Differentiability and the Tangent Plane

References are to Salas & Hille, *Calculus*, 7th Edition

Assume that $f(x,y)$ is defined in some neighborhood of the point (x_0,y_0) and let the surface S be the graph $z=f(x,y)$ of f. Alternatively, we can view S as the level surface $g=0$ of the function $g(x,y,z)=z-f(x,y)$. Let $P_0(x_0,y_0,z_0)$ be the point of S above (x_0,y_0), where $z_0=f(x_0,y_0)$. We interpret the differentiability of f analytically and geometrically.

Analytically Let $\mathbf{h}=(h_1,h_2)$ be a vector in $\mathbf{R}^2$, small enough for $f(x_0+h_1,y_0+h_2)$ to be defined.

DEFINITION 1 (cf. Defn. 15.1.1) The function f is *differentiable* at (x_0,y_0) if it has a good linear approximation near (x_0,y_0), in the sense that

$$f(x_0+h_1,y_0+h_2)=z_0+Ah_1+Bh_2+o(\mathbf{h}) \qquad \text{as } \mathbf{h}\to\mathbf{0}. \tag{2}$$

Let P be the nearby point on S

$$(x,y,z)=(x_0+h_1,y_0+h_2,f(x_0+h_1,y_0+h_2)),$$

so that $h_1=x-x_0$ and $h_2=y-y_0$. The linear approximation to f has the graph

$$z=z_0+A(x-x_0)+B(y-y_0), \tag{3}$$

which is the equation of the plane T through P_0 that is normal to the vector

$$\mathbf{n}=-A\,\mathbf{i}-B\,\mathbf{j}+\mathbf{k}. \tag{4}$$

We observe that this vector is just $\nabla g(P_0)$.

There is only one candidate for the coefficients A and B in equation (2).

LEMMA 5 *If f is differentiable at (x_0,y_0), we must have $A=\dfrac{\partial f}{\partial x}(x_0,y_0)$ and $B=\dfrac{\partial f}{\partial y}(x_0,y_0)$, and these partial derivatives do exist.*

Proof We do A only. In equation (2) we take $h_1=t$ and $h_2=0$; then

$$f(x_0+t,y_0)=f(x_0,y_0)+At+o(t) \qquad \text{as } t\to 0.$$

This shows that the limit $\dfrac{\partial f}{\partial x}(x_0,y_0)$ exists and has the value A. □

Geometrically We show that T is the *tangent plane* at P_0 to the surface S.

THEOREM 6 *If f is differentiable at (x_0,y_0), the angle between the vector $\overrightarrow{P_0P}$ and the plane T tends to 0 as $\mathbf{h}\to\mathbf{0}$.*

Proof We show instead that the angle θ between $\overrightarrow{P_0P}$ and $\mathbf{n}$ tends to $\pi/2$. It is given by

$$\cos\theta=\frac{\overrightarrow{P_0P}\bullet\mathbf{n}}{\|\overrightarrow{P_0P}\|\|\mathbf{n}\|}. \tag{7}$$

We have

$$\overrightarrow{P_0P} = h_1\,\mathbf{i} + h_2\,\mathbf{j} + (z - z_0)\mathbf{k}.$$

From equation (4), we compute and observe from equation (2) that

$$\overrightarrow{P_0P} \bullet \mathbf{n} = -Ah_1 - Bh_2 + z - z_0 = o(\mathbf{h}). \tag{8}$$

All we need to know about $\|\overrightarrow{P_0P}\|$ is that $\|\mathbf{h}\| \le \|\overrightarrow{P_0P}\|$. Substituting into equation (7), we find

$$\cos\theta \le \frac{o(\mathbf{h})}{\|\mathbf{h}\|}\frac{1}{\|\mathbf{n}\|} \to 0 \qquad \text{as } \mathbf{h} \to \mathbf{0}.$$

We apply the continuous function $\cos^{-1}(-)$ to get $\theta \to \cos^{-1}(0) = \pi/2$. □

*(Optional) The converse is also true, but somewhat trickier.

THEOREM 9 *If there exists a plane T through P_0 such that the angle between the vector $\overrightarrow{P_0P}$ and the plane T tends to 0 as $\mathbf{h} \to \mathbf{0}$, then f is differentiable at (x_0, y_0) and T is the tangent plane to S at P_0.*

Proof From the triangle inequality, we have

$$\|\overrightarrow{P_0P}\| \le \|\mathbf{h}\| + |z - z_0|. \tag{10}$$

It is first necessary to show that $z - z_0$ does not get too large compared to $\mathbf{h}$. This can be seen geometrically. Alternatively, it follows from equation (7) and the hypothesis that $\cos\theta \to 0$, that for small enough $\mathbf{h}$,

$$|z - z_0 - Ah_1 - Bh_2| \le \frac{1}{2}(\|\mathbf{h}\| + |z - z_0|).$$

We rearrange to get

$$\frac{1}{2}|z - z_0| \le |Ah_1 + Bh_2| + \frac{1}{2}\|\mathbf{h}\|$$

and hence $|z - z_0| \le M\|\mathbf{h}\|$, where we can take $M = 2|A| + 2|B| + 1$. Then from equation (10), $\|\overrightarrow{P_0P}\| \le (1 + M)\|\mathbf{h}\|$.

Now we can deduce with the help of equation (8) that

$$\frac{|z - z_0 - Ah_1 - Bh_2|}{\|\mathbf{h}\|} \le (1 + M)\frac{|z - z_0 - Ah_1 - Bh_2|}{\|\overrightarrow{P_0P}\|} = \|\mathbf{n}\|\cos\theta \to 0,$$

as required for Definition 1. □

Remark The surface S is not as special as might appear. If the function $g(x, y, z)$ is continuously differentiable in some neighborhood of P_0 and $\nabla g(P_0) \ne \mathbf{0}$, we can permute the coordinates to arrange $\dfrac{\partial g}{\partial z}(P_0) \ne 0$. A major theorem shows that near P_0, the level surface $g(x, y, z) = c$ (where $c = g(P_0)$) is the graph $z = f(x_0, y_0)$ of a certain continuously differentiable function f.