. There is also a corresponding result for slabs of width greater than π using Corollary 11.2 of [18].

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