Relations between Points and Subsets

Assume given a metric or topological space X and any subset $A \subset X$. We discuss the possible relationships between a point $x \in X$ and A in terms of neighborhoods.

First Level We note that every neighborhood N of x contains x (so is non-empty). At this level, we make no distinction between x and other points of N. This provides enough information for most of our work.

There are exactly *three* possibilities, with no overlap:

Case 1. x has a neighborhood N that is contained in A. We call x an *interior point* of A. Such points form the *interior set* $\operatorname{Int} A$, or sometimes $\mathring{A}$, of A. Since $x \in N \subset A$, we have $\operatorname{Int} A \subset A$.

Case 2. x has a neighborhood N that is contained in the complement $X - A$ of A. We call x an *exterior point* of A. Since $x \in N$, these points never lie in A. They form the *exterior set* $\operatorname{Ext} A = \operatorname{Int}(X-A)$ of A.

Case 3. Otherwise, *every* neighborhood N of x contains both a point of A and a point of $X - A$. We call x a *frontier point* or *boundary point* of A. Such points form the *frontier* or *boundary set*, $\operatorname{Fr} A$ or $\operatorname{Bd} A$, of A. These points may or may not lie in A. By symmetry, $\operatorname{Fr}(X-A) = \operatorname{Fr} A$.

To summarize, every point $x \in X$ lies in exactly one of the three sets $\operatorname{Int} A$, $\operatorname{Ext} A$, and $\operatorname{Fr} A$. The ambiguity in Case 3 motivates the next definition.

Definition 1 We call A *closed (in X)* if it contains all of its boundary points. We call A *open (in X)* if it contains none of its boundary points.

It is obvious from the symmetry that A is open if and only if $X - A$ is closed.

Lemma 2 *For any subset $A \subset X$:*

(a) *The interior* $\operatorname{Int} A$ *is open;*

(b) *If $V \subset A$ is open in X, then $V \subset \operatorname{Int} A$.* □

Thus $\operatorname{Fr} A = X - (\operatorname{Int} A \cup \operatorname{Ext} A) = X - (\operatorname{Int} A \cup \operatorname{Int}(X-A))$ is closed.

The *closure* $\operatorname{Cl} A = \overline{A}$ of A may be defined as $\operatorname{Int} A \cup \operatorname{Fr} A$ or as $A \cup \operatorname{Fr} A$. Since $\operatorname{Cl} A = X - \operatorname{Ext} A$, it is closed, and if F is any closed set that contains A, i. e. $F \supset A$, we must have $\operatorname{Cl} A \subset F$.

Second Level This is more subtle. At this level, we *do* distinguish between x and other points of N. The *limit points* and *isolated points* of A can now be defined. There are now *eight* possibilities, summarized in the table:

Case	Description	in A?	int?	limit point?
1.	x has a neighborhood N with $N - x$ empty.	$x \in A$	Int	isolated
		$x \notin A$	Ext	exterior
2.	x has a neighborhood N with non-empty $N - x \subset A$.	$x \in A$	Int	limit
		$x \notin A$	Fr	limit
3.	x has a neighborhood N with non-empty $N - x \subset X - A$.	$x \in A$	Fr	isolated
		$x \notin A$	Ext	exterior
4.	None of the above: every $N - x$ contains points of A and of $X - A$.	$x \in A$	Fr	limit
		$x \notin A$	Fr	limit