DERIVATIVES & INTEGRALS
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Derivatives

Here are a bunch of derivatives you should probably know. We highly recommend practicing with them (or creating flashcards for them) and looking at them occasionally until they are burned into your memory.

function: $f(x)$	derivative: $f'(x)$
x^a	ax^{a-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
e^x	e^x
a^x	$a^x \ln(a), \quad \text{if } a > 0$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$

Additionally, you should know the following derivative rules:

- **derivatives are linear**: $(f+g)' = f' + g', \quad (cf)' = cf'$
- **product rule**: $(f \cdot g)' = f'g + fg'$
- **quotient rule**: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- **chain rule**: $(f \circ g)' = (f' \circ g) \cdot g'$

Example 1.1. The derivative of $f(x) = \sin(\ln(1+x^2))$ is found by using the chain rule, and viewing f as a composition of functions:

$$
\begin{aligned}
f(x) &= \sin(x) \circ \Big(\ln(x) \circ (1+x^2) \Big) \\
f'(x) &= \left(\sin'(x) \circ \Big(\ln(x) \circ (1+x^2) \Big) \right) \cdot \Big(\ln(x) \circ (1+x^2) \Big)' \\
&= \Big(\cos(x) \circ \ln(1+x^2) \Big) \cdot \Big(\ln'(x) \circ (1+x^2) \Big) \cdot (1+x^2)' \\
&= \cos(\ln(1+x^2)) \cdot \Big(\tfrac{1}{x} \circ (1+x^2) \Big) \cdot (2x) \\
&= \cos(\ln(1+x^2)) \cdot \left(\frac{1}{1+x^2} \right) \cdot (2x) \\
&= \frac{2x\cos(\ln(1+x^2))}{1+x^2}
\end{aligned}
$$

Integrals

We now turn to integrals. There are two types of integrals: **indefinite integrals** (otherwise known as antiderivatives) and **definite integrals** (which represent area under the graph of a function). To make this explicit,

$$\int \frac{1}{x}\,dx$$

represents an antiderivative of $\frac{1}{x}$. That is, a function $F(x)$ such that $F'(x) = \frac{1}{x}$. While on the other hand,

$$\int_1^4 \frac{1}{x}\,dx$$

represents the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 4$. This difference cannot be overstated: the first expression represents *a function*, while the second expression represents *a number* (the area). The reason these two concepts are connected (and hence, the reason we use such similar notation for both of them) is the **Fundamental Theorem of Calculus**.

Here are a bunch of (indefinite) integrals you should probably know, or be able to compute. Notice that in almost every case, this is just the reverse of the rule you used for derivatives!

function: $f(x)$	indefinite integral: $\int f(x)\,dx$
x^a	$\frac{x^{a+1}}{a+1} + C, \quad \text{if } a \neq -1$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\csc^2(x)$	$-\csc(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\csc(x)\cot(x)$	$-\csc(x) + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln(a)} + C, \quad \text{if } a > 0$
$\frac{1}{x}$	$\ln(x) + C$
$\sinh(x)$	$\cosh(x) + C$
$\cosh(x)$	$\sinh(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$

Remark 1.2. Note that there's always a "$+C$" added to the end of every indefinite integral. This is because an indefinite integral really represents *an* antiderivative, not *the* antiderivative — antiderivatives are not unique! Indeed, not only is $\sin(x)$ an antiderivative of $\cos(x)$, but so is $\sin(x) + 25$. The "$+C$" accounts for all of the possible different antiderivatives.

Remark 1.3. Note that there are a few "famous functions" that are missing from our list. For example, what is the antiderivative of $\ln(x)$? of $\tan(x)$? It turns out that, in general, it is much harder to find antiderivatives than it is to find derivatives. We will introduce some techniques you can use to find more complicated antiderivatives.

Now, let's discuss some techniques. In each of these techniques, I find it useful to think of it as "the anti-derivative version of a derivative technique". I will try to highlight this when I can.

