Universal Coefficient Theorem for Cohomology

We present a direct proof of the universal coefficient theorem for cohomology. It is essentially dual to the proof for homology.

THEOREM 1 *Given a chain complex C in which each C_n is free abelian, and a coefficient group G, we have for each n the natural short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C), G) \longrightarrow H^n(\operatorname{Hom}(C, G)) \longrightarrow \operatorname{Hom}(H_n(C), G) \longrightarrow 0, \qquad (2)$$

which splits (but there is no natural splitting).

In particular, this applies immediately to singular cohomology.

THEOREM 3 *Given a pair of spaces (X, A) and a coefficient group G, we have for each n the natural short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X, A), G) \longrightarrow H^n(X, A; G) \longrightarrow \operatorname{Hom}(H_n(X, A), G) \longrightarrow 0,$$

which splits (but there is no natural splitting). □

We shall derive diagram (2) as an instance of the following elementary result.

LEMMA 4 *Given homomorphisms $f\colon K \to L$ and $g\colon L \to M$ of abelian groups, with a splitting homomorphism $s\colon M \to L$ such that $s \circ g = \mathrm{id}_L$, we have a split short exact sequence*

$$0 \longrightarrow \operatorname{Coker} f \xrightarrow{g'} \operatorname{Coker}(g \circ f) \longrightarrow \operatorname{Coker} g \longrightarrow 0. \qquad (5)$$

Proof We write each cokernel, such as $\operatorname{Coker} f$, as $L/\operatorname{Im} f$ etc. Then the sequence (5) appears as the upper edge of the following diagram; we shall identify it with the bottom row, which is the canonical short exact sequence formed from the triple $\operatorname{Im}(g \circ f) \subset \operatorname{Im} g \subset M$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{L}{\operatorname{Im} f} & & & & & & \\
 & & \big\downarrow & \searrow^{g'} & & & & & \\
0 & \longrightarrow & \dfrac{\operatorname{Im} g}{\operatorname{Im}(g \circ f)} & \xrightarrow{\subset} & \dfrac{M}{\operatorname{Im}(g \circ f)} & \longrightarrow & \dfrac{M}{\operatorname{Im} g} & \longrightarrow & 0
\end{array}$$

Since $g(\operatorname{Im} f) = \operatorname{Im}(g \circ f)$, g induces a homomorphism g'. Since $s(\operatorname{Im}(g \circ f)) = \operatorname{Im}(s \circ g \circ f) = \operatorname{Im} f$, $s\colon M \to L$ induces a homomorphism s' which splits g', $s' \circ g' = \mathrm{id}$; thus g' is injective. But g' clearly factors through a surjective homomorphism $L/\operatorname{Im} f \to \operatorname{Im} g/\operatorname{Im}(g \circ f)$, which is therefore an isomorphism. □

Preliminaries We consider a chain complex C as in Theorem 1. We adopt the usual notation: Z_n for the group of n-cycles, B_n for the group of n-boundaries, and $H_n = H_n(C) = Z_n/B_n$. The key idea in our proof of Theorem 1 is to express $\partial\colon C_n \to C_{n-1}$ as the composite

$$\partial\colon C_n \longrightarrow \frac{C_n}{B_n} \longrightarrow \frac{C_n}{Z_n} \xrightarrow{\cong} B_{n-1} \xrightarrow{\subset} Z_{n-1} \xrightarrow{\subset} C_{n-1}. \qquad (6)$$

By definition, we have the short exact sequence

$$0 \longrightarrow B_n \xrightarrow{\subset} Z_n \longrightarrow H_n \longrightarrow 0 \tag{7}$$

for any n. As B_n and Z_n are subgroups of the free abelian group C_n and therefore free abelian, we recognize (7) as a free resolution of H_n.

Also by definition, from diagram (6) we have the short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\subset} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0, \tag{8}$$

which splits because B_{n-1} is free abelian. Using (7) and (8), we may rewrite the canonical short exact sequence

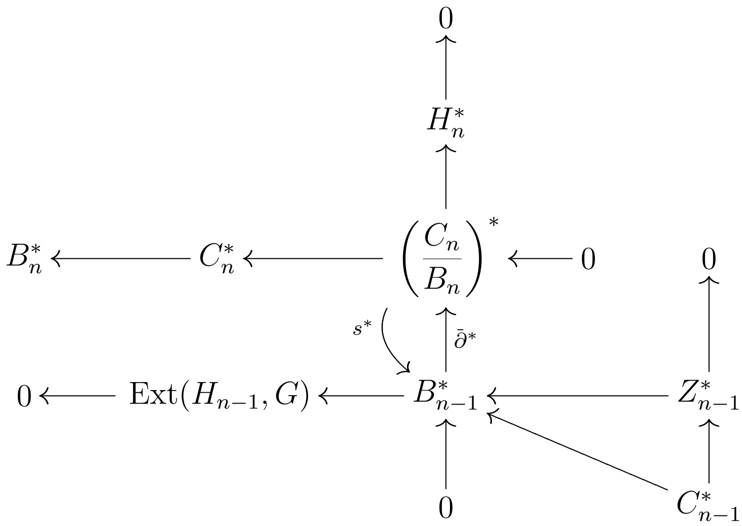
$$0 \longrightarrow Z_n/B_n \xrightarrow{\subset} C_n/B_n \longrightarrow C_n/Z_n \longrightarrow 0$$

as

$$0 \longrightarrow H_n \longrightarrow C_n/B_n \xrightarrow{\bar\partial} B_{n-1} \longrightarrow 0, \tag{9}$$

where $\bar\partial$ is a quotient of ∂. As B_{n-1} is free abelian, this short exact sequence also splits, and we may choose a splitting homomorphism $s\colon B_{n-1} \to C_n/B_n$.

Proof of Theorem 1 We dualize diagram (6) and the above short exact sequences to form the following diagram of exact sequences. To simplify, we write A^* for the G-dual $\operatorname{Hom}(A, G)$ of any abelian group A.



(10)

Both vertical sequences are exact, since (9) and (8) split. The upper horizontal sequence is exact, because the functor $\operatorname{Hom}(-, G)$ takes cokernels to kernels. The lower horizontal sequence is exact, as it defines $\operatorname{Ext}(H_{n-1}, G)$.

As usual, we write Z^n for the n-cocycles, etc., in the cochain complex C^*. Diagram (10) shows that $Z^{n-1} = \operatorname{Ker}[\partial^*\colon C_{n-1}^* \to C_n^*] = \operatorname{Ker}[C_{n-1}^* \to B_{n-1}^*]$. On replacing $n-1$ by n, we see that $Z^n = \operatorname{Ker}[C_n^* \to B_n^*] = (C_n/B_n)^*$. Also, $B^n = \operatorname{Im}\partial^* \cong \operatorname{Im}[Z_{n-1}^* \to (C_n/B_n)^*]$, so that $H^n = Z^n/B^n \cong \operatorname{Coker}[Z_{n-1}^* \to (C_n/B_n)^*]$.

We now apply Lemma 4 with $K = Z_{n-1}^*$, $L = B_{n-1}^*$ and $M = (C_n/B_n)^*$, and use the splitting $s^*\colon M \to L$. Diagram (10) identifies $\operatorname{Coker}[K \to L]$ with $\operatorname{Ext}(H_{n-1}, G)$ and $\operatorname{Coker}[L \to M]$ with H_n^*. These identifications reduce the split short exact sequence (5) to the desired (2). □