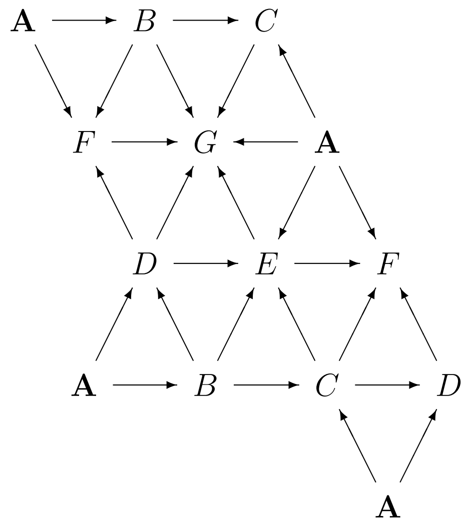
The Torus Triangulated

Triangulation The most efficient way to triangulate the (2-dimensional) torus T is the following:



(1)

This is a (distorted) rectangle with all four vertices A, and two pairs of opposite edges $ABCA$ and $ADFA$ identified. (In this picture, opposite edges are equal and parallel, though they are not straight lines. The pattern may be repeated over the whole plane; if you wish to tile a floor using hexagonal tiles in seven colors, this is a good way to do it, by centering each tile over a vertex of the triangulation.)

We *order* the seven vertices alphabetically.

Cellular chain complex As a cell decomposition, we use one 0-cell A, two 1-cells $ABCA$ and $ADFA$, and the whole rectangle as the 2-cell. The two edges give obvious simplicial 1-cycles

$$w_1 = [AB] + [BC] - [AC], \qquad z_1 = [AD] + [DF] - [AF],$$

where we take care to keep the vertices in the correct order, and introduce signs as necessary. For the 2-cycle, we add all 14 triangles in (1), oriented anti-clockwise, with signs as necessary,

$$d_2 = -[ABF] + [BFG] + \dots \tag{2}$$

This is a 2-cycle because all interior edges in (1) cancel, giving

$$\begin{aligned} \partial d_2 &= [AF] - [DF] - [DA] + [AB] + [BC] - [AC] \\ &\qquad + [AD] + [DF] - [FA] + [AC] - [CB] - [BA] \\ &= -z_1 - w_1 + z_1 + w_1 = 0. \end{aligned}$$

Thus the cellular chain complex is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}, \tag{3}$$

and the homology groups coincide with the cellular chain groups.

Cocycles We take $\mathbb{Z}$ as the coefficient group (although the calculations are essentially the same for any coefficient group). From (3), the cellular cochain complex is

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{0} \mathbb{Z},$$

and the cohomology groups coincide with the cellular cochain groups. One can check that the simplicial 1-cochains

$$w^1 = [AB]^* - [BF]^* + [FG]^* + [DG]^* + [DE]^* - [BD]^*, \tag{4}$$

where $[v_0v_1]^*$ denotes the cochain dual to the chain $[v_0v_1]$, and

$$z^1 = [AD]^* + [BD]^* + [BE]^* + [CE]^* + [CF]^* + [CD]^* \tag{5}$$

are in fact 1-cocycles, and it is easy to see that they are indeed dual to w_1 and z_1 in the sense that

$$\langle w^1, w_1 \rangle = 1, \quad \langle w^1, z_1 \rangle = 0, \quad \langle z^1, w_1 \rangle = 0, \quad \langle z^1, z_1 \rangle = 1.$$

You may wish to color these cocycles in the picture (1).

Let $\alpha \in H^1(T;\mathbb{Z})$ and $\beta \in H^1(T;\mathbb{Z})$ be the classes of w^1 and z^1, and let $\gamma \in H^2(T;\mathbb{Z})$ be dual to d_2. Specifically, we could take γ to be the class of $[ABD]^*$, since $\langle [ABD]^*, d_2 \rangle = 1$. (All 2-cochains are 2-cocycles.)

Cup products The purpose of finding the representative simplicial cocycles (4) and (5) is to enable us to use the formula

$$\langle x \smile y, [v_0v_1v_2] \rangle = \langle x, [v_0v_1] \rangle \langle y, [v_1v_2] \rangle$$

to compute cup products, where v_0, v_1, and v_2 must be in order. In computing $w^1 \smile z^1$ for example, $[v_0v_1]^*$ must appear in w^1 and $[v_1v_2]^*$ must appear in z^1 in order to produce anything; only one term survives, namely

$$w^1 \smile z^1 = [ABD]^*.$$

Similarly,

$$z^1 \smile w^1 = [BDE]^*,$$

which represents $-\gamma$ (as BDE is taken clockwise),

$$w^1 \smile w^1 = -[ABD]^* - [BDE]^* - [BFG]^* - [ABF]^*, \tag{6}$$

and $z^1 \smile z^1 = 0$ (exactly). One can show directly that $w^1 \smile w^1$ is a coboundary, but it is simpler to calculate from (6) that

$$\langle w^1 \smile w^1, d_2 \rangle = -1 + 1 - 1 + 1 = 0.$$

We assemble all this information.

THEOREM 7 *The cohomology $H^*(T;\mathbb{Z})$ is a free abelian group with basis elements $1 \in H^0(T;\mathbb{Z})$, $\alpha \in H^1(T;\mathbb{Z})$, $\beta \in H^1(T;\mathbb{Z})$, and $\gamma \in H^2(T;\mathbb{Z})$. The ring structure is given by $\alpha \smile \alpha = 0$, $\alpha \smile \beta = \gamma$, $\beta \smile \alpha = -\gamma$, and $\beta \smile \beta = 0$.*

Remark If we take advantage of the anticommutativity of the cup product in cohomology and the fact that $H^2(T;\mathbb{Z}) = \mathbb{Z}$, we only need to compute $\alpha \smile \beta$; the other three products are automatic.