- **Substitution**. This is the reverse of the chain rule! It involves recognizing that some (complicated-looking) functions can be written in the form

$$f'(u(x)) \cdot u'(x).$$

We then use our knowledge of the chain rule to recognize that its antiderivative should be

$$F(x) = f(u(x)) + C.$$

It is called "substitution" because you usually begin by identifying a portion of the function that you can use to "substitute" in for the function $u(x)$. **The trickiest part of substitution is finding out what your u should be!**

Example 1.4. Suppose we want to find an antiderivative of the function

$$h(x) = \frac{\cos(x)}{\sin^5(x)}$$

We can begin by identifying that there is a good candidate to represent $u(x)$: namely, the function

$$u(x) = \sin(x)$$

This is a good candidate because not only is $u(x)$ present as part of h, but so is a single copy of $u'(x) = \cos(x)$. Therefore, if we substitute in $u(x) = \sin(x)$ and $u'(x) = \cos(x)$, we get

$$h(x) = \frac{u'(x)}{u^5(x)} = (u(x))^{-5} u'(x) = (u)^{-5} \cdot u'$$

This looks like the composition of functions we mentioned above, with the "outer function" being $f'(u) = u^{-5}$. Its antiderivative is $f(u) = -\frac{1}{4}u^{-4}$. Therefore, our total antiderivative is

$$H(x) = \int h(x)\, dx = \frac{-1}{4}(\sin(x))^{-4} + C.$$

A good way to check your work is to verify that indeed $H'(x) = h(x)$.

Example 1.5. We can now find the antiderivative of $\tan(x)$. Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Try to use substitution. What should you let your $u(x)$ equal?

- **Integration by Parts**. This is the reverse of the product rule! Recall that the product rule says that
$$(fg)' = f'g + fg'.$$
In other words, fg is an antiderivative of $f'g + fg'$. In the language of indefinite integrals, this can be rewritten as
$$fg = \int f'g + \int fg',$$
or, written another way,
$$\int f'g = fg - \int fg'.$$
This allows us to take one integral that we're not terribly sure how to solve (say, $\int f'g$) and write it in the form of a function (fg) minus a new integral which might be easier to solve ($\int g'f$). **The trickiest part of IBP is figuring out which parts of your original integral should equal f', and which should equal g, in order to make $\int g'f$ easier to solve!**

Example 1.6. Suppose we want to find an antiderivative of xe^x. It looks tricky, it isn't on our list, and after a bit of work you can see that substitution will fail. However, we can use integration by parts!

We set $f'(x) = e^x$ and $g(x) = x$. Notice that by doing so, the IBP formula will yield (remember that $f = \int f' \, dx = e^x$ and $g' = 1$)
$$\int xe^x \, dx = xe^x - \int e^x \cdot 1 \, dx.$$
The integral on the right hand side is now much easier to compute, and we find the antiderivative to be
$$xe^x - e^x + C.$$

Example 1.7. We can now compute the integral of $\ln(x)$! Begin by looking at $\int \ln(x) \, dx$, and cleverly rewriting it as $\int 1 \ln(x) \, dx$. Let $f' = 1$ and $g = \ln(x)$. What happens?

Practice Problems

Compute the derivatives of the following functions:

1. $34x^2 + 62x^{-2} + 4x^{1/4}$
2. $e^{\sin(x)}$
3. $\ln(4x + 5)$
4. $\sin(x)\cos(5x)$
5. $\arcsin(\tan(45x^2))$
6. $x^{4/5}(4/5)^x$
7. $\ln\left(\frac{x}{\sin x}\right)$
8. $\sin(\cos(\tan(\ln(x)))) + 5$

Compute the following antiderivatives:

1. $\int \frac{1}{x^4}\, dx$
2. $\int x\sqrt{2x}\, dx$
3. $\int x + \sin(3x)\, dx$
4. $\int x\cos(x^2)\, dx$
5. $\int e^x(1 + e^x)^5\, dx$
6. $\int \frac{1}{\sqrt{1-x^2}}\, dx$

7. $\int \frac{2x}{1+x^2}\,dx$

8. $\int e^{\sin(x)} \cos x\,dx$

9. $\int \sec^5(x)\tan(x)\,dx$

10. $\int \tan^4(x)\sec^2(x)\,dx$

11. $\int \ln(x)\,dx$

12. $\int x^2 \sec(x^3)\tan(x^3)\,dx$

13. $\int x\sin(x)\,dx$

14. $\int x^2\sin(x)\,dx$

15. $\int x\sin(x^2)\,dx$

16. $\int \sin(x)e^{\sin(x)}\cos(x)\,dx$

17. $\int x^2\sqrt{x^3+1}\,dx$

18. $\int \ln(\sin(x))\cos(x)\,dx$

19. $\int \ln(\sin(x))\cot(x)\,dx$

20. $\int e^x\sin(x)\,dx$