Relations between Points and Sets

Assume given a fixed metric space or topological space X and any subset $E \subset X$. We list the possible relationships between a typical point $p \in X$ and E in terms of neighborhoods. Every neighborhood N of p contains p (so is non-empty).

First Level At this level, we make no distinction between p and other points of N. This provides enough information for most of our work. There are exactly three possibilities, with no overlap:

Case 1. p has a neighborhood N that is contained in E. We call p an *interior point* of E. Such points form the *interior* $\operatorname{Int} E$ of E. Since $p \in N$, we have $\operatorname{Int} E \subset E$.

Case 2. p has a neighborhood N that is contained in the complement $E^c = X - E$ of E. We call p an *exterior point* of E. Since $p \in N$, these points never lie in E. They form the set $\operatorname{Int}(X - E)$.

Case 3. Otherwise, *every* neighborhood N of p contains both a point of E and a point of $X - E$. We call p a *boundary point* or *frontier point* of E. Such points form the *boundary* or *frontier* $\operatorname{Bd} E$ of E. These points may or may not lie in E. By symmetry, $\operatorname{Bd}(X - E) = \operatorname{Bd} E$.

To summarize, every point $p \in X$ lies in exactly one of the three sets $\operatorname{Int} E$, $\operatorname{Int}(X - E)$, and $\operatorname{Bd} E$. The ambiguity in Case 3 motivates the main definition.

Definition 1 We call E *closed (in X)* if it contains all of its boundary points. We call E *open (in X)* if it contains none of its boundary points.

It is obvious from the symmetry that E is open if and only if $X - E$ is closed.

Lemma 2 *For any subset $E \subset X$:*

(a) *The interior* $\operatorname{Int} E$ *is open;*

(b) *If $V \subset E$ is open in X, then $V \subset \operatorname{Int} E$.* □

Thus $\operatorname{Bd} E = X - (\operatorname{Int} E \cup \operatorname{Int}(X - E))$ is closed.

The *closure* $\operatorname{Cl} E = \overline{E}$ of E may be defined as $\operatorname{Int} E \cup \operatorname{Bd} E$ or as $E \cup \operatorname{Bd} E$. Since $\overline{E} = X - \operatorname{Int}(X - E)$, it is closed, and if F is any closed set that contains E, i. e. $F \supset E$, we must have $\overline{E} \subset F$.

Second Level This is more subtle. At this level, we *do* distinguish between p and other points of N. The *limit points* and *isolated points* of E can now be defined. There are now *eight* possibilities, summarized in the table:

Case	Description	in E?	int?	limit point?
1.	p has a neighborhood N with $N - p$ empty.	$p \in E$	int	isolated
		$p \notin E$	ext	exterior
2.	p has a neighborhood N with non-empty $N - p \subset E$.	$p \in E$	int	limit
		$p \notin E$	bd	limit
3.	p has a neighborhood N with non-empty $N - p \subset X - E$.	$p \in E$	bd	isolated
		$p \notin E$	ext	exterior
4.	None of the above: every $N - p$ contains points of E and of $X - E$.	$p \in E$	bd	limit
		$p \notin E$	bd	limit