The Riemann Zeta Function

The *Riemann zeta function* is defined by the p-series

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots, \quad \text{valid for } p > 1, \tag{1}$$

which converges for $p > 1$ by the Integral Test (and diverges for $p \leq 1$). One interesting special value [though hard to prove] is

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6} \ . \tag{2}$$

Convergence Suppose we try to use equation (2) to compute $\pi^2/6$ with an error of less than 10^{-6}. Question: *How many terms do we need?* Explicitly, how large must k be to bring $\sum_{n=1}^{k} 1/n^2$ within 10^{-6} of $\pi^2/6$?

We note first that $1/n^2 < 10^{-6}$ whenever $n > 1000$. This suggests taking $k = 1000$. But consider the next thousand terms, each of which is at least $1/2000^2$; their sum is therefore greater than $1000 \times 1/2000^2 = 0.000250$, which is far larger than 10^{-6}. [The error is actually 0.0009995, according to `maple`. In fact, `maple` knows about the series equation (1), and the remainder after k terms is a built-in function.]

The correct answer is (exactly) 1,000,000, with error 9.999995×10^{-7}. Thus equation (1) is highly impractical for computation.

An alternating series We introduce the variant

$$f(p) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots, \quad \text{valid for } p > 0, \tag{3}$$

which converges for $p > 0$ by the Alternating Series Test.

Rearrangement The function $f(p)$ is easily expressed in terms of $\zeta(p)$, when $p > 1$. Consider the odd and even terms separately, by defining

$$g(p) = 1 + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \dots, \quad \text{valid for } p > 1,$$

and

$$h(p) = \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \dots, \quad \text{valid for } p > 1.$$

Clearly, $g(p) + h(p) = \zeta(p)$. This rearrangement is valid for $p > 1$. Also, multiplication of each term of $h(p)$ by 2^p gives $2^p h(p) = \zeta(p)$.

We have enough information to deduce that

$$h(p) = \frac{1}{2^p}\zeta(p), \qquad g(p) = \frac{2^p - 1}{2^p}\zeta(p) \ . \tag{4}$$

Then $f(p) = g(p) - h(p)$ gives

$$f(p) = \left(\frac{2^p - 1}{2^p} - \frac{1}{2^p} \right) \zeta(p) = \frac{2^p - 2}{2^p}\zeta(p) \ . \tag{5}$$

More examples For $p = 2$, equation (4) gives two more well-known series,

$$g(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8} \tag{6}$$

and

$$f(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}\ . \tag{7}$$

By Theorem 8, we only need 1000 terms of the series equation (7) to guarantee that the error is less than 10^{-6}. [The actual error is only 4.997×10^{-7}.] This is a big improvement over equation (2), but still hardly practical.

Extension Nevertheless, equation (3) is useful for more than just computation. We can turn equation (5) around to the form

$$\zeta(p) = \frac{2^p}{2^p - 2} f(p) \quad \text{for } p > 1. \tag{8}$$

However, the right side of this equation is defined for all $p > 0$ (provided we exclude the case $p = 1$, so as not to divide by 0). This suggests that it is reasonable to *extend* the definition of $\zeta(p)$ to all $p > 0$ ($p \neq 1$) by means of this equation. [This is in fact true, but requires more justification than this.]